Spectral Analysis of Dynamical Systems

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Preface

This dissertation, except where otherwise stated, is the result of my own work and contains nothing which is the outcome of work done in collaboration. Part of Chapter 5 is based on material presented in the author's M.Phil dissertation, which was approved for the degree of Master of Philosophy at the University of Cambridge. No other part of this dissertation has been submitted for a degree or diploma at this or any other university.

I would like to thank my supervisor, Dr. Peter Coveney, for providing me with an excellent introduction to the ideas of the Brussels school, as well as for his constant encouragement and guidance. His criticism, generosity, and patience have been present throughout the process of writing this dissertation.

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Chapter 1

Introduction

One of the fundamental tasks of physics is the study of time-dependent phenomena. At the heart of this endeavour is the notion of a dynamical system serving as a mathematical model for any object or process which evolves deterministically as time passes. In the present thesis we shall study dynamical systems which, although they are deterministic, exhibit random behaviour. This may be the case, for example, if the system comprises a large number of particles or if the system is chaotic. A fertile method of investigating these systems is to adopt a probabilistic point of view and study the associated evolution operators which implement a probabilistic description of the system.

One of the main themes of the present thesis is the idea that the fine structure of the probabilistic evolution can be analysed by applying the methods of spectral theory to the evolution operators of the system. Of central importance to this analysis is the notion of a test function space. We shall see that these test function spaces implement a mode of description for the dynamical system, and that information about the collective temporal behaviour of the system can be obtained from the spectrum of the associated evolution operator.

The same idea plays an important role in the so-called 'Brussels formalism', a framework concerned with the foundations of non-equilibrium statistical mechanics developed by large groups of researchers in Brussels and Austin, Texas. Most of the present thesis will be concerned with two versions of this formalism which we refer to as the 'spectral decomposition algorithm' and 'subdynamics'. Our main task will be to analyse these two versions of the formalism from a mathematically rigorous point of view. Using the notion of a test function space we shall construct examples of dynamical systems to which both versions of the formalism can be applied.

The background to the theory of dynamical systems and a brief description of the aims and the history of the Brussels formalism are presented in Chapter 2. Most of the subsequent chapters will deal with systems with a discrete time parameter, the exception being Chapter 6 where we consider systems with a continuous time parameter.

The subject of Chapters 3 and 4 is the spectral decomposition algorithm put forward by the Brussels/Austin groups. This method aims at providing a systematic procedure to obtain a physical representation of a system by means of a 'spectral decomposition' of the evolution operator of the system. In Chapter 3 we shall introduce the notion of a test function space for the evolution operator of a dynamical system. Using this concept we shall investigate to what extent the formal spectral decompositions of the Brussels/Austin groups qualify as a physical representation of the system. Throughout the chapter, a simple dynamical system will serve as an example to illustrate the basic ideas. In the following chapter we take the idea of a spectral decomposition further and show that the evolution operator of every exact dynamical system admits a spectral decomposition. We also show that the same system may have infinitely many different spectral decompositions.

In the final two chapters we shall study the older version of the Brussels formalism known as subdynamics. Chapter 5 will deal with the discrete time variant of subdynamics. Spectral methods will be employed to analyse those steps in the formalism which lead to the construction of the fundamental objects of the theory. Using the notion of a test function space in connection with the results of the previous two chapters we shall show that these steps can be justified rigorously for certain non-invertible dynamical systems. In the final chapter we shall consider the continuous time variant of subdynamics, and show that most of the results of the discrete time case carry over to the continuous time scenario.

The research presented in the main part relies heavily on the spectral theory of operators on locally convex spaces. This theory is developed in Appendix A. A shorter second appendix deals with $\mathcal{Z}$-transform techniques, which are used in Chapter 5. Appendix A is self-contained and may be read independently of the other parts. For the reader's convenience we included an index containing the less familiar terms and symbols used throughout the present thesis.

Chapter 2

Dynamical systems and the Brussels formalism

This chapter provides the background for the research to be presented in subsequent chapters. We introduce the general concept of a dynamical system and discuss the relationship between two important subclasses of dynamical systems: continuous time and discrete time dynamical systems. Probabilistic dynamical systems and their associated evolution operators implementing a probabilistic description are defined. After a description of the ergodic hierarchy we briefly discuss correlation functions and the related concept of a resonance. The last section contains a brief survey of the development of the Brussels formalism.

2.0.1 Notation Throughout, the symbol $\mathbb{Z}$ ($\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$) will denote the set of all integers (positive integers, real numbers, positive real numbers, complex numbers). For $z \in \mathbb{C}$, the symbol $\Re z$ ($\Im z$) will be used to denote the real (imaginary) part of z. We also use the abbreviations $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. Finally, the symbol $\mathbb{T} := \left\{ z \in \mathbb{C} \,\middle|\, |z| = 1 \right\}$ will denote the unit circle and the symbol $\mathbb{D} := \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1 \right\}$ the open unit disk.

2.1 Dynamical systems

The study of dynamics is the study of how things change as time passes. It began with the investigation of the motion of planets in ancient astronomy and reached its first climax at the end of the 17th century, when Newton laid the foundation for a quantitative theory of motion, by means of which Kepler's laws and Galileo's observations could be explained theoretically. The very concept of a dynamical system originated from the geometrisation of Newtonian mechanics due to Poincaré and Birkhoff, and now appears as a new paradigm yielding a common mathematical description for the time-dependent phenomena encountered in the various branches of natural science, such as physics, chemistry, biology, or economy.

By a dynamical system we mean any object or process for which it is possible to define the concept of state as an instantaneous description, the temporal evolution of the system being given by the temporal evolution of the state. In its present form the theory of dynamical systems constitutes a body of ideas and techniques drawn from various mathematical disciplines such as analysis and topology, probability and number theory, which aims at treating the major types of behaviour or evolution without direct reference to the actual matter through which they are manifested. The basic objects of this mathematical framework are an abstract space which represents the collection of states of the system, and a group or semigroup of transformations acting on the space which implements the temporal evolution of the system.

Groups arise in the description of reversible processes for which the dynamical rule furnishes the entire history of the system from the distant 'past' to the distant 'future', while irreversible or dissipative processes for which the time-symmetry is broken need to be modelled by semigroups. Accordingly, one may distinguish between dynamical systems and semidynamical systems. Since every dynamical system is also a semidynamical system, there is certainly no loss of generality by considering semidynamical systems only, and in the present thesis we shall use the term dynamical system in this broader sense. At the same time we note that the transition between these two types of description is at the heart of the controversial issue of the foundations of non-equilibrium statistical mechanics. We shall briefly return to this problem at the end of this chapter when we discuss the Brussels formalism (Section 2.6).

The abstract space on which the semigroup of transformation acts, the so-called *state space* or *phase space* of the system, usually carries some additional structure, which is preserved by each element of the semigroup of transformations. The space may, for example, be a measure space, a topological space, or a smooth manifold, while the elements of the semigroup are measurable transformations, continuous transformations, or smooth transformations, respectively. Accordingly, one speaks of *measurable dynamics* (or *ergodic theory*), *topological dynamics*, or *differentiable dynamics*.

To illustrate the language of the theory, we consider the motion of n particles in a region of three-dimensional space. Adopting a classical point of view, we assume that the positions and momenta of each of the n particles suffice to provide a complete instantaneous description of the system. In this case, the state of the system may be identified with a point in $\mathbb{R}^{6n}$. We also assume that the temporal evolution is governed by a system of first-order differential equations known as Hamilton's equations. Provided that these equations can be uniquely solved for a given initial condition the solution curve or *trajectory* in $\mathbb{R}^{6n}$ determines a transformation T_t on the phase space of the system sending the state at time zero into the state at time t:

$$x_t = T_t(x_0).$$

In mathematical terms, the dynamical rule $t \mapsto T_t$ embodying the law of evolution of the system is an action of the group $\mathbb{R}$ on the state space, that is, T_t ($t \in \mathbb{R}$) is a transformation on the state space with T_0 being the identity on $\mathbb{R}^{6n}$ and $T_{t+s} = T_t \circ T_s$. Since the temporal evolution of the system is governed by a Hamiltonian, a theorem of Liouville asserts that each transformation T_t preserves the natural $6n$-dimensional measure on the phase space. We may thus regard the system as a measurable dynamical system. The topological and differentiable set-ups arise by restricting T_t to a specified energy surface. In this case, the restriction is continuous and even smooth if the energy surface is smooth. Depending on one's point of view, one may now use the methods of measurable, topological or differentiable dynamics to analyse this system.

Apart from serving as an example to illustrate the language of the theory of dynamical systems, this model of n particles moving classically in three-dimensional space, played a decisive role in the history of this field. In at least two instances — the case '$n = 3$', the so-called three-body problem, and the case 'n very large' — the analysis of this model turned out to be instrumental for the subsequent shift in the focus of attention which marks the advent of dynamical systems theory. When Poincaré found that the general solution of the three-body problem could not be obtained by quadrature, which triggered off the subsequent crisis of analytical dynamics, he himself pointed out the importance of a qualitative theory based on topological methods. This observation resulted in the development of topological dynamics, which received the first important contributions from Birkhoff and Lyapunov at the beginning of the century.

The second case arose from the attempts of Boltzmann and Gibbs to reduce thermodynamics to classical mechanics. Using our example with n very large as a model for a piece of matter, they tried to relate the quantities and laws of thermodynamics to the underlying microscopic behaviour of the system by making as few assumptions as possible concerning the nature of the microscopic dynamics. Their work started two new disciplines: statistical mechanics as the direct continuation of their original programme, and the abstract study of group-actions on measure spaces, or measurable dynamics.

2.2 Discrete time and continuous time dynamical systems

Dynamical systems may be classified according to the semigroup which acts on the system space. In practice, only two types of actions are studied: $\mathbb{R}_0^+$-actions, which are usually referred to as *continuous time* systems or *flows*, and $\mathbb{N}_0$-actions, known as *discrete time* systems or *maps*. The former are closer to our intuitive notion of time as a continuum, and thus more physical, while the latter are more easily analysed, since their study reduces to that of a single transformation $T : X \to X$ acting on the state space X of the system and its iterates $\{T^n\}_{n\in\mathbb{N}_0}$. Discrete time dynamical systems are studied not only for their intrinsic beauty, for their relevance to other branches of mathematics, and because they serve as toy-models in certain problems of physics, but also since many problems concerning continuous time systems can be reduced to discrete time systems. This reduction can be effected in a number of ways.

Given a flow $\{T_t\}_{t\in\mathbb{R}_0^+}$, a somewhat crude procedure is based on choosing a sufficiently small 'unit of time' t_0, and considering the 'discretised' flow $\{T_{t_0}^n\}_{n\in\mathbb{N}_0}$. Since $T_{t_0}^n = T_{nt_0}$ for $n \in \mathbb{N}_0$, we expect the discretised flow to be a good approximation to the original flow, provided that t_0 is of the order of the intrinsic time scale of the system, for example, the mean free time between collisions in a hard-sphere model of a gas.

A better way of coding a flow by a discrete time system can sometimes be achieved by considering the *Poincaré* or *first return map* of the system (see, for example, [AP2]). Let us assume that the state space of the flow is a smooth manifold, and that the flow is smooth, or at least continuous. If we can find a hypersurface V transversal to the flow, a so-called surface of section, then we may associate a mapping $T : V \to V$, the so-called first return map, with the flow as follows. Given a point $v_0 \in V$ we follow the trajectory of the flow starting at v_0 forward in time, until it again hits the hypersurface V for the first time at a point, say v_1. The mapping T is then defined by

$$Tv_0 = v_1$$

for all $v_0 \in V$ for which a v_1 given by the above procedure exists. If all the forward trajectories intersect V, then the Poincaré map determines the behaviour of the system to a considerable extent. It is not always possible to find such a 'global' surface of section, although in the vicinity of a periodic point we can always find local sections. This procedure discretises the flow, and additionally reduces the dimension of the state space by one. Thus, the analysis of a three-dimensional continuous time system may be reduced to studying a two-dimensional map.

Often a further reduction of the problem is possible, which leads to one-dimensional interval transformations. A famous example where this is possible is the Lorenz model of thermal convection in a horizontal fluid layer. The model is described by three coupled ordinary differential equations, which provide a relation between three components of an infinite number of spatial Fourier components of the velocity and temperature fields. To a good approximation, the first return map of this three-dimensional continuous time dynamical system, determined by a plane containing two unstable fixed points of the flow, can be described by a piece-wise monotonic one-dimensional map with a cusp-like shape. Details about the Lorenz model and this reduction procedure can be found in the book by Bergé *et. al.* [BPV] or the article by Yorke and Yorke [YY].

In certain models of statistical mechanics, the so-called *billiards*, the possibility of a discrete description is already inherent in the very formulation of the continuous time evolution. Here one considers the inertial motion of one or more particles on a two-dimensional billiard table with elastically scattering boundary. The billiard is studied by means of a mapping, which sends the position and angle of a collision of the particle with the boundary into the position and angle of the next [Bum].

Whatever the reason or mechanism leading to a discrete time description, maps and their iterates offer a convenient tool to simplify the analysis and to reduce the complexity of continuous time systems. Most of the present thesis will deal with discrete time systems (Chapters 3–5), the exception being Chapter 6 where continuous time systems will be considered.

2.3 Probabilistic description of dynamical systems

The discovery of deterministic chaos in the late sixties generated a new interest in dynamical systems. Although a universally accepted definition of the term 'chaos' is still lacking to date, it is commonly used to describe the phenomenon that deterministic systems can exhibit unpredictable and stochastic behaviour. Chaotic motion is usually thought to originate from the fact that the trajectories of certain systems depend crucially on their starting point, a property which is often referred to as 'sensitive dependence on initial conditions'. Experimentally indistinguishable trajectories of systems with this property will follow quite different paths in time and the observer, limited in his precision, will come to the conclusion that he is observing an irregular and unpredictable process. One may thus argue that chaotic systems call for a probabilistic description complementing the usual description in terms of trajectories. We shall adopt this idea and focus attention on systems which carry a probabilistic structure.

The background in measure and probability theory may be found in the classic book by Halmos [Hal]. We note that a probability space $(X, \mathcal{A}, \mu)$, which we always assume to be a Lebesgue space, is given by the following data: a set X, a σ-algebra $\mathcal{A}$ of measurable subsets of X, and a probability measure μ.

In the following definition, G^+ denotes either the semigroup $(\mathbb{R}_0^+, +, 0)$ or the semigroup $(\mathbb{N}_0, +, 0)$. The symbol $\mathcal{B}^+$ will be used for the restriction of the Borel σ-algebra of the topological group $\mathbb{R}$ (respectively, $\mathbb{Z}$) to $\mathbb{R}_0^+$ (respectively, $\mathbb{N}_0$).

2.3.1 Definition Let $(X, \mathcal{A}, \mu)$ be a probability space. A family of transformations $\left\{ T_g : X \to X \,\middle|\, g \in G^+ \right\}$ satisfying the conditions

(i) T_g is measure preserving for every $g \in G^+$, that is,

$$\mu(T_g^{-1} A) = \mu(A) \text{ for } A \in \mathcal{A},\ g \in G^+;$$

(ii) $T_{g_1+g_2} = T_{g_1} \circ T_{g_2}$ for $g_1,\ g_2 \in G^+$;

(iii) $T_0 x = x$ for μ-a.e. $x \in X$;

(iv) the mapping $(x, g) \mapsto T_g x$ from $(X \times G^+, \mathcal{A} \times \mathcal{B}^+)$ into $(X, \mathcal{A})$ is measurable;

is called a *probabilistic dynamical system* and will be denoted by

$$(X, \mathcal{A}, \mu; \{T_g\}_{g \in G^+}).$$

The probabilistic dynamical system is said to be *invertible* if T_g is invertible (mod 0) for every $g \in G^+$.

2.3.2 Remark

(i) If $T : X \to X$ is a measure preserving transformation of the probability space $(X, \mathcal{A}, \mu)$, then $(X, \mathcal{A}, \mu; \{T^n\}_{n \in \mathbb{N}_0})$ is clearly a (discrete time) probabilistic dynamical system, and we write $(X, \mathcal{A}, \mu; T)$ in this case.

(ii) An invertible probabilistic dynamical system can be extended to a group-action of $\mathbb{R}$ or $\mathbb{Z}$ by setting

$$T_{-g} := (T_g)^{-1} \text{ for every } g \in G^+.$$

Our definition of a probabilistic dynamical system requires the existence of an invariant measure. In other words, by restricting attention to probabilistic dynamical systems, we assume that the system can be described by means of a *stationary* stochastic process. Over the last twenty years the problem of finding suitable invariant measures, which describe relevant aspects of the temporal evolution of a dynamical system, has been intensively investigated, and, by now, there exists an impressive body of results which guarantee the existence of a suitable invariant measure. We shall not touch this subject here, and refer the reader to the books by Lasota and Mackey [LM] and Beck and Schlögl [BS], which provide an easily accessible account of these techniques. Subsequently, we shall deal with probabilistic dynamical systems only, and suppress the adjective 'probabilistic' if no confusion is possible.

