Queen Mary
University of London

MTH716U / MTHM007 **Measure Theory & Probability**

Problems 1 **November 2011**

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1 *Prove that if* $m^*(A) = 0$ *then* $m^*(A \cup B) = m^*(B)$ *for all* B.

2 *Prove that if* $m^*(A \Delta B) = 0$ *then* $m^*(A) = m^*(B)$.

3 *Prove that outer measure is translation-invariant, i.e. that* $m^*(A) = m^*(A+t)$ *for every* A *and* t, *where* $A+t := \{a+t : a \in A\}$.

4 *Find an infinite collection of subsets of* $\mathbb{R}$ *which contains* $\mathbb{R}$, *is closed under countable unions, and is closed under countable intersections, but is not a* σ*-field.*

5 *Let* Ω *be a set, and* $\mathscr{F}$ *a* σ*-field on* Ω. *If* $x, y \in \Omega$, *show that the (Dirac delta) measures* δ_x *and* δ_y *(defined on* $\mathscr{F}$*) are equal if and only if* x *and* y *belong to exactly the same subsets in* $\mathscr{F}$.

6 *Prove that every monotone function is measurable.*

7 *Prove that if* $f : \mathbb{R} \to \mathbb{R}$ *is a measurable function, then the set* $\{x : f(x) = c\}$ *is measurable for every* $c \in \mathbb{R}$.

8 *Given* $f : \mathbb{R} \to \mathbb{R}$, *show that the condition "*$\{x : f(x) = c\}$ *is measurable for all* $c \in \mathbb{R}$*" is* NOT *enough to guarantee that* f *is measurable.*

9 *Give an example of a non-measurable function* f *such that* f^2 *is measurable (where we define* f^2 *by* $f^2(x) = f(x)^2$*).*

10 *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable. Show that its derivative* f' *is a measurable function.*