Having fixed the scenario we need some idea of how probability distributions evolve under the dynamics of a system. Before we proceed let us recall that with any probability space $(X, \mathcal{A}, \mu)$ we can associate the complex Banach spaces $L^p(X, \mathcal{A}, \mu) = \left\{ f : X \to \mathbb{C} \,\middle|\, \int |f|^p \, d\mu < \infty \right\}$ $(p \geq 1)$ with the usual norms denoted by $\|.\|_p$. We use $L^0(X, \mathcal{A}, \mu)$ to denote the space of all complex-valued measurable functions on $(X, \mathcal{A}, \mu)$, and $L^\infty(X, \mathcal{A}, \mu)$ for the space of all complex-valued essentially bounded measurable functions on $(X, \mathcal{A}, \mu)$. A *probability density* on $(X, \mathcal{A}, \mu)$ is a positive integrable function f with $\int f \, d\mu = 1$.

Suppose now that we are given a measure preserving transformation T on the probability space $(X, \mathcal{A}, \mu)$. Our aim is to find a transformation U_T which yields the image of a probability density under the action of the map T. To do this we note that a probability density f defines a probability measure ν on X given by

$$\nu_f(A) = \int_A f \, d\mu \text{ for } A \in \mathcal{A}.$$

The image $\nu_f T^{-1}$ of this measure under the measure preserving transformation T given by

$$\nu_f T^{-1}(A) = \nu_f(T^{-1}A) \text{ for } A \in \mathcal{A}$$

is again a probability measure, which is absolutely continuous with respect to μ. An application of the Radon-Nikodým theorem now yields the transformation U_T

$$U_T f := \frac{d}{d\mu} \nu_f T^{-1},$$

where $\frac{d}{d\mu}$ denotes the Radon-Nikodým derivative of the measure $\nu_f T^{-1}$. Since the probability densities span $L^1(X, \mathcal{A}, \mu)$, the transformation U_T extends to a linear operator on all of $L^1(X, \mathcal{A}, \mu)$. This operator is called the Frobenius-Perron operator of the transformation T.

2.3.3 Definition Let $(X, \mathcal{A}, \mu)$ be a probability space and let $T : X \to X$ be a measure-preserving transformation. The *Frobenius-Perron operator* U_T *of* T is defined by

$$U_T f = \frac{d}{d\mu} \nu_f T^{-1} \text{ for } f \in L^1(X, \mathcal{A}, \mu),$$

where the complex measure $\nu_f T^{-1}$ is given by

$$\nu_f T^{-1}(A) = \int_{T^{-1}A} f \, d\mu \text{ for } A \in \mathcal{A}.$$

A close relative of the Frobenius-Perron operator is the Koopman operator, which describes the backward evolution of probability densities.

2.3.4 Definition Let $(X, \mathcal{A}, \mu)$ be a probability space. If $T : X \to X$ is a measure-preserving transformation, then the *Koopman operator* $\bar{U}_T$ *of* T is given by

$$\bar{U}_T f = f \circ T \text{ for } f \in L^0(X, \mathcal{A}, \mu).$$

The following proposition describes the basic properties of these two operators.

2.3.5 Proposition *Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $T : X \to X$ be a measure-preserving transformation.*

(i) The Frobenius-Perron operator U_T is a contraction on $L^p(X, \mathcal{A}, \mu)$ for $p \geq 1$, that is,

$$\|U_T f\|_p \leq \|f\|_p \text{ for } f \in L^p(X, \mathcal{A}, \mu);$$

(ii) the Koopman operator is an isometry on $L^p(X, \mathcal{A}, \mu)$ for $p \geq 1$, that is,

$$\left\|\bar{U}_T f\right\|_p = \|f\|_p \text{ for } f \in L^p(X, \mathcal{A}, \mu);$$

(iii) if T is invertible, then

$$U_{T^{-1}} = U_T^{-1} \text{ and } \bar{U}_{T^{-1}} = \bar{U}_T^{-1};$$

(iv) if $1 \leq p < \infty$ then U_T is the Banach space adjoint of $\bar{U}_T$ on $L^p(X, \mathcal{A}, \mu)$;

(vi) U_T is the Hilbert space adjoint of $\bar{U}_T$ on $L^2(X, \mathcal{A}, \mu)$, that is,

$$\int \bar{U}_T f \cdot \overline{g} \, d\mu = \int f \cdot \overline{U_T g} \, d\mu \text{ for } f, g \in L^2(X, \mathcal{A}, \mu).$$

Proof See [LM, 3.2 and 3.3].

QED

The following proposition describes the properties of the Frobenius-Perron operator and the Koopman operator of a dynamical system.

2.3.6 Proposition *Let $(X, \mathcal{A}, \mu; \{T_g\}_{g \in G^+})$ be a dynamical system. Then*

(i) $U_{T_0} = I$ and $\bar{U}_{T_0} = I$;

(ii) $U_{T_{g_1+g_2}} = U_{T_{g_1}} U_{T_{g_2}}$ *and* $\bar{U}_{T_{g_1+g_2}} = \bar{U}_{T_{g_1}} \bar{U}_{T_{g_2}}$ *for* $g_1,\, g_2 \in G^+$*;*

(iii) $\lim_{g\to g_0} \left\| U_{T_g} f - U_{T_{g_0}} f \right\|_2 = 0$ *for* $f \in L^2(X, \mathcal{A}, \mu)$.

Proof Assertions (i) and (ii) follow immediately from the definition of U_T and $\bar{U}_T$. The last assertion is trivial if $G^+ = \mathbb{N}_0$. Suppose now that $G^+ = \mathbb{R}_0^+$. Condition (iii) of the definition of a dynamical system implies that the function $\mathbb{R}_0^+ \ni t \mapsto U_{T_t}$ is weakly measurable, that is, for every $f,\, g \in L^2(X, \mathcal{A}, \mu)$, the function

$$\mathbb{R}_0^+ \ni t \mapsto \int g \circ T_t \cdot \overline{f}\, d\mu$$

is measurable. This in turn implies that $t \mapsto U_{T_t}$ is strongly continuous (this follows, for example, from the proof of theorem V.4 in [Yos]).

QED

We saw that with a dynamical system we can associate two semigroups of *evolution operators*. The semigroup of Frobenius-Perron operators implements the forward evolution of probability densities under the action of the system, and the semigroup of Koopman operators the backward evolution of probability densities. If the dynamical system is invertible, both semigroups extend to groups of isometric isomorphisms on $L^p(X, \mathcal{A}, \mu)$ $(p \geq 1)$, and both semigroups extend to strongly continuous groups of unitary operators on $L^2(X, \mathcal{A}, \mu)$. This follows from the last two propositions. Since the two evolution operators map $L^p(X, \mathcal{A}, \mu)$ into itself for every p, we may choose any of them.

The most natural choice for the Frobenius-Perron operator is $L^1(X, \mathcal{A}, \mu)$, as we have a clear interpretation of its positive elements as probability densities. On the other hand, it is more convenient to consider the restriction of the Frobenius-Perron operator to $L^2(X, \mathcal{A}, \mu)$, since we may then use its powerful structure as a Hilbert space. We shall see later that the domain of the Frobenius-Perron operator has to be restricted even further, if we want to obtain information about the relaxation time of the corresponding dynamical system. A large part of Chapter 3 is devoted to setting up a mathematical framework enabling a systematic study of these restrictions.

We conclude this section with a characterisation of the L^2-spectrum[1] of the Frobenius-Perron operator of a discrete time dynamical system.

[1] By this we mean the spectrum of the Frobenius-Perron operator acting on $L^2(X, \mathcal{A}, \mu)$.

2.3.7 Proposition *Let $(X, \mathcal{A}, \mu; T)$ be a discrete time dynamical system, and let $U := U_T$ denote the associated Frobenius-Perron operator.*

(i) If T is invertible then U is unitary; in particular, $\sigma(U|L^2) \subset \mathbb{T}$.

(ii) If T is non-invertible then the L^2-spectrum of U is the closed unit disk $\sigma(U|L^2) = \mathrm{clo}\,\mathbb{D}$; each $\lambda \in \mathbb{D}$ is an eigenvalue of infinite algebraic multiplicity. If, in addition, $(X, \mathcal{A}, \mu)$ is non-atomic then the geometric multiplicity of each eigenvalue is infinite as well.

Proof The proof relies on an application of the Wold decomposition [RR, 1.3] to the isometric Koopman operator $\bar{U}_T =: \bar{U}$ as follows.

Letting F denote the orthogonal complement of

$$\bar{U} L^2(X, \mathcal{A}, \mu) = L^2(X, T^{-1}\mathcal{A}, \mu)$$

in $L^2(X, \mathcal{A}, \mu)$ we have

$$L^2(X, \mathcal{A}, \mu) = E_1 \oplus E_2,$$

where $E_1 = \bigcap \left\{ \bar{U}^n L^2(X, \mathcal{A}, \mu) \,\middle|\, n \geq 0 \right\}$ and $E_2 = \sum_{n\geq 0} \bar{U}^n F$ are reducing subspaces for $\bar{U}$ with the property that $\bar{U}|E_1$ is unitary and $\bar{U}|E_2$ is the unilateral right-shift of multiplicity $\dim F$ (see Section 4.1 for the definition).

(i) If T is invertible then $T^{-1}\mathcal{A} = \mathcal{A}$. Thus, $\dim F = 0$ and $\bar{U}$ is unitary.

(ii) If T is non-invertible then $T^{-1}\mathcal{A} \subsetneq \mathcal{A}$. Thus, $\dim F > 0$. Since

$$U = \bar{U}^* = (\bar{U}|E_1)^* \oplus (\bar{U}|E_2)^*$$

and $(\bar{U}|E_2)^*$ is the unilateral left-shift, each $\lambda \in \mathbb{D}$ is an eigenvalue of infinite algebraic multiplicity by Beurling's lemma (see lemma 4.1.6). Finally, if $\mathcal{A}$ has no atoms, then $T^{-1}\mathcal{A} \subset \mathcal{A}$ has no atoms either. Thus, F is infinite-dimensional by virtue of Rohlin's lemma (see [Par, 3.3.5 lemma]).

QED

The basic assertion of this proposition that either $\sigma(U) \subset \mathbb{T}$ or $\sigma(U) = \mathrm{clo}\,\mathbb{D}$ is well-known (see, for example, [Kel2]). However, as far as we are aware, neither the simple proof based on the Wold decomposition nor the fine structure of the spectrum in (ii) has yet appeared in the literature.

2.4 The ergodic hierarchy

Probabilistic dynamical systems may be classified according to their degree of 'randomness'. This classification establishes a hierarchy in which each level is occupied by a class of systems with a certain property. These properties are

(i) ergodic;

(ii) weakly mixing;

(iii) mixing (or strongly mixing);

(iv) (a) countable Lebesgue spectrum;
 (b) countable one-sided Lebesgue spectrum;

(v) (a) K-property;
 (b) exact;

(vi) Bernoulli.

The subdivision of properties (iv) and (v) refers to invertible (a) and non-invertible (b) systems. We shall briefly discuss these properties for discrete time systems and indicate the necessary modifications for continuous time systems.

Let $(X, \mathcal{A}, \mu; T)$ be a discrete time dynamical system. The system is *ergodic* if the only sets which are invariant under T are the trivial subsets of X, that is, the space itself and the null-set. This means that an ergodic system is 'irreducible' in the sense that it is impossible to subdivide the phase space into smaller parts on which the system can be studied separately. An important consequence of ergodicity is that averages over time equal averages over the phase space, a property which is frequently used to link experimental data of a system with quantities of the underlying theoretical model.

The system is said to be *mixing* if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B) \text{ for } A,\, B \in \mathcal{A},$$

and is said to be *weakly mixing* if the above limit exists in the Cesàro sense. Both properties can be interpreted to mean that a small 'patch' in the phase space is eventually uniformly dispersed over all of the phase space, implying that an observer with limited precision will lose any information about the system in the long run.

The next level in this hierarchy differs from the previous ones in that it amounts to a spectral characterisation of the Koopman operator of T in terms of 'shift operators'. Systems belonging to this class will play an important role in the present thesis, as we shall exploit the spectral properties of the associated evolution operators to study the decay of correlation functions. The reader is referred to Chapter 4 for more details, in particular to Section 4.1, where the definition of the various shift operators can be found.

2.4.1 Definition Let $(X, \mathcal{A}, \mu; T)$ be a discrete time dynamical system, and let $\bar{U}$ be the Koopman operator of T.

(a) T has *countable Lebesgue spectrum* if the restriction of $\bar{U}$ to the orthocomplement of the constant functions is unitarily equivalent to the bilateral right-shift of infinite multiplicity $\bar{\mathcal{S}}_F$, that is,

$$(\bar{U}|L^2(X, \mathcal{A}, \mu) \ominus \mathbb{C}) \cong (\bar{\mathcal{S}}_F|L^2(\mathbb{T}; F))$$

with $\dim F = \infty$.

(b) T has *countable one-sided Lebesgue spectrum* if $\bar{U}$ restricted to the orthocomplement of the constant functions is unitarily equivalent to the unilateral right-shift of infinite multiplicity $\bar{S}_F$, that is,

$$(\bar{U}|L^2(X, \mathcal{A}, \mu) \ominus \mathbb{C}) \cong (\bar{S}_F|H^2(\mathbb{D}; F))$$

with $\dim F = \infty$.

2.4.2 Remark
The above definition may be cast in the following form, which is perhaps more familiar in the context of ergodic theory. It characterises T in terms of a 'translation representation' of its Koopman operator.

(a) T has countable Lebesgue spectrum if there exists a countable family of functions $\left\{ f_k \in L^2(X, \mathcal{A}, \mu) \,\middle|\, k \in \mathbb{N}_0 \right\}$ such that

$$\{1\} \cup \left\{ \bar{U}^n f_k \,\middle|\, n \in \mathbb{Z}, k \in \mathbb{N}_0 \right\}$$

is an orthonormal basis for $L^2(X, \mathcal{A}, \mu)$.

(b) T has countable one-sided Lebesgue spectrum if there exists a countable family of functions $\left\{ f_k \in L^2(X, \mathcal{A}, \mu) \,\middle|\, k \in \mathbb{N}_0 \right\}$ such that

$$\{1\} \cup \left\{ \bar{U}^n f_k \,\middle|\, n \in \mathbb{N}_0, k \in \mathbb{N}_0 \right\}$$

is an orthonormal basis for $L^2(X, \mathcal{A}, \mu)$.

Here, 1 denotes the constant function $1 : X \to \mathbb{C}$ with $1(x) = 1$.

The next level in this hierarchy is defined as follows.

2.4.3 Definition Let $(X, \mathcal{A}, \mu; T)$ be a dynamical system.

(a) T is is said to be a *K-system*, or is said to have the *K-property* if there exists a sub-σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$ such that

(i) $T^{-1}\mathcal{A}_0 \subset \mathcal{A}_0$;

(ii) $\bigwedge_{n=0}^{\infty} T^{-n}\mathcal{A}_0 = \mathcal{N}$;

(iii) $\bigvee_{n=0}^{\infty} T^{n}\mathcal{A}_0 = \mathcal{A}$,

where $\mathcal{N}$ denotes the trivial algebra.

(b) T is *exact* if

$$\bigwedge_{n=0}^{\infty} T^{-n}\mathcal{A} = \mathcal{N}.$$

K-systems (respectively, exact systems) may be interpreted to be abstractions of regular identically distributed stochastic processes indexed by $\mathbb{Z}$ (respectively, $\mathbb{N}_0$).

The last property corresponds to an abstraction of an independent identically distributed stochastic process, and represents an even higher level of randomness. It shall not concern us further.

The term ‘hierarchy’ is justified in view of the following implications:

$$\text{(vi)} \Rightarrow \text{(v)(a)} \Rightarrow \text{(iv)(a)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)} \tag{2.1}$$

for invertible systems, and

$$\text{(vi)} \Rightarrow \text{(v)(b)} \Leftrightarrow \text{(iv)(b)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)} \tag{2.2}$$

for non-invertible systems. We note that a system is exact if and only if it has countable one-sided Lebesgue spectrum, but that all other implications cannot be reversed in general.

The properties giving rise to the ergodic hierarchy are reflected in the L^2-spectrum of the associated evolution operators. More precisely, all properties are 'spectral invariants', that is, the Koopman operators of two systems belonging to the same level are unitarily equivalent. For example, a system is ergodic if and only if 1 is a simple eigenvalue of $\bar{U}$ with geometric multiplicity 1, and a system is weakly mixing if and only if 1 is the only eigenvalue of $\bar{U}$. Details of these results and concrete examples of systems belonging to these classes may be found in the standard texts on ergodic theory such as [CFS, Par, Pet].

The continuous time analogues of all levels of randomness apart from (iv) can be obtained by simply replacing the discrete time parameter $n \in \mathbb{Z}$ (or $n \in \mathbb{N}_0$) by the continuous time parameter $t \in \mathbb{R}$ (or $t \in \mathbb{R}_0^+$), the interpretation of these properties being the same.

To our knowledge, the continuous time version of (iv)(b) has not received much attention in the literature. We are not aware of a proof that exactness implies countable one-sided Lebesgue spectrum[2]. For this reason we shall state this level of randomness for invertible systems only.

For $t \in \mathbb{R}$ the symbol $\{e^{itw}\}$ denotes the operator of multiplication

$$(e^{itw} f)(x) = e^{itx} f(x)$$

on $L^2(\mathbb{R}; F)$, F a separable Hilbert space (see [RR] for the definition of this space).

2.4.4 Definition Let $(X, \mathcal{A}, \mu; \{T_t\})$ be an invertible continuous time dynamical system, and let $\{\bar{U}_t\}$ denote the associated group of Koopman operators. Then $\{T_t\}_{t\in\mathbb{R}}$ has *countable Lebesgue spectrum* if there is a unitary transformation V from $L^2(X, \mathcal{A}, \mu) \ominus \mathbb{C}$ onto $L^2(\mathbb{R}; F)$ such that

$$\bar{U}_t | (L^2(X, \mathcal{A}, \mu) \ominus \mathbb{C}) = V^{-1} e^{itw} V \text{ for every } t \in \mathbb{R}.$$

As is well known, it is difficult to show that concrete continuous time systems are ergodic. However, over the past thirty years a variety of systems of

[2] Using the Nagy-Foiaş dilation theory [NF] one may readily show that exactness implies that the associated semigroup has homogeneous Lebesgue spectrum. The difficult part is to establish that the multiplicity is infinite.

physical interest have in fact been proven to enjoy the K-property. These include the infinite ideal gas [CFS], the hard rods system [AGL], the motion of n billiard balls on a table with convex boundaries (see [GO] for $n = 1$ and [BLPS] for n arbitrary), the hard sphere gas [Sin], and the geodesic flow on a manifold of negative curvature [Ano, Orn].

2.5 Correlation functions and resonances

Time-correlation functions are studied with a variety of motivations: to detect a transition from ordered to chaotic motion; to test whether a system relaxes to equilibrium; or to determine the transport coefficients for dynamical variables (see, for example, [Rei, GG]). As they are often the only experimentally accessible quantities they provide an important link between theory and experiment. Suppose, for simplicity, that $(X, \mathcal{A}, \mu; T)$ is a discrete time dynamical system. We define the *correlation function* $C_{fg}(n)$ for the L^2-functions f and g by

$$C_{fg}(n) = \int f \cdot g \circ T^n \, d\mu - \int f \, d\mu \cdot \int g \, d\mu \text{ for } n \in \mathbb{N}_0. \tag{2.3}$$

In physical terms, the correlation function could be interpreted to represent the successive outcomes of measurements of the variable g performed on a system initially prepared in the probabilistic state f.

In practice, the correlation function is calculated from an experimental time-series by assuming that the system is ergodic, since then $C_{fg}(n)$ is equal to

$$C_{fg}(n) = \lim_{m\to\infty} \frac{1}{m} \sum_{l=0}^{m-1} f(T^l x) \cdot g(T^{l+n} x) - \lim_{m\to\infty} \frac{1}{m} \sum_{l=0}^{m-1} f(T^l x) \cdot \lim_{m\to\infty} \frac{1}{m} \sum_{l=0}^{m-1} g(T^l x)$$

for μ-almost every initial point $x \in X$.

If additionally the system is strong-mixing then the correlation function decays to zero for all square-integrable functions f and g:

$$\lim_{n\to\infty} C_{fg}(n) = 0.$$

This property is usually interpreted as a 'memory loss' of the system, that is, the system quickly forgets how it was set up. An important element in the study of correlation functions is the analysis of the asymptotics of this decay. Complete analytical treatment for systems of physical interest is a formidable task, as the solution will depend on intrinsic properties of the system as well as on the way the system is observed. The asymptotics of the correlation functions for billiards, for example, is still subject to debate [Bum, Che].

Systems which have been extensively studied from this point of view include a class of smooth dynamical systems known as 'Axiom A systems' (see [Bow1, Rue1]). For these systems it has been shown that the correlation functions of smooth observables are modulated by a sum of decreasing exponentials. Separating the contributions of the decaying exponentials by taking the Fourier transform of the correlation functions, that is, by considering the *power spectrum*, gave rise to the notion of a 'resonance' [Eck, Rue2].

In a series of papers, Ruelle showed that in the case of Axiom A diffeomorphisms with smooth observations, the power spectrum is meromorphic in a strip containing the real axis, that is, the singularities in this region can at worst be poles [Rue2, Rue3, Rue4]. He called these poles, suggestive of the pronounced peaks in the power spectra of certain experimental time series, *resonances*, and showed that the location of the resonances does not depend on the choice of observables. He also showed that the resonances are given by the reciprocals of the poles of the Ruelle ζ-function. This function encodes information about the distribution of periodic orbits of the system and is defined as

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x=T^n(x)} g_n(x)\right).$$

Here, the second sum extends over the fixed points of the iterates of the system T, and g_n is a certain weight function.

The relationship between the resonances of a system and the poles of the Ruelle ζ-function has also been established for non-invertible systems[3], including:

- piecewise monotonic expanding interval transformations with observations of bounded variation [BK];
- piecewise analytic Markov maps with analytic observables [May] (extensions to the non-Markovian case are considered in [Kel2]);

[3]For interval transformations T the weight function g_n takes the form $g_n(x) := \prod_{k=0}^{n-1} \frac{1}{|T'(T^k x)|}$ with T' denoting the derivative of T (see [May]).

- expanding maps with differentiable observables [Rue5].

We shall see later (Chapter 4) that the power spectrum of a system may have poles which are not related to the Ruelle ζ-function. In order to distinguish between arbitrary poles in the power spectrum and those poles which can be obtained from the Ruelle ζ-function, we shall call the former *resonances* and the latter *Ruelle resonances.*

Once the resonances of a system are determined quite a lot is known about the fine structure of the approach to the asymptotic distribution of the system. The resonance closest to the real axis determines the characteristic decay rate of correlation functions, while the subdominant resonances contain information about the modulation of this decay on shorter time scales. We shall investigate these relationships and the difference between resonances and Ruelle resonances in more detail in Chapters 3 and 4.

2.6 The Brussels formalism

As in the theory of chaotic dynamical systems, dynamical and statistical considerations combine in the field of non-equilibrium statistical mechanics, which aims to derive the phenomenological laws describing the behaviour of matter in bulk from the underlying fundamental laws of nature governing the motion of the constituting particles. The epistemological background of this subject is still very much disputed, the most prominent conceptual difficulty being the origin of irreversibility. There are at least three conflicting schools of thought concerned with the general foundations[4] of this subject as exemplified by the writings of Lebowitz [Leb], Jaynes [Jay], and Prigogine [Pri2]. Few authors have sought to compare the different approaches, the notable exception being Dougherty [Dou], who studied the relationship between the 'Maxent school' associated with Edwin Jaynes, and the 'Brussels school' associated with Ilya Prigogine.

The approach of the latter school is also known as the *Brussels formalism*, a term which refers to a set of mathematical tools and ideas, developed by large groups of researchers in Brussels and Austin, Texas under the direction

[4]General accounts may be found in the paper by Penrose [Pen], or in the books by Davies [Dav4] and Zeh [Zeh]. The philosophical perspective is explored in a highly recommended recent book by Price [Pri]. Also recommended from a different perspective is the best-selling popular account of Coveney and Highfield [CH].

of Ilya Prigogine. The conceptual *leitmotiv* of this approach is the idea that macroscopic behaviour should be explained without appealing to any 'extra-dynamical' considerations, such as coarse-graining or the use of information theory [Pri2]. Over the past thirty years the formalism has undergone considerable development giving rise to three major 'sub-formalisms', which we refer to as:

- subdynamics;
- the Misra-Prigogine-Courbage theory;
- the spectral decomposition algorithm.

All versions of the formalism employ a probabilistic description of dynamical systems. The temporal behaviour is usually assumed to be given by the Liouville-von Neumann equation,

$$i\frac{d}{dt}\varrho = L\varrho, \tag{2.4}$$

which governs the evolution of a classical probability density or a quantal density matrix ϱ under the action of the Liouville operator L of the system.

Subdynamics, the earliest version of the formalism, is concerned with the derivation of kinetic equations, that is, equations which govern the temporal evolution of reduced descriptions of a system. The main elements of this approach can be traced back to the work of Prigogine and collaborators in the early sixties ([Pri1] and references therein), who studied large systems of interacting particles employing a method known as 'dynamics of correlations'.

At the heart of subdynamics is the idea that there should be a non-trivial class of probability distributions on the phase-space of a system which simultaneously satisfy a kinetic equation and the law of evolution imposed by the underlying microscopic dynamics. This class is selected by constructing an idempotent operator Π which projects onto an invariant subspace of L such that the elements of this subspace obey an autonomous evolution equation, which is the desired kinetic equation. The evolution of the system thus splits into two independently evolving parts each following its own 'subdynamics'. This version of the formalism is explained in detail in the book by Balescu [Bal] and in some of the original articles, for example, [PGH, GPR, PGHR, GGG].

Further development of subdynamics in the early seventies led to the idea of a 'transformation theory' (see, for example, [PGHR]), in which an invertible transformation Λ implements the transition to a 'kinetic' description of the system. Subsequently, the Λ-transformation became the focus of attention resulting in a reformulation of the formalism. In a series of papers Misra, Prigogine, and Courbage investigated the possibility of obtaining a Markov process from a deterministic dynamical system through a 'change of representation' involving no loss of information [MPC1, MPC2, MP, CM]. The change of representation is effected by an invertible positivity-preserving transformation Λ, which intertwines the unitary group associated with a dynamical system with a semigroup of contractions associated with a Markov process. For this reason, dynamical systems for which Λ can be constructed are called 'intrinsically random'. There is a close connection with ergodic theory: strong-mixing is necessary [Mis] and the K-property sufficient [GMC] for a dynamical system to be intrinsically random.

At the heart of this version of the formalism is the idea that reversible dynamical evolution and irreversible thermodynamic evolution can be reconciled owing to the high delocalisation of trajectories in phase space exhibited by certain dynamical systems, such as K-systems. The Misra-Prigogine-Courbage theory differs from subdynamics in that it seeks to attribute the thermodynamic behaviour of large systems to the 'full' dynamics of the system, rather than to the effect of reducing the level of description.

In recent years, the Brussels/Austin groups have turned their attention to the study of 'large Poincaré systems', a special class of non-integrable dynamical systems characterised by Liouvillians, the spectra of which become continuous in the large system limit [PH, PPT, AP1, PP2], and chaotic maps [HS1, HS2, HS3, AT1, AT2, AT3, AQ]. The methods employed in the investigation of these systems are much closer to subdynamics than to the Misra-Prigogine-Courbage theory. However, emphasis has shifted from deriving kinetic equations to obtaining 'generalised spectral decompositions' of the evolution operator[5] of the system. Various procedures to construct generalised spectral decompositions have been suggested. The method of Hasegawa and Saphir [HS3], for example, differs from the one outlined by Antoniou and Tasaki [AT3], which in turn differs from the approach employed by Petrosky and Prigogine [PP2]. In the concrete examples discussed by these authors, however, the different procedures yield the same result. For want of a better name we shall refer to this version of the Brussels formalism as the

[5]That is, the Liouvillian for continuous time systems and the Frobenius-Perron operator for discrete time systems.

spectral decomposition algorithm.

The idea of a generalised spectral decomposition originated from the observation that formal solutions of the eigenvalue problem for models of unstable systems, such as the Friedrichs model (see, for example, [AP1]), give rise to eigenvalues with a non-vanishing imaginary part. To give meaning to the corresponding formal eigenvectors, which may be interpreted as unstable states, the original Hilbert space is discarded for a less conventional mathematical structure, known as a 'rigged Hilbert space' [AP1, AT3]. Within this structure the formally constructed eigenvalues and eigenvectors can be assembled to yield a generalised spectral decomposition of the evolution operator. The new spectral representation is understood to provide a physical and intrinsically irreversible representation of the system. Here, 'physical' means that dynamical properties of the system are reflected in the spectrum[6], while 'intrinsically irreversible' refers to the fact that, as a result of the construction, the group of evolution operators splits into two semigroups, one decaying in the 'future', the other in the 'past'. Irreversibility is then seen to arise through the choice of the semigroup which is compatible with future observations (see [AP1, AT3]).

We shall avoid the discussion as to what extent the Brussels formalism is able to explain the origin of irreversibility, and limit ourselves to the following remark. Irreversibility is by no means explained, if, through some procedure, one is able to split the dynamical group into *two* semigroups[7]. Explaining irreversibility requires an additional selection rule distinguishing between these two semigroups, as is pointed out by the Brussels school. It is, however, not entirely satisfactory to base this selection procedure on the compatibility with future observations, as what needs to be explained is *why* only one semigroup is compatible with future observations.

[6] The lifetimes of the Friedrichs model, for example, are related to the spectral values of the generalised decomposition.

[7] The existence of *two* 'intrinsically irreversible' representations, one 'physical', the other 'unphysical', is not a prerogative of the spectral decomposition algorithm. Within the framework of the Misra-Prigogine-Courbage theory, for example, one may construct two Λ-transformations, one describing the transition to a 'physical' Markov semigroup, the other describing the transition to an 'unphysical' Markov semigroup (see, for example, Kumičák and Brändas [KB1], who also offer a critique of the Misra-Prigogine-Courbage theory and suggest a more 'conventional' explanation of irreversibility).

Chapter 3

Discrete time systems: test function spaces and spectral decompositions

In this chapter, we shall investigate the temporal evolution of discrete time dynamical systems by analysing the spectral properties of a restriction of the evolution operator. This means that instead of considering the evolution operator on the original Hilbert space we shall restrict it to a subspace and then study its spectral properties. From a technical point of view this idea plays an important role in establishing the existence of suitable invariant measures for interval transformations [HK] and smooth dynamical systems [Rue1]. From a more philosophical perspective Penrose and Coveney [PC] have argued that some restriction needs to be placed on the L^2 universe as not every L^2 function corresponds to a physically realisable preparation of the system. A mixture of both the technical and the conceptual aspect of this idea can be found in the investigations dealing with the decay properties of correlation functions [GO].

This idea has also been used in recent work of the Brussels/Austin groups to provide a mathematical framework for the generalised spectral decomposition algorithm put forward by Hasegawa and Saphir [HS1, HS2, HS3, SH] and Antoniou and Tasaki [AT1, AT2, AT3]. Their method aims at providing a systematic procedure to construct eigenvalues and eigenfunctions of the evolution operator of a chaotic dynamical system. The algorithm amounts to a successive triangularisation of a matrix representation of the evolution operator. A suitable biorthogonal system or 'basis' is chosen and the corresponding matrix elements of the operator are calculated. The matrix is then transformed into an upper triangular matrix through an iterative procedure. The triangularised matrix is referred to as the *(generalised) spectral decomposition* of the evolution operator. The procedure suggested by Hasegawa and Saphir is different from the algorithm employed by Antoniou and Tasaki. We shall not discuss these differences here as we are mostly concerned with the interpretation of the spectral decomposition.

In the concrete examples discussed by these authors, which include both invertible and non-invertible systems such as the baker's transformation or the Rényi map, the diagonal entries of the spectral decomposition are found to be located in the unit disk and are interpreted as eigenvalues of the operator. These eigenvalues are shown to correspond to the Ruelle resonances of the system, and the spectral decomposition is referred to as a 'physical representation' of the system as it incorporates the characteristic time scales of the system.

We saw earlier that the L^2-spectrum of the Frobenius-Perron operator of a dynamical system is either a subset of the unit circle or the whole unit disk (see proposition 2.3.7). Thus, the question arises as to how the spectral decomposition is to be understood. Hasegawa and Saphir [HS3] and Antoniou and Tasaki [AT3] point out that their algorithm and the spectral decomposition are meaningless in the original Hilbert space L^2, and suggest that it should be replaced by a 'rigged Hilbert space'. In this framework, the formal spectral decomposition can be understood by restricting the domain of the Frobenius-Perron operator of the system to suitable dense subspaces of L^2 carrying a locally convex topology, the so-called *test function spaces*.

In this chapter we shall take up this idea of studying the restrictions of evolution operators associated with dynamical systems to suitable test function spaces. Our main task will be to interpret the notion of a spectral decomposition used in [HS1] and [AT3], and to investigate to what extent spectral decompositions qualify as physical representations of dynamical systems. In doing so we shall limit ourselves to non-invertible systems. Our analysis is based on the idea that these concepts should be understood in terms of operator spectral theory in locally convex spaces. This theory, which is presented in Appendix A, will enable us to use the notion of spectrum for the non-normable test function spaces considered in [HS3, AT2, AT3], which in turn makes it possible to link the holomorphic structure of the resolvent of the evolution operator with observable quantities.

In the first two sections we take a fresh look at the Rényi map, which throughout the chapter will serve as an example to illustrate the basic concepts. We present a number of test function spaces for this map and calculate the corresponding spectra of the Frobenius-Perron operator. We shall then state the general definition of a test function space and suggest a possible ontology in terms of its relevance for the study of the long-time behaviour of a dynamical system. After a discussion of resonances in the locally convex setting, we shall present a definition of the term 'spectral decomposition' and show that the Frobenius-Perron operator of the Rényi map meets our conditions.

3.1 The Rényi map

Let β be a real number greater than one and define a transformation T_β on the closed unit interval $[0,1]$ by

$$T_\beta : [0,1] \to [0,1];$$

$$T_\beta(x) := \beta x \bmod 1.$$

The map T_β is called the *β-adic Rényi map* after Alfréd Rényi, who first studied the ergodic properties of this family of maps [Ren]. He showed that, for each $\beta > 1$, there is a unique T_β-invariant measure ν_β equivalent to the Lebesgue measure, such that $([0,1], \mathcal{B}, \nu_\beta; T_\beta)$[1] is ergodic. Not much later, Rohlin [Roh] proved that $([0,1], \mathcal{B}, \nu_\beta; T_\beta)$ is also exact.

The map T_β takes a particularly simple form if β is an *integer*. In this case, which has been considered by several authors [Roe, HS1, AT2, Hat], the invariant measure is the Lebesgue measure, which we shall denote by m, and the associated Frobenius-Perron operator is easily calculated (see [LM, remark 3.2.2]):

$$U_{T_\beta} : L^2([0,1], \mathcal{B}, m) \to L^2([0,1], \mathcal{B}, m),$$

$$(U_{T_\beta} f)(x) = \frac{1}{\beta} \sum_{k=0}^{\beta-1} f(\phi_k(x)), \tag{3.1}$$

where ϕ_k denotes the inverse of T_β on its k-th interval of monotonicity,

$$\phi_k(x) := \frac{x+k}{\beta}.$$

[1] $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$.

From now on, we shall assume that $\beta > 1$ is an integer, and we shall also drop the subscript T_β to avoid cluttered notation.

Since the Rényi maps are exact, the associated Frobenius-Perron operators are unitarily equivalent to the unilateral left-shift. In our case the unitary operator effecting the shift representation is essentially the Fourier transform. As in [AT2], this may be seen by considering the functions

$$\psi_n^s(x) := \exp(2\pi i s \beta^n x), \tag{3.2}$$

where $n \in \mathbb{N}_0$ and $s \in \mathcal{Z}_\beta$ with

$$\mathcal{Z}_\beta := \left\{ s \in \mathbb{Z} \,\middle|\, s/\beta \notin \mathbb{Z} \right\}.$$

The set $\left\{ \psi_n^s \,\middle|\, n \in \mathbb{N}_0, s \in \mathcal{Z}_\beta \right\}$ is simply the Fourier basis of $L^2 \ominus \mathbb{C}$ and the action of U on this basis is given by

$$U\psi_n^s = \begin{cases} \psi_{n-1}^s & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \end{cases}. \tag{3.3}$$

Another special feature of the Rényi map is that the Frobenius-Perron operator has polynomial eigenfunctions, the so-called Bernoulli polynomials. This was first pointed out by Roepstorff in an unpublished paper [Roe], where he showed that

$$(UB_n)(x) = \beta^{-n} B_n(x) \text{ for } n \in \mathbb{N}_0;$$

here, B_n denotes the n-th order Bernoulli polynomial (see [AS, Chapter 23]). These polynomial eigenfunctions were also obtained by Gaspard [Gas], Hasegawa and Saphir [HS3], and Antoniou and Tasaki [AT2], and these authors also point out that the Frobenius-Perron operator has a 'spectral decomposition' of the form

$$(Uf)(x) = \int_0^1 f(y)\,dy + \sum_{n=1}^{\infty} \beta^{-n} \frac{1}{n!}[f^{(n-1)}(1) - f^{(n-1)}(0)]B_n(x), \tag{3.4}$$

where $f^{(k)}$ denotes the k-th derivative of f. The above expression is obviously defined for infinitely differentiable f only. Moreover, it is known to converge for certain entire functions only. Thus, expression (3.4) cannot be a representation of the Frobenius-Perron operator on L^2 .

On the other hand, the Frobenius-Perron operator U has a number of invariant subspaces. Since U is a finite sum of operators of the form $f \mapsto f \circ \phi$

with ϕ being a linear function, U maps polynomials (entire functions, differentiable functions) into polynomials (entire functions, differentiable functions). It turns out that if one considers the Frobenius-Perron operator not as an operator on L^2 but as an operator on a 'smaller' space, for example, the space of polynomials, then expression (3.4) does represent the Frobenius-Perron operator of the Rényi map, and may be called a spectral decomposition. To get a better understanding of the phenomenon that the indecomposable L^2-Frobenius-Perron operator of the Rényi map admits a spectral decomposition on a smaller subspace, we shall study the restriction of U to a sequence of dense subspaces of L^2. This will be the content of the next section.

3.2 Test function spaces for the Rényi map

We shall now investigate the spectral properties of the Frobenius-Perron operator U of the Rényi map. More precisely, we shall consider the restriction of U to the following spaces:

- the inductive limit $\mathcal{P}$ of polynomials in one variable;
- the inductive limit $\tilde{\mathcal{E}}_c$ of entire functions of exponential type less than c;
- the Fréchet space $\mathcal{C}^\infty$ of infinitely differentiable functions on the closed unit interval;
- the Banach space $\mathcal{C}^n$ of n-times continuously differentiable functions on the closed unit interval;
- the Banach space $\mathcal{C}_\alpha$ of Hölder-continuous functions with exponent α.

Apart from $\mathcal{C}^n$ and $\mathcal{C}_\alpha$ none of these spaces is normable. The analysis of the spectral properties of U on these spaces relies heavily on the spectral theory of operators on locally convex spaces. The reader is referred to Appendix A for an account of this theory as well as for the notation to be used throughout this chapter.

The space of polynomials

A closer look at the Frobenius-Perron operator U of the Rényi map reveals that it maps polynomials into polynomials, and that it even preserves the degree of a polynomial. In other words

$$U(\mathcal{P}_n) \subset \mathcal{P}_n, \tag{3.5}$$

where $\mathcal{P}_n$ denotes the space of polynomials with complex coefficients and degree not exceeding n. We may thus regard U as an operator on $\mathcal{P}_n$, that is, we shall consider the restriction of U to $\mathcal{P}_n$, which we denote by $U|\mathcal{P}_n$. The spectrum of $U|\mathcal{P}_n$ is easily calculated using the canonical basis $\left\{ x \mapsto x^i \,\middle|\, 0 \leq i \leq n \right\}$ of $\mathcal{P}_n$. By (3.5), the matrix representation $(U|\mathcal{P}_n)_{ij}$ of $U|\mathcal{P}_n$ with respect to this basis is upper triangular, and a simple calculation shows that the diagonal entries are given by $(U|\mathcal{P}_n)_{ii} = \beta^{-i}$. Thus, the spectrum of $U|\mathcal{P}_n$ is given by

$$\sigma(U|\mathcal{P}_n) = \left\{ \beta^{-i} \,\middle|\, 0 \leq i \leq n \right\}. \tag{3.6}$$

We note that, since the diagonal entries are all distinct, the geometric and algebraic multiplicity of each eigenvalue is 1.

These observations allow us to calculate the spectrum of U on the space of all polynomials $\mathcal{P}$. Before we proceed we need to equip this space with a topology. Let π_{nm} denote the canonical embedding $\pi_{nm} : \mathcal{P}_n \hookrightarrow \mathcal{P}_m$ for $n \leq m$. Then $\{\mathcal{P}_n; \pi_{nm}\}$ is an inductive system, and the definition

$$\mathcal{P} := \underrightarrow{\mathrm{Ind}}\, \mathcal{P}_n$$

turns $\mathcal{P}$ into a complete and barrelled locally convex space (see A.7.6).

3.2.1 Proposition *The Frobenius-Perron operator of the β-adic Rényi map is a continuous transformation on $\mathcal{P}$. The spectrum is given by*

$$\sigma(U|\mathcal{P}) = \left\{ \beta^{-k} \,\middle|\, k \in \mathbb{N}_0 \right\} \cup \{0\}; \tag{3.7}$$

each non-zero element of the spectrum is an eigenvalue with geometric and algebraic multiplicity equal to 1.

Proof Observe that (3.5) implies that $\{U|\mathcal{P}_n\}$ is a map of the inductive system $\{\mathcal{P}_n; \pi_{nm}\}$ into itself. Thus, $U|\mathcal{P}$ is continuous by theorem (A.7.8). The characterisation of the spectrum (3.7) follows from corollary A.7.11, and the last assertion is obvious.

QED

Spaces of entire functions of exponential growth

Let c be a positive real number, and let f be an entire function, that is, f is holomorphic on $\mathbb{C}$. If there is a $K > 0$, such that

$$|f(z)| \leq Ke^{c|z|} \text{ for } z \in \mathbb{C},$$

then f is called an *entire function of exponential growth c*. The collection of all entire functions of exponential growth c is a vector space, which we shall denote by $\mathcal{E}_c$. It is not difficult to see that $\mathcal{E}_c$ equipped with the norm

$$\|f\|_c := \sup_{z \in \mathbb{C}} |f(z)|\, e^{-c|z|}$$

is a Banach space[2]. Obviously, every entire function of growth c is also of growth c', whenever $c < c'$. Thus, $\mathcal{E}_c$ can be embedded into $\mathcal{E}_{c'}$. This embedding is not only continuous, but even nuclear (see [Sch, III.7.1] for the definition).

3.2.2 Lemma *For $c < c'$ the canonical embedding $\mathcal{E}_c \hookrightarrow \mathcal{E}_{c'}$ is nuclear.*

Proof Let $f \in \mathcal{E}_c$, and expand f into its Taylor series at 0

$$f(z) = \sum_{n=0}^{\infty} \alpha_n(f) z^n.$$

The functions $z \mapsto z^n$, $n \in \mathbb{N}_0$, are obviously entire functions of exponential growth c', and satisfy

$$\|z^n\|_{c'} = \sup_{r \geq 0} r^n e^{-cr} = \left(\frac{n}{ec'}\right)^n \quad \text{for } n \in \mathbb{N}_0, \tag{3.8}$$

with the convention '$0^0 = 1$'. For $r > 0$, an application of Cauchy's formula yields the estimate

$$|\alpha_n(f)| \leq \frac{1}{2\pi} \oint_{|\zeta|=r} \frac{\|f\|_c \, e^{cr}}{r^{n+1}} \, d\zeta = \|f\|_c \frac{e^{cr}}{r^n}.$$

[2]Entire functions of exponential growth are closely related to entire functions of exponential type (see [BB]). In fact, f is an entire function of exponential type τ if and only if $f \in \bigcap \left\{ \mathcal{E}_c \,\middle|\, \tau < c \right\}$.

By specifying $r = n/c$, we obtain

$$|\alpha_n(f)| \leq \|f\|_c \left(\frac{ec}{n}\right)^n \text{ for } n \in \mathbb{N}_0, \tag{3.9}$$

with the convention '$\left(\frac{1}{0}\right)^0 = 1$'. Choose c_1 and c_2 with $c < c_1 < c_2 < c'$, and let $f_n \in \mathcal{E}_{c'}$ and $a_n \in \mathcal{E}_c'$ be defined as

$$f_n(z) := \left(\frac{ec_2}{n}\right)^n z^n \text{ for } n \in \mathbb{N}_0,$$

$$a_n(f) := \left(\frac{n}{ec_1}\right)^n \alpha_n(f) \text{ for } n \in \mathbb{N}_0.$$

The estimates (3.8) and (3.9) yield

$$\|f_n\|_{c'} \leq (c_2/c')^n \leq 1 \text{ for } n \in \mathbb{N}_0, \tag{3.10}$$

$$|a_n(f)| \leq \|f\|_c (c/c_1)^n \leq \|f\|_c \text{ for } n \in \mathbb{N}_0. \tag{3.11}$$

Define $i : \mathcal{E}_c \to \mathcal{E}_{c'}$ by

$$i(f) := \sum_{n=0}^{\infty} (c_1/c_2)^n a_n(f) f_n = \sum_{n=0}^{\infty} (c_1/c_2)^n \langle f, a_n \rangle f_n.$$

By construction, i is the canonical embedding of $\mathcal{E}_c$ into $\mathcal{E}_{c'}$. Moreover, i is nuclear by virtue of (3.10) and (3.11).

QED

Before we show that the Frobenius-Perron operator is nuclear on $\mathcal{E}_c$ we require another lemma.

3.2.3 Lemma *Let $c > 0$. For every c' with $c \leq \beta c'$, the Frobenius-Perron operator U of the β-adic Rényi map is continuous as a transformation from $\mathcal{E}_c$ into $\mathcal{E}_{c'}$.*

Proof Observe that the Frobenius-Perron operator is a finite sum of operators of the form $f \mapsto f \circ \phi_k$. Clearly, each ϕ_k extends to an entire function on the complex plane. Consequently, U maps entire functions into entire functions. The lemma is proved, if we are able to show that there is a $K > 0$ such that

$$\|Uf\|_{c'} \leq K \|f\|_c \text{ for every } f \in \mathcal{E}_c.$$

Let $f \in \mathcal{E}_c$. Then

$$\|f \circ \phi_k\|_{c'} = \sup_{z\in\mathbb{C}} \left| f\left(\frac{z+k}{\beta}\right)\right| e^{-c'|z|} = \sup_{z\in\mathbb{C}} |f(z)|\, e^{-c'|\beta z - k|}.$$

Now $|\beta z - k| \geq \beta\,|z| - \beta$, and, therefore,

$$-c'\,|\beta z - k| \leq -c\,|z| + \beta c'.$$

Thus

$$\|f \circ \phi_k\|_{c'} \leq e^{\beta c'} \sup_{z\in\mathbb{C}} |f(z)|\, e^{-c|z|} = e^{\beta c'}\, \|f\|_c\,,$$

and the desired inequality is proved, since the Frobenius-Perron operator is just a finite sum of operators of this kind.

QED

3.2.4 Proposition *If $c > 0$ then the Frobenius-Perron operator U of the Rényi map is a nuclear operator on $\mathcal{E}_c$. In particular, $\sigma(U|\mathcal{E}_c)$ is totally disconnected.*

Proof Choose c' with $\beta^{-1}c \leq c' < c$. By the preceding lemma, U maps $\mathcal{E}_c$ continuously into $\mathcal{E}_{c'}$. Let $\hat{U}$ be the Frobenius-Perron operator viewed as a transformation from $\mathcal{E}_c$ into $\mathcal{E}_{c'}$. Then $U|\mathcal{E}_c$ factors into

$$U|\mathcal{E}_c = i \circ \hat{U},$$

where i is the embedding of $\mathcal{E}_{c'}$ into $\mathcal{E}_c$. Since this embedding is nuclear, and $\hat{U}$ is continuous, $U|\mathcal{E}_c$ is nuclear by a well-known property of nuclear transformations (see [Sch, III.7.1 corollary 2]). To prove the last assertion we note that a nuclear operator is compact. Compact operators, however, are known to have completely disconnected spectra (see [Gro, Chapter 5 Part 2.4]).

QED

Spaces of entire functions of exponential type

Let c be a positive real number, and put

$$\tilde{\mathcal{E}}_c := \bigcup \left\{ \mathcal{E}_{c'} \,\middle|\, c' < c \right\}.$$

The space $\tilde{\mathcal{E}}_c$ is the space of all entire functions of exponential type strictly less than c. The space $\tilde{\mathcal{E}}_c$ can also be generated by a countable family. To see this, put $c_n := c - \frac{1}{n}$, and note that we have $\tilde{\mathcal{E}}_c = \bigcup \left\{ \mathcal{E}_{c_n} \,\middle|\, n \in \mathbb{N} \right\}$. If we denote by π_{nm} the canonical embedding $\pi_{nm} : \mathcal{E}_{c_n} \hookrightarrow \mathcal{E}_{c_m}$ $(n \leq m)$, then $\{\mathcal{E}_{c_n}; \pi_{nm}\}$ is an inductive system, which, by lemma 3.2.2, is also compact. Putting

$$\tilde{\mathcal{E}}_c := \underrightarrow{\operatorname{Ind}}\, \mathcal{E}_{c_n}$$

turns $\tilde{\mathcal{E}}_c$ into a complete and barrelled locally convex space[3] by virtue of theorem A.7.6. A factorisation argument similar to the one employed in the last section yields the following.

3.2.5 Proposition *If $c > 0$ then the Frobenius-Perron operator U of the Rényi map is a nuclear operator on $\tilde{\mathcal{E}}_c$. In particular, $\sigma(U|\tilde{\mathcal{E}}_c)$ is totally disconnected.*

Proof By lemma 3.2.3, we can choose a space $\mathcal{E}_{c_m}$ from the generating sequence $\{\tilde{\mathcal{E}}_{c_n}\}_{n\in\mathbb{N}}$ such that the Frobenius-Perron operator maps $\mathcal{E}_c$ continuously into $\mathcal{E}_{c_m}$. Let $\hat{U}$ denote the Frobenius-Perron operator viewed as a transformation from $\mathcal{E}_c$ into $\mathcal{E}_{c_m}$. If i denotes the embedding $i : \tilde{\mathcal{E}}_c \to \mathcal{E}_c$, and π_m denotes the canonical map $\pi_m : \mathcal{E}_m \to \tilde{\mathcal{E}}_c$, then $U|\tilde{\mathcal{E}}_c$ factors into

$$U|\tilde{\mathcal{E}}_c = \pi_m \circ \hat{U} \circ i.$$

Now, i is continuous by proposition A.7.8, and π_m is nuclear since all connecting maps π_{nm} are nuclear. Thus, $U|\tilde{\mathcal{E}}_c$ is nuclear.

QED

Spaces of differentiable functions

Let $n \in \mathbb{N}$, and let $\mathcal{C}^n$ denote the Banach space of all complex-valued differentiable functions f on $[0,1]$ with the usual norm

$$\|f\|_n := \sup_{0\leq k\leq n} \sup_{x\in[0,1]} \left| f^{(k)}(x) \right|.$$

We note that the canonical embedding $\mu_{mn} : \mathcal{C}^m \hookrightarrow \mathcal{C}^n$ $(n \leq m)$ is continuous. The projective limit $\mathcal{C}^\infty$ of the projective system $\{\mathcal{C}^n; \mu_{mn}\}$,

$$\mathcal{C}^\infty := \underleftarrow{\operatorname{Proj}}\, \mathcal{C}^n,$$

[3] This space is even nuclear, since the connecting maps are nuclear. An alternative proof and more details about these spaces may be found in [SATB].

can be identified with the space of all infinitely differentiable functions on $[0,1]$.

It is easily verified that U maps $\mathcal{C}^n$ continuously into itself, that is,

$$U|\mathcal{C}^n \in \mathcal{L}(\mathcal{C}^n). \tag{3.12}$$

The characterisation of $\sigma(U|\mathcal{C}^n)$ relies on the notion of the 'essential spectrum', the definition of which may be found in Appendix A p. 160.

3.2.6 Proposition *Let $n \in \mathbb{N}$. Then*

(i) the essential spectral radius satisfies $r_e(U|\mathcal{C}^n) = \beta^{-n}$;

(ii) $\left\{ \lambda \,\middle|\, |\lambda| \le \beta^{-n} \right\} \subset \sigma(U|\mathcal{C}^n)$.

Proof Part of the proof of (i), that is, $r_e(U|\mathcal{C}^n) \le \beta^{-n}$, is based on the Ionescu-Tulcea Marinescu theorem [IM] which provides an upper bound for the essential spectral radius[4].

Let $n \in \mathbb{N}$ and observe that

$$\sup_{x\in[0,1]} \left|(Uf)^{(k)}(x)\right| \le \beta^{-k} \sup_{x\in[0,1]} \left|f^{(k)}(x)\right| \text{ for } k \le n.$$

This inequality implies that

$$\|Uf\|_n \le \|f\|_{n-1} + \beta^{-n} \|f\|_n .$$

Since the embedding $\mathcal{C}^n \hookrightarrow \mathcal{C}^{n-1}$ is compact by the Arzela-Ascoli theorem (see, for example, [Rud1, 11.28]), the above inequality implies that $r_e(U|\mathcal{C}^n) \le \beta^{-n}$ by virtue of the Ionescu-Tulcea Marinescu theorem in the form given by Hennion [Hen].

This estimate is sharp since each λ with $\lambda < \beta^{-n}$ belongs to the $\mathcal{C}^n$-spectrum of U. This was already pointed out in [AT2] and [Hat], and may be seen as follows. Let ψ_m be the functions defined earlier (3.2):

$$\psi_m(x) := \psi_m^1(x) = \exp(2\pi i \beta^m x) \text{ for } m \in \mathbb{N}_0 .$$

[4]This theorem is frequently invoked to establish ergodic properties of interval transformations (see, for example, [HK]).

We note that $\psi_m \in \mathcal{C}^n$ for every $m \in \mathbb{N}_0$ and that $\|\psi_m\|_n = (2\pi\beta^m)^n$. Thus, the series $f_\lambda := \sum_{m=0}^{\infty} \lambda^m \psi_m$ converges in $\mathcal{C}^n$ for all complex λ with $|\lambda| < \beta^{-n}$. Using (3.3) we obtain

$$Uf_\lambda = \sum_{m=0}^{\infty} \lambda^m U\psi_m = \lambda \sum_{m=0}^{\infty} \lambda^m \psi_m = \lambda f_\lambda.$$

Hence

$$\left\{ \lambda \in \mathbb{C} \,\middle|\, |\lambda| \leq \beta^{-n} \right\} \subset \sigma(U|\mathcal{C}^n),$$

which proves assertion (ii) and the remaining part of assertion (i).

QED

This proposition states that the radius of the essential spectrum shrinks to zero as the 'smoothness' of the functions increases. We may now apply proposition A.7.20 to get a characterisation of the spectrum of U on the 'limit space', that is the space $\mathcal{C}^\infty$.

3.2.7 Proposition *The spectrum of the Frobenius-Perron operator on $\mathcal{C}^\infty$ is totally disconnected. Each non-zero point in the spectrum is a pole with finite rank residue.*

Proof Observe that equation (3.12) means that $\{U|\mathcal{C}^n\}$ is a map of the projective system $\{\mathcal{C}^n; \mu_{mn}\}$. Moreover, $\mathcal{C}^m$ is dense in $\mathcal{C}^n$, whenever $n \leq m$. The assertion now follows from proposition A.7.20.

QED

Hölder-continuous functions

Let $\alpha \in (0, 1]$ and let $\mathcal{C}_\alpha$ denote the Banach space of all complex-valued Hölder-continuous functions on the unit interval with exponent α under the usual norm

$$\|f\|_\alpha = \sup_{x\in[0,1]} |f(x)| + \sup_{\substack{x,y\in[0,1] \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

It is easily verified that

$$U|\mathcal{C}_\alpha \in \mathcal{L}(\mathcal{C}_\alpha).$$

Methods similar to those employed in the discussion of spaces of differentiable functions yield the following characterisation of the spectrum of $U|\mathcal{C}_\alpha$.

3.2.8 Proposition *Let* $\alpha \in (0,1]$. *Then*

(i) $r_e(U|\mathcal{C}_\alpha) = \beta^{-\alpha}$;

(ii) $\left\{ \lambda \in \mathbb{C} \,\middle|\, |\lambda| \leq \beta^{-\alpha} \right\} \subset \sigma(U|\mathcal{C}_\alpha)$.

Proof For $f \in \mathcal{C}_\alpha$ we have the following inequality

$$\|Uf\|_\alpha \leq \sup_{x\in[0,1]} |f(x)| + \beta^{-\alpha} \|f\|_\alpha .$$

Since the embedding $\mathcal{C}_\alpha \to \mathcal{C}^0$ is compact (see [EE, V1.1]) the above inequality yields the estimate $r_e(U|\mathcal{C}_\alpha) \leq \beta^{-\alpha}$, again by virtue of the Ionescu-Tulcea Marinescu theorem. Assertion (ii), which is proved in [AT2], implies that this estimate is sharp.

QED

Synopsis

So far, we have seen that the Frobenius-Perron operator of the Rényi map is a continuous operator on each element of the chain of spaces

$$\mathcal{P} \hookrightarrow \mathcal{E}_c \hookrightarrow \tilde{\mathcal{E}}_c \hookrightarrow \mathcal{E}_{c'} \hookrightarrow \tilde{\mathcal{E}}_{c'} \hookrightarrow \mathcal{C}^\infty \ (c < c') \tag{3.13}$$

and that the spectrum of the Frobenius-Perron operator is totally disconnected on each of these spaces. However, apart from $\mathcal{P}$, we were not able to determine the spectra precisely. On the other hand, it is not difficult to verify that each embedding in the chain (3.13) is continuous. Moreover, each space is dense in the following one, since the polynomials are dense in each of them. The embeddings intertwine the restrictions of the Frobenius-Perron operator to the spaces, and we may apply the intertwining theorem A.6.2 to conclude that the spectra coincide.

3.2.9 Theorem *Let E be any of the spaces in the chain (3.13). The spectrum of the Frobenius-Perron operator of the β-adic Rényi map on E is given by*

$$\sigma(U|E) = \left\{ \beta^{-k} \,\middle|\, k \in \mathbb{N}_0 \right\} \cup \{0\}. \tag{3.14}$$

Each β^{-k} is a simple pole of the resolvent, the residue of which is a one-dimensional projection.

Proof Since the assertion is true for $E = \mathcal{P}$ by proposition 3.2.1, the theorem holds for all other spaces by virtue of the intertwining theorem A.6.2 and proposition A.6.5.

QED

For the remaining classes of spaces $\mathcal{C}^n$ and $\mathcal{C}_\alpha$ we have the embeddings

$$\mathcal{P} \hookrightarrow \mathcal{C}^n \hookrightarrow \mathcal{C}_\alpha \text{ with } n \in \mathbb{N} \text{ and } \alpha \in (0, 1]. \tag{3.15}$$

Once again, each embedding is continuous and has dense range. Using the same arguments as before we get the following.

3.2.10 Theorem

(i) $\sigma(U|\mathcal{C}^n) = \left\{ \beta^{-k} \,\middle|\, k \in \mathbb{N}_0 \right\} \cup \left\{ \lambda \,\middle|\, |\lambda| \leq \beta^{-n} \right\}$*; each isolated point in the spectrum is a simple pole of the resolvent, the residue of which is a one-dimensional projection.*

(ii) $\sigma(U|\mathcal{C}_\alpha) = \left\{ \lambda \,\middle|\, |\lambda| \leq \beta^{-\alpha} \right\} \cup \{1\}$.

Having obtained a precise characterisation of the spectra of the evolution operator of a specific dynamical system we need to pause and ask what can be learnt from this exercise in operator spectral theory. Firstly, we have used an alternative methodology. Apart from $\mathcal{C}^\infty$ and $\tilde{\mathcal{E}}_c$ the precise structure of the spectra has been obtained previously by other authors [Gas, HS1, AT2] on the basis of the expansion (3.4). Our approach relied on using compactness arguments to conclude that the spectra are of a certain form (totally disconnected). The intertwining theorem then enabled us to obtain the precise structure of these spectra from the spectrum on the space of polynomials, which could be calculated explicitly. In practice it is often easier to show that the spectra are of a particular form rather than to calculate them explicitly. If, additionally, one is able to find a certain space on which the spectrum is known exactly, the intertwining theorem can be used to obtain knowledge of the spectrum on the other spaces.

Secondly, our discussion showed that the spectrum of the Frobenius-Perron operator may change dramatically, when the Frobenius-Perron operator is considered as an operator on certain dense subspaces of L^2. Each of the spaces in the chains (3.13) and (3.15) is comprised of functions with a certain degree of 'smoothness'. As the smoothness increases, that is, as we move to the left in the chain, the essential spectrum shrinks and exposes isolated eigenvalues, which remain stable once a certain degree of smoothness is reached. Apart from being a mathematical curiosity, this phenomenon has some physical consequences for the approach to equilibrium. We shall explore these consequences in the next section, in a more general context.

3.3 Test function spaces and the approach to equilibrium

In this section we shall define the concept of a 'test function space' for the evolution operator of a discrete time dynamical system, and study the relationship between the spectral properties of the evolution operator and the rate of approach to equilibrium.

3.3.1 Definition Let $(X, \mathcal{A}, \mu; T)$ be a dynamical system and let

$$U : L^2 \to L^2$$

be the associated Frobenius-Perron operator. A subspace Φ of L^2 is called a *test function space for U* if the following conditions are satisfied:

(i) Φ carries a topology τ with respect to which it becomes a locally convex topological vector space;

(ii) (Φ, τ) is sequentially complete and barrelled;

(iii) (Φ, τ) is continuously and densely embedded in L^2, that is, the topology τ on Φ is stronger than the topology induced by L^2 and Φ is dense in L^2;

(iv) $U|\Phi \in \mathcal{L}(\Phi)$, that is, $U(\Phi) \subset \Phi$ and U is continuous as an operator on Φ.

The function spaces listed in the previous section clearly meet the requirements of this definition. It is similar to the corresponding definition in [AT3], the difference being that we require Φ to be sequentially complete and barrelled, so that the spectral theory presented in Appendix A can be applied. We note that (i) and (iii) above imply that the topological dual of

L^2, and hence L^2, can be identified with a subspace of the topological dual of Φ,

$$\Phi \subset L^2 \simeq (L^2)' \subset \Phi'. \tag{3.16}$$

The identification of elements belonging to L^2 with elements of Φ' should be understood in the following sense. If i denotes the embedding of Φ in L^2 then the adjoint i' of i,

$$i' : \Phi' \to \Phi',$$

is injective as a consequence of (i) and (iii). Elements of L^2 are thus in one-to-one correspondence with elements of Φ'. Using this identification the integral over the product of $f \in \Phi$ and $g \in L^2$ may be interpreted in the duality $\langle \Phi, \Phi' \rangle$ as follows:

$$\int (if) \cdot g \, d\mu = \langle if, g \rangle_{L^2} = \langle f, i'g \rangle_{\langle \Phi, \Phi' \rangle} .$$

To avoid cluttered notation we shall drop reference to i and i' in the following.

The structure (3.16) is known as a 'rigged Hilbert space' or a 'Gelfand triplet'. Elements of Φ' are usually referred to as generalised functions. The concept of a rigged Hilbert space arose in the study of eigenfunction expansions of selfadjoint operators. The basic idea is to extend a given selfadjoint operator to a larger space (the space Φ'), so that every spectral value of the operator becomes an eigenvalue of the extension. Under certain additional conditions it can be shown that the corresponding eigenvectors form a complete set, giving rise to a 'generalised spectral theorem' for the selfadjoint operator (details may be found in the books by Gelfand *et. al.* [GS, GV] and Maurin [Mau]).

The test function spaces for the Rényi map have the remarkable property that the eigenvalue 1, which corresponds to the equilibrium density, is a pole of the resolvent. Moreover, 1 is the only singularity on the unit circle. In this situation the Riesz projection theorem furnishes a decomposition of the Frobenius-Perron operator of the form

$$U = P_{\{1\}} + V,$$

where $P_{\{1\}}$ is the spectral projection associated with the spectral set $\{1\}$ and V is a continuous operator with $r(V) < 1$ (see corollary A.4.4). The spectral radius $r(V)$ of the 'remainder operator' V determines how fast the densities in the test function spaces approach equilibrium. For members of the test function spaces of the Rényi map this approach is 'exponentially' fast. We shall now prove this in a more general setting.

3.3.2 Proposition *Let U be the Frobenius-Perron operator of the dynamical system T, and let Φ be a test function space for U. Assume furthermore that the resolvent of $U|\Phi$ is meromorphic on the unit circle. If T is weakly mixing, then there is $r < 1$ such that*

$$\left\| U^n f - P_{\{1\}} f \right\|_{L^2} = O(r^n) \text{ for } f \in \Phi, \tag{3.17}$$

where 'O' is Landau's order symbol.

Proof First we show that our assumptions imply that 1 is the only singularity of the resolvent on the unit circle $\mathbb{T}$. To see this, note that each singularity of the resolvent of $U|\Phi$ is an eigenvalue by theorem A.4.3. However, since T is weakly mixing, 1 is the only eigenvalue lying on $\mathbb{T}$, and thus the only singularity of the resolvent of $U|\Phi$.

As before, we may now write

$$U|\Phi = P_{\{1\}} + V,$$

where $V \in \mathcal{L}(\Phi)$ with $r(V) < 1$, by virtue of corollary A.4.4. Since the embedding $\Phi \hookrightarrow L^2$ is continuous there is a continuous seminorm p on Φ such that

$$\left\| U^n f - P_{\{1\}} f \right\|_{L^2} = \|V^n f\|_{L^2} \leq p(V^n f) \text{ for } f \in \Phi \text{ and } n \in \mathbb{N}. \tag{3.18}$$

Choose r with $r(V) < r < 1$. The spectral radius formula (see theorem A.1.9) implies that there is $K > 0$ such that

$$p(V^n f) \leq K r^n \text{ for every } n \in \mathbb{N} \tag{3.19}$$

Putting (3.18) and (3.19) together, the assertion follows.

QED

3.3.3 Corollary *Under the assumptions made in the proposition the dynamical system T is exact.*

Proof Since Φ is dense in L^2, and U is a contraction, equation (3.17) implies

$$\left\| U^n f - P_{\{1\}} f \right\|_{L^2} \to 0 \text{ as } n \to \infty \tag{3.20}$$

for every $f \in L^2$. This, however, implies exactness by a well-known criterion [LM, 4.4.1 corollary].

QED

3.3.4 Remark
The convergence to equilibrium in (3.20), that is, convergence with respect to the strong operator topology, is peculiar to non-invertible systems (more narrowly, exact systems). For invertible systems this type of convergence can never arise, since the associated Frobenius-Perron operator is unitary. A discussion of this problem for continuous time systems can be found in a paper by Kumičák [Kumi], where other types of convergence (weak, coarse-grained) are investigated.

This proposition is only useful for non-invertible systems. Indeed, the premises of the proposition imply that the system is exact and hence not invertible. It is interesting to see that the same is true not only for weakly mixing systems but also for ergodic systems.

3.3.5 Proposition *Let $(X, \mathcal{A}, \mu)$ be a probability space such that $L^2(X, \mathcal{A}, \mu)$ is infinite-dimensional*[5] *and let $(X, \mathcal{A}, \mu; T)$ be ergodic. If there exists a test function space Φ for U such that the resolvent of $U|\Phi$ is meromorphic on $\mathbb{T}$ then T is not invertible.*

Proof Suppose to the contrary that T is invertible. Then the associated Frobenius-Perron operator U is unitary. Since the resolvent of $U|\Phi$ is meromorphic on $\mathbb{T}$ the operator $U|\Phi$ has a representation of the form

$$U|\Phi = \sum_{i \in I} \lambda_i P_{\{\lambda_i\}} + V.$$

Here, I is a finite set, λ_i $(i \in I)$ belongs to $\mathbb{T}$, and $V \in \mathcal{L}(\Phi)$ with $r(V) < 1$. Since $(X, \mathcal{A}, \mu; T)$ is ergodic the spectral projections $P_{\{\lambda_i\}}$ are rank 1 projections. We shall consider two cases.

(i) If $V \neq 0$ then $\sigma(V)$ is non-empty and must intersect $\sigma(U|L^2) \subset \mathbb{T}$ by virtue of the intertwining theorem A.6.2. This, however, is a contradiction, since $\sigma(V)$ is contained in the open unit disk.

(ii) If $V = 0$ then $U|\Phi$ is a finite rank operator. Since Φ is dense in L^2 the operator U must be a finite rank operator on L^2 as well. Since L^2 was assumed to be infinite-dimensional the operator $U|L^2$ cannot be invertible, which furnishes the desired contradiction.

QED

[5] This is the case, for example, if $(X, \mathcal{A}, \mu)$ is non-atomic.

Returning to the Rényi map we see that the approach to equilibrium is exponentially fast for densities f belonging to the test function spaces. Letting $\gamma(f)$ denote the decay rate of such an f, that is,

$$\gamma(f) = \inf \left\{ r > 0 \,\middle|\, \left\| U^n f - P_{\{1\}} f \right\|_{L^2} = O(r^n) \right\},$$

or equivalently

$$\gamma(f) = \limsup_{n \to \infty} \left\| U^n f - P_{\{1\}} f \right\|_{L^2}^{1/n},$$

we see that for a fixed test function space Φ the decay rates of *all* densities f belonging to Φ are bounded from above. This bound is just the spectral radius of the remainder operator V. Thus, the collective bound for the decay of α-Hölder continuous densities under the action of the Rényi map is $\beta^{-\alpha}$. The other test function spaces yield a collective decay rate β^{-1}.

It is interesting to see that this decay rate is 'typical' for the test function spaces in which the remainder operator V has a largest (in modulus) eigenvalue, that is, for $\mathcal{C}^2$ and all smaller spaces. In this case the Frobenius-Perron operator may be written

$$U|\Phi = P_{\{1\}} + \beta^{-1} P_{\{\beta^{-1}\}} + W,$$

with $W \in \mathcal{L}(\Phi)$ and $r(W) < \beta^{-1}$ by another application of the Riesz theorem. Before we explain what we mean by 'typical' we require the following proposition.

3.3.6 Proposition *Let Φ be a test function space for U such that U has a decomposition of the form*

$$U|\Phi = P_{\{1\}} + \lambda P_{\{\lambda\}} + W$$

with $W \in \mathcal{L}(\Phi)$ and $r(W) < \lambda < 1$. If $f \in \Phi$ then $\gamma(f) < \lambda$ if and only if $P_\lambda f = 0$.

Proof The 'if' clause is trivial. To prove the 'only if' clause let $P_{\{\lambda\}} f \neq 0$. Then there is $h \in L^2$ such that $\left\langle P_{\{\lambda\}} f, h \right\rangle = 1$ and $\|h\|_{L^2} = 1$. Choose r with $r(W) < r < \lambda$ and observe that

$$|\langle W^n f, h \rangle| \leq r^n,$$

for n suitably large, by virtue of the spectral radius formula (see theorem A.1.9). Now

$$\begin{aligned}\left\| U^n f - P_{\{1\}} \right\|_{L^2} &= \left\| \lambda^n P_{\{\lambda\}} f + W^n f \right\|_{L^2} \\ &\geq \lambda^n \left\langle P_{\{\lambda\}} f, h \right\rangle + \langle W^n f, h \rangle \\ &\geq \lambda^n - r^n\end{aligned}$$

for n large enough. Thus

$$\gamma(f) \geq \limsup_{n\to\infty} (\lambda^n - r^n)^{1/n} = \lambda.$$

QED

The proposition asserts that the decay rate $\gamma(f)$ is strictly less than the collective decay rate λ only for f belonging to the kernel of $P_{\{\lambda\}}$. For $P_{\{\lambda\}}$ with finite rank, however, this is a topologically small set in Φ, that is, it is nowhere dense. Thus, 'typical' means that a randomly chosen $f \in \Phi$ will most likely (in the topological sense) decay at rate λ. For the Rényi map the typical decay rate for $\mathcal{C}^2$-functions is β^{-1}.

We have seen that the speed of the approach to equilibrium depends crucially on how the system is set up. Different initial distributions will relax towards equilibrium at different rates[6]. Test function spaces offer the possibility of making collective statements about entire classes of probability distributions. The spectral radius of the remainder operator, or the second largest (in modulus) eigenvalue of the evolution operator acting on the selected class of probability distributions has a physical meaning in terms of collective or typical decay rates. In this respect, a test function space implements a specified mode of description, as it preselects a class of probability distributions corresponding to admissible preparations of the system.

3.4 Eigenstates and resonances

We saw in the previous section that a good deal of knowledge about the probabilistic long-time behaviour of a dynamical system can be obtained

[6] The relation between the smoothness of the distribution and the speed of convergence to equilibrium is also observed for other dynamical systems, such as Axiom A systems [Rue3, Rue4].

from the spectrum of the evolution operator. In fact, only the part closest to the unit circle contributed to our analysis. The existence of a gap in the spectrum allowed us to decompose the evolution operator as a sum

$$U = P_{\{1\}} + V. \tag{3.21}$$

We then used the fact that the contribution of V to the dynamics becomes negligible in the long run. In other words, the decomposition (3.21) is sufficient if we are only interested in the asymptotics of the evolution.

We may gain some insight into the behaviour of the system on shorter time scales if the remainder operator V itself has a decomposition. In the case of the Rényi map this is possible provided that the test function spaces are small enough. For example, if $n \geq 2$ we may write

$$U|\mathcal{C}^n = \sum_{k=0}^{n-1} \beta^{-k} P_{\{\beta^{-k}\}} + W$$

with $r(W) \leq \beta^{-n}$. The spectral projections $P_{\{\beta^{-k}\}}$ are of course just the projections onto the one-dimensional eigenspaces. Borrowing terminology from quantum mechanics the Brussels/Austin groups refer to the corresponding eigenfunctions f_k as 'eigenstates' of the system. It should be noted that these eigenstates do not permit a probabilistic interpretation in general. Indeed, an eigenstate f_k with $k \in \mathbb{N}$ cannot be strictly non-negative, that is, it cannot be a probability density, since $U^n f_k = \beta^{-kn} f_k \to 0$ as $n \to \infty$ violates conservation of probability. A physical interpretation may, however, be supplied for the eigenvalues associated with these eigenstates. Indeed, the eigenvalues correspond to the resonances of the system, that is, the poles of the power spectrum. The resonances yield information about the modulation of correlation functions and as such describe the behaviour of the system on shorter time scales.

To establish the correspondence between the resonances of a system and the eigenvalues of the evolution operator for the non-normable, locally convex test function spaces considered by the Brussels/Austin groups[7], we rewrite the correlation function

$$C_{fg}(n) = \int f \cdot g \circ T^n \, d\mu - \int f \, d\mu \cdot \int g \, d\mu \text{ for } n \in \mathbb{N}_0$$

[7]For normable test function spaces see, for example, [Eck].

in terms of the Frobenius-Perron operator U

$$C_{fg}(n) = \int U^n f \cdot g \, d\mu - \int f \, d\mu \cdot \int g \, d\mu = \langle U_T^n f, g \rangle - \langle P_1 f, g \rangle \,. \tag{3.22}$$

Here P_1 denotes the projection onto the equilibrium distribution of the system

$$P_1 f := \langle f, 1 \rangle \, 1 \text{ for } f \in L^2.$$

A short calculation shows that the power spectrum $\hat{C}_{fg}$, that is, the Fourier transform[8] of the correlation function

$$\hat{C}_{fg}(\omega) := \sum_{n=0}^{\infty} e^{-in\omega} C_{fg}(n)$$

can be expressed in terms of the resolvent of the Frobenius-Perron operator. Setting $z = e^{i\omega}$ and using $U^n - P_1 = (U - P_1)^n$ $(n > 0)$ we obtain

$$\begin{aligned} \hat{C}_{fg}(\omega) &= \sum_{n=0}^{\infty} z^{-n} (\langle U^n f, g \rangle - \langle P_1 f, g \rangle) \\ &= \left\{ \sum_{n=0}^{\infty} z^{-n} \, \langle (U - P_1)^n f, g \rangle \right\} - \langle P_1 f, g \rangle \\ &= \left\langle z[z - (U - P_1)]^{-1} f, g \right\rangle - \langle P_1 f, g \rangle \,. \end{aligned}$$

The above reasoning, which is based on the Neumann series of the resolvent (see theorem A.1.8), shows that the resolvent of $U - P_1$ yields the analytic continuation of the power spectrum of the system. Suppose now that f belongs to a test function space Φ for which the resolvent of $U|\Phi$ is meromorphic on the unit circle. In this case, the power spectrum extends meromorphically into a strip containing the real axis, and the poles or resonances in this region are up to a factor i identical with the logarithms of the eigenvalues of $U|\Phi$ by virtue of theorem A.4.3.

Returning to the Rényi map we see that the poles of the power spectrum are simple and are located at $\omega_k = ik \ln \beta$ provided that $f \in \mathcal{C}^\infty$. It turns out that these poles coincide with the Ruelle resonances. This may be verified by an explicit calculation of the Ruelle ζ-function or by means of the following argument. For piecewise analytic Markov maps, Mayer [May] has shown that the eigenvalues of $U|H^\infty(D_r)$, that is, the restriction of U to the Banach

[8] We shall work with the one-sided Fourier transform as we are dealing with non-invertible systems.

space $H^\infty(D_r)$ $(r > 1)$ of bounded holomorphic functions on the disk D_r with radius r, coincide with the Ruelle resonances of the system. Since the Rényi map belongs to this class and $H^\infty(D_r)$ is densely and continuously embedded in $\mathcal{C}^\infty$, the spectra coincide as a consequence of the intertwining theorem.

3.5 Spectral decomposition of the Frobenius-Perron operator

A particularly interesting situation occurs when the Frobenius-Perron operator of a dynamical system has a test function space Φ for which the spectrum is discrete, that is, a finite set or a sequence converging to zero. In this case, which is of particular importance for the approach of the Brussels/Austin groups, one might expect that U admits a 'spectral decomposition', that is, U can be represented as a sum

$$U|\Phi = \sum_n \lambda_n P_{\{\lambda_n\}}, \tag{3.23}$$

where λ_n are the points of the spectrum and $P_{\{\lambda_n\}}$ are the associated Riesz projections. If zero is not an isolated point of the spectrum then $P_{\{0\}}$ denotes the projection onto the kernel of U.

A spectral decomposition of the Frobenius-Perron operator of a system may be called a 'physical' representation of the system: the evolution is completely described as a superposition of the eigenstates and the associated decay rates λ_n corresponding to the resonances of the system are explicitly incorporated in the probabilistic description. In order to be a 'representation' of the system we need to interpret the formal sum in (3.23) by specifying a notion of convergence. It is natural to require that the sum should converge for every element in Φ with respect to the topology of Φ.

3.5.1 Definition Let Φ be a test function space for U. We shall say that U admits a *spectral decomposition* if

$$U|\Phi = \sum_n \lambda_n P_{\{\lambda_n\}} \text{ with the sum converging in } \mathcal{L}_s(\Phi).$$

The spectral decomposition is called *complete* if

$$I = \sum_n P_{\{\lambda_n\}} \text{ with the sum converging in } \mathcal{L}_s(\Phi).$$

The existence of a complete spectral decomposition of the evolution operator of a dynamical system is interesting from a practical point of view since the calculation of the iterates of a density in Φ reduces to calculating the expansion coefficients of the density with respect to $P_{\{\lambda_n\}}$. We shall now show that the Frobenius-Perron operator of the Rényi map admits a complete spectral decomposition on some of the spaces $\tilde{\mathcal{E}}_c$. Incidentally, we shall see that the discreteness of the spectrum is by no means sufficient to guarantee the existence of a spectral decomposition.

We know from theorem 3.2.9 that the spectrum of the Frobenius-Perron operator of the Rényi map is discrete on $\mathcal{C}^\infty$ and the test function spaces contained in it. We also know that the associated eigenfunctions (or the ranges of the spectral projections) coincide with the eigenfunctions of $U|\mathcal{P}$, and, consequently, must be polynomials. We recall that these 'eigenpolynomials' are the Bernoulli polynomials B_n, which are defined by the generating function (see [AS, Chapter 23])

$$\frac{w}{e^w - 1} e^{zw} = \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n = \sum_{n=0}^{\infty} p_n(z) w^n. \tag{3.24}$$

We shall mostly work with the p_ns, which we shall also refer to as 'Bernoulli polynomials', since they arise from a simple rescaling of the 'classical' Bernoulli polynomials B_n.

The fact that the Bernoulli polynomials are eigenfunctions of the Frobenius-Perron operator of the Rényi map follows immediately from the shifting theorem ([AS, 23.1.10]):

$$p_n(\beta z) = \beta^{n-1} \sum_{i=0}^{\beta-1} p_n(z + \frac{i}{\beta}).$$

In order to get an explicit representation of the projections $P_{\{\beta^{-n}\}}$ we note that every entire function f of type less than 2π has an expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f) p_n(z), \tag{3.25}$$

where

$$a_n(f) := \begin{cases} \int_0^1 f(x)\, dx & \text{for } n = 0 \\ f^{(n-1)}(1) - f^{(n-1)}(0) & \text{for } n \geq 1 \end{cases}.$$

This expansion is the so-called Euler-Maclaurin summation formula, which is known to converge uniformly on compact subsets of the complex plane (see [BB, II.9 (i)]). The Euler-Maclaurin formula implies

$$a_n(p_m) = \delta_{nm},$$

so that the spectral projections $P_{\{\beta^{-n}\}}$ are given by

$$P_{\{\beta^{-n}\}}f = a_n(f)p_n \text{ for } n \in \mathbb{N}_0 \text{ and } f \in \mathcal{C}^\infty.$$

We may thus write

$$Uf = \sum_{n=0}^{\infty} \frac{1}{\beta^n} a_n(f)p_n = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \langle f, a_n \rangle \, p_n = \sum_{n=0}^{\infty} \frac{1}{\beta^n} P_{\{\beta^{-n}\}}f \tag{3.26}$$

The above expression is certainly correct if $f \in \mathcal{P}$, since then the sum has only finitely many non-zero terms. Moreover, the Bernoulli polynomials span the space of polynomials, which implies that

$$f = \sum_{n=0}^{\infty} P_{\{\beta^{-n}\}}f \text{ for every } f \in \mathcal{P}.$$

In other words $U|\mathcal{P}$ admits a complete spectral decomposition. The same holds for some of the larger test function spaces. To prove this we require two lemmata.

3.5.2 Lemma *Let $c < 2\pi$ and $z \in \mathbb{C}$. Then there are constants K and K', such that*

(i) $\|p_n\|_c \leq Kc^{-n}$ *for every* $n \in \mathbb{N}_0$*;*

(ii) $\left|f^{(n)}(z)\right| \leq K'\sqrt{n}\,\|f\|_c\, c^n$ *for every* $n \in \mathbb{N}$ *and every* $f \in \mathcal{E}_c$*.*

Proof

(i) Taking the derivative of the generating function (3.24) of the Bernoulli polynomials with respect to z, and comparing coefficients, yields

$$p'_{n+1} = p_n \text{ for } n \geq 0.$$

Thus

$$p_n^{(k)} = p_{n-k} \text{ for } k \leq n,$$

and the Bernoulli polynomials may be written

$$p_n(z) = \sum_{k=0}^{n} \frac{p_n^{(k)}(0)}{k!} z^k = \sum_{k=0}^{n} \frac{p_{n-k}(0)}{k!} z^k.$$

Therefore

$$\|p_n\|_c \leq \sup_{r \geq 0} \sum_{k=0}^{n} \frac{|p_{n-k}(0)|}{k!} r^k e^{-cr}.$$

Observing that $\left(\frac{k}{ce}\right)^k$ is the the maximum of the function $r \mapsto r^k e^{-cr}$ on $\mathbb{R}_0^+$, and using Stirling's formula to estimate $k!$,

$$k! \geq k^k e^{-k},$$

we arrive at

$$\|p_n\|_c \leq \sum_{k=0}^{n} |p_{n-k}(0)|\, c^{-k} = c^{-n} \sum_{k=0}^{n} |p_k(0)|\, c^k.$$

Putting $z = 0$ in (3.24)

$$\frac{w}{e^w - 1} = \sum_{k=0}^{\infty} p_k(0) w^k,$$

we see that $\sum_{k=0}^{\infty} p_k(0) w^k$ is a power series, the radius of convergence of which is 2π, because $2\pi i$ is the pole of $\frac{w}{e^w-1}$ closest to zero. Thus, $\sum_{k=0}^{\infty} p_k(0) c^k < \infty$ for all $c < 2\pi$, and the assertion is proved.

(ii) Now let $z \in \mathbb{C}$ and $f \in \mathcal{E}_c$. Cauchy's formula implies

$$\left|f^{(n)}(z)\right| \leq \frac{n!}{2\pi} \oint_{\mathcal{C}_r} \frac{\|f\|_c\, e^{c|\zeta|}}{|\zeta - z|^{n+1}}\, d\zeta \text{ for } n \in \mathbb{N},$$

where $\mathcal{C}_r$ is a circle of radius r centred at z. Evaluating the integral, we obtain

$$\left|f^{(n)}(z)\right| \leq n! e^{c|z|}\, \|f\|_c\, \frac{e^{cr}}{r^n}.$$

Specifying $r = n/c$ and using Stirling's formula in the form

$$n! = n^n e^{-n} \sqrt{2\pi n} e^{\delta_n/(12n)} \text{ with } 0 < \delta_n < 1,$$

we see that the assertion is true with $K' = \sqrt{2\pi} e^{c|z|+1}$.

QED

3.5.3 Lemma

(i) If $c < 2\pi$*, then* $\sum_{n=0}^{\infty} \frac{1}{\beta^n} \langle f, a_n \rangle p_n$ *converges in* $\mathcal{E}_c$ *for every* $f \in \mathcal{E}_c$*.*

(ii) If $c \leq 2\pi$*, then* $\sum_{n=0}^{\infty} \frac{1}{\beta^n} \langle f, a_n \rangle p_n$ *converges in* $\tilde{\mathcal{E}}_c$ *for every* $f \in \tilde{\mathcal{E}}_c$*.*

Proof
(i) Let $c < 2\pi$ and $f \in \mathcal{E}_c$. The previous lemma implies that

$$|\langle f, a_{n+1} \rangle| \leq 2K' \|f\|_c \sqrt{n} c^n \text{ for } n \in \mathbb{N}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \|\langle f, a_{n+1} \rangle p_{n+1}\|_c \leq \frac{2KK'}{c} \|f\|_c \sum_{n=1}^{\infty} \frac{\sqrt{n}}{\beta^{n+1}} < \infty.$$

Since $\mathcal{E}_c$ is complete, the sum $\sum_{n=0}^{\infty} \langle f, a_n \rangle p_n$ converges in $\mathcal{E}_c$ for every $f \in \mathcal{E}_c$.
(ii) Let $c \leq 2\pi$. If $f \in \tilde{\mathcal{E}}_c$, then $f \in \mathcal{E}_{c'}$ for some $c' < c$. By (i), the sum $\sum_{n=0}^{\infty} \langle f, a_n \rangle p_n$ converges in $\mathcal{E}_{c'}$, and hence also in $\tilde{\mathcal{E}}_c$ by virtue of the continuity of the embedding $\mathcal{E}_{c'} \hookrightarrow \tilde{\mathcal{E}}_c$.

QED

We are now able to prove the existence of the spectral decomposition.

3.5.4 Proposition *Let* $c \leq 2\pi$*. Then the Frobenius-Perron operator of the* β*-adic Rényi map admits the following representation on* $\tilde{\mathcal{E}}_c$*:*

$$U|\tilde{\mathcal{E}}_c = \sum_{n=0}^{\infty} \frac{1}{\beta^n} P_{\{\beta^{-n}\}}, \tag{3.27}$$

with the sum converging in $\mathcal{L}_s(\tilde{\mathcal{E}}_c)$*.*

Proof Since $\mathcal{L}_s(\tilde{\mathcal{E}}_c)$ is sequentially complete (see [Sch, III.4.4]) the expression on the right hand side of (3.27) is a continuous operator on $\tilde{\mathcal{E}}_c$. Since this operator coincides with $U|\tilde{\mathcal{E}}_c$ on the dense subset $\mathcal{P}$, they must be equal on all of $\tilde{\mathcal{E}}_c$.

QED

The next proposition shows that the spectral decomposition in $\tilde{\mathcal{E}}_c$ ($c \leq 2\pi$) is complete. Incidentally we prove that the Euler-Maclaurin formula, which is known to converge uniformly on compact subsets of the complex plane, even converges with respect to the (stronger) topology of $\tilde{\mathcal{E}}_c$.

3.5.5 Proposition *Let $c \leq 2\pi$. Then*

$$I = \sum_{n=0}^{\infty} P_{\{\beta^{-n}\}},$$

with the sum converging in $\mathcal{L}_s(\tilde{\mathcal{E}}_c)$.

Proof Fix $0 < c \leq 2\pi$, and let $f \in \tilde{\mathcal{E}}_c$. First we show that

$$\sum_{n=0}^{\infty} \langle f, a_n \rangle \, p_n \tag{3.28}$$

converges in $\tilde{\mathcal{E}}_c$. Since $f \in \tilde{\mathcal{E}}_c$ there is $c' < c$ such that $f \in \mathcal{E}_{c'}$. Choose c'' with $c' < c'' < c$. Then by lemma 3.5.2

$$\|p_n\|_{c''} \leq K(c'')^{-n} \text{ for } n \in \mathbb{N},$$

and

$$|\langle f, a_{n+1} \rangle| \leq K' \sqrt{n} \, \|f\|_{c'} \, (c')^n \text{ for } n \in \mathbb{N}.$$

Therefore,

$$\sum_{n=1}^{\infty} \|\langle f, a_{n+1} \rangle \, p_{n+1}\|_{c''} \leq \frac{KK'}{c''} \, \|f\|_{c'} \sum_{n=1}^{\infty} (c'/c'')^n \sqrt{n} < \infty,$$

since $(c'/c'') < 1$. Thus, the expansion (3.28) converges in $\mathcal{E}_{c''}$ and consequently also in $\tilde{\mathcal{E}}_c$ by virtue of the continuity of the embedding $\mathcal{E}_{c''} \hookrightarrow \tilde{\mathcal{E}}_c$. Since $\tilde{\mathcal{E}}_c$ is barrelled this expansion is a continuous operator, which coincides with the identity on the dense subspace $\mathcal{P}$.

QED

The last two propositions show that the Frobenius-Perron operator of the β-adic Rényi map admits a complete spectral decomposition on the test function spaces $\tilde{\mathcal{E}}_c$ with $c \leq 2\pi$, and trivially on $\mathcal{P}$. What about the other spaces? It turns out that the value 2π cannot be replaced by anything larger. To see this we consider the functions

$$\psi_0(x) := \exp(2\pi i x),$$

$$\psi_1(x) := \exp(2\pi i \beta x),$$

where β is the parameter of the Rényi map. We note that $\psi_0 \in \tilde{\mathcal{E}}_c$ for every $c > 2\pi$ and that $\psi_1 \in \tilde{\mathcal{E}}_c$ for every $c > 2\pi\beta$. Using (3.3) we obtain

$$\sum_{n=0}^{\infty} P_{\{\beta^{-n}\}}\psi_0 = 0 \neq \psi_0; \tag{3.29}$$

$$\sum_{n=0}^{\infty} \beta^{-n} P_{\{\beta^{-n}\}}\psi_1 = 0 \neq \psi_0 = U\psi_1. \tag{3.30}$$

By (3.29), the spectral decomposition (if it exists) cannot be complete in any of the spaces containing $\tilde{\mathcal{E}}_{2\pi}$. Relation (3.30) implies that even if the weighted sum over the spectral projections converges it need not represent the Frobenius-Perron operator of the Rényi map. The space $\tilde{\mathcal{E}}_{2\pi}$ thus appears to be the largest test function space in our chain on which the Frobenius-Perron operator admits a complete spectral decomposition.

3.6 Discussion

We saw that the temporal behaviour of non-invertible systems can be studied quite profitably by restricting the domain of the associated Frobenius-Perron operator of the system. Once a mode of description is chosen by fixing a test function space, information about the collective or typical long time behaviour can be obtained from the spectrum of the restricted Frobenius-Perron operator. The domain of the evolution operator is thus at least as important as the operator itself. Or in other words, the 'physics' is not only in the operator but also in the domain. Varying the domain, as in the discussion of the Rényi map, can yield a more detailed picture of the fine structure of the probabilistic evolution of the system.

As we mentioned in the introduction to this chapter, this idea is not new. Densely embedded subspaces carrying a norm topology have been used for similar purposes, and most of the results in Sections 3.3 and 3.4 have their analogues in the familiar Banach space setting. The more general spaces suggested by the Brussels/Austin groups, however, necessitate an extension of the Banach space reasoning to locally convex spaces, and, in particular, require elements of an operator spectral theory in locally convex spaces. This theory enabled us to analyse the spectral properties of the Frobenius-Perron operator of the Rényi map on a sequence of test function spaces. In this analysis the intertwining theorem A.6.2 played an important role, as it allowed us to relate the spectra of the Frobenius-Perron operator on different test function spaces. The spaces $\mathcal{P}$ and $\mathcal{C}^\infty$ are interesting in that they were constructed from a sequence of 'smaller' and 'larger' spaces by taking inductive and projective limits, respectively. The discussion of the spectral properties of the Frobenius-Perron operator on these spaces relied on the characterisation of the spectra of operators on inductive and projective limits (see theorem A.7.10 and theorem A.7.19). Remembering that test function spaces preselect a class of admissible preparations of the system, these two theorems could turn out to be useful in a more general context as a constructive method to pass from one mode of description to another.

Spectral theory in locally convex spaces also entered our discussion of the notion of a spectral decomposition. It furnished the relation between the eigenvalues of the restricted Frobenius-Perron operator and the resonances of the system, and enabled us to formalise the concept of a spectral decomposition used by the Brussels/Austin groups as a convergent sum over the spectral projections of the operator. As a result we argued that spectral decompositions qualify as *physical* representations of the system to the extent that all resonances of the system are explicitly incorporated in the probabilistic description of the system.

Let us mention that the spectral decompositions sought by the Brussels/Austin groups are physical *representations* only if they represent the Frobenius-Perron operator of the system. In the case of the Rényi map the class of functions for which this could be shown turned out to be rather restrictive (entire functions of exponential type strictly less than 2π). It is interesting to compare the spectral decomposition of the Frobenius-Perron operator on $\tilde{\mathcal{E}}_{2\pi}$,

$$U|\tilde{\mathcal{E}}_{2\pi} = \sum_{n=0}^{\infty} \beta^{-n} P_{\{\beta^{-n}\}},$$

with the partial spectral decomposition on $\mathcal{C}^2$,

$$U|\mathcal{C}^2 = P_{\{1\}} + \beta^{-1} P_{\{\beta^{-1}\}} + W.$$

The latter also yields a good description of the system at least as far as the long-time behaviour of the system is concerned, and is valid for a much larger class of functions. The possibility of having a description including all resonances of the system has the cost of excluding a number of possible preparations of the system.

In this context, one might wonder which other dynamical systems, apart from the Rényi map discussed in this chapter, admit a complete spectral decomposition. This will be the subject of the next chapter.

Chapter 4

Spectral decomposition of exact systems

In this chapter we shall prove that every exact system admits a complete spectral decomposition. More precisely, we shall show that with every exact system and any given sequence $\{\lambda_n\}$ converging to zero we may associate a test function space Φ such that:

- the spectrum of the Frobenius-Perron operator restricted to Φ is given by $\{\lambda_n\}$;
- the Frobenius-Perron operator admits a complete spectral decomposition on Φ.

The proof relies on the unitary equivalence between the Frobenius-Perron operator of exact systems and the unilateral left-shift, and amounts to showing that the above assertions are true for the unilateral left-shift.

In the first section we introduce the various shift operators and collect some basic results we require later. We shall then prove the main result for the unilateral left-shift (Section 4.2). The application to exact systems will be discussed in the final two sections.

4.1 The shift operator

The basic object we shall be concerned with is the operator of multiplication by the independent variable in spaces of holomorphic functions, the so-called shift operator

$$f \mapsto zf.$$

This operator plays an important role in operator spectral theory as it serves as a universal model for operators on Hilbert spaces. The shift operator is also important in the context of the present thesis, since the evolution operators of K-systems and exact systems are, up to isometric isomorphism, identical with this operator, provided that the space of functions on which it acts is suitably chosen. Our first task will be to describe these function spaces.

4.1.1 Notation In the following, we use m to denote the normalised Lebesgue measure on $\mathbb{T}$, that is, $m(\mathbb{T}) = 1$. The symbol z designates the identity map on a portion of the complex plane, $z(\zeta) = \zeta$, and we use $\bigvee$ for the linear hull of elements of a vector space. Finally, F will denote a separable Hilbert space with inner product $(.,.)_F$ and norm $\|.\|_F$.

4.1.2 Definition

(i) We denote by $H^2(\mathbb{D}; F) = H^2(F)$ the space of all F-valued holomorphic functions $f(\zeta) = \sum_{k=0}^{\infty} f_k \zeta^k$, $f_k \in F$ on the disk $\mathbb{D}$, for which the quantities

$$\frac{1}{2\pi} \int_0^{2\pi} \left\| f(re^{i\theta}) \right\|_F^2 \, d\theta = \sum_{k=0}^{\infty} \|f_k\|_F^2 \, r^{2k}$$

remain finite for $0 \leq r < 1$.

(ii) We use $L^2(\mathbb{T}; F) = L^2(F)$ for the space of all weakly measurable functions $f : \mathbb{T} \to F$ (see [Yos, V.4]) for which

$$\int_{\mathbb{T}} \|f(t)\|_F^2 \, dm(t)$$

remains finite.

It is not difficult to see that $H^2(F)$ is a Hilbert space with inner product given by

$$(f, g)_{H^2(F)} = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \left(f(re^{i\theta}), g(re^{i\theta}) \right)_F \, d\theta = \sum_{k=0}^{\infty} (f_k, g_k)_F \,,$$

for any $f(\zeta) = \sum_{k=0}^{\infty} f_k \zeta^k$ and $g(\zeta) = \sum_{k=0}^{\infty} g_k \zeta^k$ in $H^2(F)$.

The space $L^2(F)$ is a Hilbert space with inner product

$$(f,g)_{L^2(F)} = \int_{\mathbb{T}} (f(t), g(t))_F \; dm(t) \text{ for } f,\, g \in L^2(F).$$

$H^2(\mathbb{C})$ and $L^2(\mathbb{C})$ are of course just the classical H^2 and L^2 spaces, which are treated in some detail in [Hof] and [Rud1]. As in the classical case one may show that $H^2(F)$ can be identified with a closed subspace of $L^2(F)$: the space $H^2(F)$ consists of all functions in $L^2(F)$ which can be analytically continued into the disk $\mathbb{D}$.

We note that if $\left\{ b_j \,\middle|\, j \in \mathcal{J} \right\}$ is an orthonormal basis for F, then the family of functions $\left\{ z^n b_j \,\middle|\, n \in \mathbb{N}_0, j \in \mathcal{J} \right\}$ (respectively, $\left\{ z^n b_j \,\middle|\, n \in \mathbb{Z}, j \in \mathcal{J} \right\}$) is an orthonormal basis for $H^2(F)$ (respectively, $L^2(F)$). Using this basis it is not difficult to see that $H^2(F)$ can be identified with the Hilbert sum of $\dim F$ copies of H^2, that is $H^2(F) = \sum_{k=1}^{\dim F} H^2$, and similarly for $L^2(F)$.

4.1.3 Definition

(i) The operator of multiplication by z on $H^2(F)$

$$\bar{S}_F : H^2(F) \to H^2(F)$$
$$\bar{S}_F f = zf$$

is called the *unilateral right-shift of multiplicity* $\dim F$.

(ii) The operator of multiplication by z on $L^2(F)$

$$\bar{\mathcal{S}}_F : L^2(F) \to L^2(F)$$
$$\bar{\mathcal{S}}_F f = zf$$

is called the *bilateral right-shift of multiplicity* $\dim F$.

Clearly, $\bar{\mathcal{S}}_F$ is a unitary operator and $\bar{S}_F$ is a non-surjective isometry. Moreover, the unilateral right-shift is just the restriction of $\bar{\mathcal{S}}_F$ to $H^2(F)$. The term 'shift' for these two operators may be understood as follows. If we denote by $\{f_k\}$ the Taylor (Fourier) coefficients of $f \in H^2(F)$ ($f \in L^2(F)$) so that $f = \sum_{k \in \mathbb{N}_0} f_k z^k$ ($f = \sum_{k \in \mathbb{Z}} f_k z^k$) then

$$(\bar{S}_F f)_k = \begin{cases} 0 & \text{for } k = 0 \\ f_{k-1} & \text{for } k \geq 1 \end{cases},$$

and

$$(\bar{\mathcal{S}}_F f)_k = f_{k-1} \text{ for } k \in \mathbb{Z}.$$

Thus, the action of $\bar{S}_F$ (respectively, $\bar{\mathcal{S}}_F$) amounts to shifting the sequence of the Taylor (Fourier) coefficients of a function to the *right* by one unit. The operators effecting the shift in the opposite direction are given by the Hilbert space adjoints of these operators.

4.1.4 Definition

(i) The Hilbert space adjoint of the unilateral right-shift

$$S_F : H^2(F) \to H^2(F)$$

$$S_F := \bar{S}_F^*$$

is called the *unilateral left-shift of multiplicity* $\dim F$.

(ii) The Hilbert space adjoint of the bilateral right-shift

$$\mathcal{S}_F : L^2(F) \to L^2(F)$$

$$\mathcal{S}_F := \bar{\mathcal{S}}_F^*$$

is called the *bilateral left-shift of multiplicity* $\dim F$.

It is not difficult to see that

$$S_F f = z^{-1}(f - f(0)) \text{ for } f \in H^2(F),$$

$$\mathcal{S}_F f = z^{-1} f \text{ for } f \in L^2(F),$$

and that the action of S_F (respectively, $\mathcal{S}_F$) amounts to shifting the sequence of the Taylor (Fourier) coefficients of a function to the *left* by one unit.

The rich structure of shift operators is explored and treated in detail in the books by Nikolskii [Nik] and Rosenblum and Rovnyak [RR] to which we refer the interested reader. We do not require any of the deeper results presented there, apart from a simple characterisation of the spectrum of the unilateral left-shift originally due to Beurling [Beu]. Before we state and prove his result we mention that Cauchy's integral formula holds in H^2 in the following form.

4.1.5 Lemma *Let $\lambda \in \mathbb{D}$, $k \in \mathbb{N}_0$ and define $g_{\lambda,k} \in H^2$ by*

$$g_{\lambda,k} := (1 - \lambda z)^{-(k+1)} z^k.$$

Then

$$(f, g_{\lambda,k})_{H^2} = \frac{1}{k!} f^{(k)}(\overline{\lambda}) \text{ for any } f \in H^2.$$

Proof Observe that

$$\begin{aligned}
(f, g_{\lambda,k})_{H^2} &= \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})(1 - \overline{\lambda} re^{-i\theta})^{-(k+1)} (re^{-i\theta})^k \, d\theta \\
&= \lim_{r \uparrow 1} \frac{r^{2k}}{2\pi} \int_0^{2\pi} f(re^{i\theta})(re^{i\theta} - \overline{\lambda} r^2)^{-(k+1)} re^{i\theta} \, d\theta \\
&= \lim_{r \uparrow 1} \frac{r^{2k}}{k!} f^{(k)}(\overline{\lambda} r^2) = \frac{1}{k!} f^{(k)}(\overline{\lambda}).
\end{aligned}$$

The last equality follows from Cauchy's integral formula and the fact that f is holomorphic on $\mathbb{D}$.

QED

4.1.6 Lemma (Beurling) *The spectrum of the unilateral left-shift of multiplicity* 1 *coincides with the closed unit disk. For $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}$ we have*

$$\mathcal{N}((S_{\mathbb{C}} - \lambda)^n) = \bigvee \left\{ (1 - \lambda z)^{-k} z^{k-1} \,\middle|\, 1 \le k \le n \right\}. \tag{4.1}$$

In particular, $(1 - \lambda z)^{-1} = \sum_{k=0}^{\infty} (\lambda z)^k$ is an eigenfunction for the eigenvalue λ. The corresponding algebraic eigenspace consists of all rational functions with the only pole at λ^{-1}.

Proof Since $\|S_{\mathbb{C}}\|_{\mathcal{L}(H^2)} = \|S_{\mathbb{C}}^*\|_{\mathcal{L}(H^2)} = 1$ we only need to prove (4.1). Now let $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}$, and note that

$$\mathcal{N}((S_{\mathbb{C}} - \lambda)^n) = H^2 \ominus \mathcal{R}((S_{\mathbb{C}}^* - \overline{\lambda})^n).$$

We have

$$\mathcal{R}((S_{\mathbb{C}}^* - \overline{\lambda})^n) = \left\{ f \in H^2 \,\middle|\, f(\overline{\lambda}) = f^{(1)}(\overline{\lambda}) = \cdots = f^{(n-1)}(\overline{\lambda}) = 0 \right\},$$

since $(S_{\mathbb{C}}^* - \overline{\lambda})^n g = (z - \overline{\lambda})^n g = f$ holds for $f, g \in H^2$ if and only if f has a zero of order n at $\overline{\lambda}$. Thus, by the previous lemma,

$$\mathcal{R}((S_{\mathbb{C}}^* - \overline{\lambda})^n) = \left\{ f \in H^2 \,\middle|\, (f, g_{\lambda,0})_{H^2} = (f, g_{\lambda,1})_{H^2} = \cdots = (f, g_{\lambda,n-1})_{H^2} = 0 \right\}.$$

QED

The generalisation of this lemma to the unilateral left-shift S_F of multiplicity $\dim F$ is straightforward.

4.1.7 Corollary *The spectrum of* $S_F : H^2(F) \to H^2(F)$ *coincides with the closed unit disk. If* $\lambda \in \mathbb{D}$ *and* $n \in \mathbb{N}$*, then*

$$\mathcal{N}((S_F - \lambda)^n) = \bigvee \left\{ (1 - \lambda z)^{-k} z^{k-1} e \,\middle|\, 1 \le k \le n, e \in F \right\}. \tag{4.2}$$

In particular, each $\lambda \in \mathbb{D}$ *is an eigenvalue of geometric multiplicity* $\dim F$ *and infinite algebraic multiplicity.*

Proof Since $\|S_F\|_{\mathcal{L}(H^2(F))} = \|S_F^*\|_{\mathcal{L}(H^2(F))} = 1$ we only need to prove (4.2). Let $\left\{ b_j \,\middle|\, j \in \mathcal{J} \right\}$ be an orthonormal basis for F. Using the identification $H^2(F) = \sum_{j \in \mathcal{J}} H^2$ we expand $f \in H^2(F)$ into $f = \sum_{j \in \mathcal{J}} f_j b_j$, where $f_j \in H^2$. Since

$$(S_F - \lambda)^n f = \sum_{j \in \mathcal{J}} (S_{\mathbb{C}} - \lambda)^n f_j = 0$$

if and only if

$$f_j \in \mathcal{N}((S_{\mathbb{C}} - \lambda)^n) \text{ for every } j \in \mathcal{J},$$

equation (4.2) follows from the previous lemma.

QED

4.2 Spectral decomposition of the unilateral left-shift

We know from Beurling's lemma that the unilateral left-shift has plenty of invariant subspaces, namely the kernels of $(S_F - \lambda)^n$. We shall now show that it is possible to 'glue' these spaces together in such a way that they form a complete and barrelled space Φ which enjoys the following properties:

- Φ is continuously and densely embedded in $H^2(F)$;
- $S_F|\Phi$ has poles of preassigned order at preassigned locations.

The construction proceeds in three steps.

(i) First we show that certain families of algebraic eigenvectors are dense in $H^2(F)$ (see lemma 4.2.1).

(ii) Next we construct auxiliary spaces $\Phi(\lambda, K, N, M)$ with the property that $S_F|\Phi(\lambda, K, N, M)$ has one and only one pole at λ, of order determined by K, and residue determined by N and M. The meaning of K, N, and M will become clear during the construction, which is presented in the proof of lemma 4.2.2.

(iii) Finally we show that the auxiliary spaces $\Phi(\lambda, K, N, M)$ can be assembled such as to yield the desired space Φ. This result will be presented in theorem 4.2.3.

From now on we assume that F is an infinite-dimensional Hilbert space; furthermore, we let $\left\{ e_n \,\middle|\, n \in \mathbb{N} \right\}$ be a fixed family of linearly independent elements of F which are *total in* F, that is, $\bigvee \left\{ e_n \,\middle|\, n \in \mathbb{N} \right\}$ is dense in F. In order to avoid cluttered notation we shall suppress the subscript 'F' we previously used to distinguish the shift operators of different multiplicity. Finally, we fix 'canonical' algebraic eigenfunctions of the unilateral left-shift:

$$g_{\lambda,n,k} := (1 - \lambda z)^{-k} z^{k-1} e_n \text{ for } \lambda \in \mathbb{D},\ n \in \mathbb{N},\ k \in \mathbb{N}.$$

Algebraic eigenfunctions $g_{\lambda,n,k}$ associated with distinct λ, n, and k are clearly linearly independent and satisfy

$$(S - \lambda) g_{\lambda,n,k} = \begin{cases} 0 & \text{for } k = 1 \\ g_{\lambda,n,k-1} & \text{for } k > 1 \end{cases} . \tag{4.3}$$

First we establish conditions to guarantee that certain families of algebraic eigenfunctions are dense in $H^2(F)$.

4.2.1 Lemma

(i) If $\left\{ \lambda_n \,\middle|\, n \in \mathbb{N} \right\}$ is a sequence of complex numbers in $\mathbb{D}$ having a limit point in $\mathbb{D}$, then $\left\{ (1 - \lambda_n z)^{-1} \,\middle|\, n \in \mathbb{N} \right\}$ is total in H^2.

(ii) Let $\left\{ h_{n,m} \in H^2 \,\middle|\, n, m \in \mathbb{N} \right\}$ be a family of functions such that

$$\left\{ h_{n,m} \,\middle|\, n \in \mathbb{N} \right\} \text{ is total in } H^2 \text{ for each } m \in \mathbb{N}.$$

Then $\left\{ h_{n,m} e_m \,\middle|\, n, m \in \mathbb{N} \right\}$ is total in $H^2(F)$.

Proof (i) It suffices to show that if $f \in H^2$ and $((1-\lambda_n z)^{-1}, f)_{H^2} = 0$ for every $n \in \mathbb{N}$, then $f = 0$.

Now let $f \in H^2$ and define a function $F_f : \mathbb{D} \to \mathbb{C}$ by

$$F_f(\lambda) := \sum_{k=0}^{\infty} \lambda^k \left(z^k, f\right)_{H^2}.$$

Since $f \in H^2$, we have $\limsup_{k\to\infty} \left|\left(z^k, f\right)_{H^2}\right|^{1/k} \leq 1$, so that F_f is holomorphic on $\mathbb{D}$. If

$$\left((1-\lambda_n z)^{-1}, f\right)_{H^2} = \sum_{k=0}^{\infty} \lambda_n^k \left(z^k, f\right)_{H^2} = F_f(\lambda_n) = 0 \text{ for every } n \in \mathbb{N},$$

then F_f must vanish identically on all of $\mathbb{D}$, since we assumed that $\{\lambda_n\}$ has a limit point in $\mathbb{D}$. This implies $\left(z^k, f\right)_{H^2} = 0$ for every $k \in \mathbb{N}_0$, which forces $f = 0$.

(ii) Let $f = \sum_{k\geq 0} f_k z^k \in H^2(F)$ and assume that

$$(f, h_{n,m} e_m)_{H^2(F)} = \left(\sum_{k\geq 0} (f_k, e_m)_F \, z^k, h_{n,m}\right)_{H^2} = 0 \text{ for every } n,\, m \in \mathbb{N}.$$

Since $\left\{ h_{n,m} \,\middle|\, n \in \mathbb{N} \right\}$ is total in H^2 we have

$$\sum_{k\geq 0} (f_k, e_m)_F \, z^k = 0 \text{ for every } m \in \mathbb{N},$$

which implies

$$(f_k, e_m)_F = 0 \text{ for every } m \in \mathbb{N} \text{ and } k \geq 0.$$

Since $\left\{ e_m \,\middle|\, m \in \mathbb{N} \right\}$ is total in F we must have $f_k = 0$ for every $k \geq 0$, so that $f = 0$.

QED

We now proceed to the construction of the spaces $\Phi(\lambda, N, K, M)$. In what follows we shall adopt the convention that a *pole of order* ∞ is an essential singularity. Similarly, we shall say that an operator is *nilpotent of order* ∞ if it is not nilpotent. In the lemma below 'card' denotes the cardinality of a set.

4.2.2 Lemma *Let $\lambda \in \mathbb{D}$, and let N and K be subsets of $\mathbb{N}$. Assume that there is $M \subset N \times K$ such that*

(a) $(n, 1) \in M$ *for every* $n \in N$*;*

(b) *if* $k \geq 2$ *and* $(n, k) \in M$ *then* $(n, k-1) \in M$*;*

(c) *there is* $n_0 \in N$ *such that* $\operatorname{card}\{(n_0, k) \in M\} = \operatorname{card} K$*.*

Then there is a complete and barrelled space $\Phi(\lambda, N, K, M) =: \Phi$ *such that*

(i) *the embedding* $\Phi \hookrightarrow H^2(F)$ *is continuous;*

(ii) $S|\Phi \in \mathcal{L}(\Phi)$*;*

(iii) $\sigma(S|\Phi) = \{\lambda\}$*;*

(iv) λ *is a pole of order* $\operatorname{card} K$ *of* $S|\Phi$*, and* $(S|\Phi - \lambda)$ *is nilpotent of order* $\operatorname{card} K$*;*

(v) λ *is an eigenvalue of* $S|\Phi$ *with geometric multiplicity* $\operatorname{card} N$ *and algebraic multiplicity* $\operatorname{card} M$*.*

Proof First we turn M into a directed set by introducing an order '$\preceq$' on M by

$$(n_1, k_1) \preceq (n_2, k_2) :\Longleftrightarrow n_1 + k_1 \leq n_2 + k_2 \text{ or } (n_1 + k_1 = n_2 + k_2 \text{ and } n_1 \leq n_2).$$

Next we define spaces $\Phi_{(n,k)}$ by

$$\Phi_{(n,k)} := \bigvee \left\{ g_{\lambda,\nu,\kappa} \,\middle|\, (\nu, \kappa) \preceq (n, k) \right\} \text{ for } (n, k) \in M.$$

Each $\Phi_{(n,k)}$ is finite-dimensional, since each (n, k) has only finitely many predecessors. Moreover, each embedding

$$\pi_{(n_1,k_1)(n_2,k_2)} : \Phi_{(n_1,k_1)} \hookrightarrow \Phi_{(n_2,k_2)} \text{ for } (n_1, k_1) \preceq (n_2, k_2)$$

is trivially continuous. The desired space is now defined as the inductive limit

$$\Phi(\lambda, N, K, M) := (M, \preceq)\text{-}\varinjlim \Phi_{(n,k)}$$

of the inductive system $\{\Phi_{(n_1,k_1)}; \pi_{(n_1,k_1)(n_2,k_2)}\}$. By theorem A.7.6, this inductive limit is complete and barrelled.

In order to get a better understanding of the construction it is useful to think of $\Phi(\lambda, N, K, M)$ as the linear hull of the elements of an ordered $\operatorname{card} N \times \operatorname{card} K$ matrix with $\operatorname{card} M$ entries:

$$\operatorname{card} N \left\{ \overbrace{\begin{array}{cccccc} g_{\lambda,n_1,1} & \rightarrow & g_{\lambda,n_1,2} & & g_{\lambda,n_1,3} & \cdots \\ & \swarrow & & \swarrow & & \\ g_{\lambda,n_2,1} & & g_{\lambda,n_2,2} & & g_{\lambda,n_2,3} & \cdots \\ & \swarrow & & \swarrow & & \\ g_{\lambda,n_3,1} & & g_{\lambda,n_3,2} & & & \\ & & & & & \\ \vdots & & \vdots & & \vdots & \ddots \end{array}}^{\operatorname{card} K} \right.$$

The arrows represent the order of the indices in M. Some entries (indexed by (n, k)) of the matrix are missing if (n, k) does not belong to M. Condition (a) means that the first column, which contains the eigenfunctions of S, is completely filled. Condition (b) states that if a row has an entry, then all other entries to the left are present as well. Finally, condition (c) implies that there is at least one row which is completely filled. Each $\Phi_{(n,k)}$ is the linear hull of the entry $g_{\lambda,n,k}$ and all other entries which are connected with it through the arrows. Let us recall that

$$(S - \lambda)g_{\lambda,\nu,\kappa} = \begin{cases} 0 & \text{for } \kappa = 1 \\ g_{\lambda,\nu,\kappa-1} & \text{for } \kappa > 1 \end{cases} .$$

The definition of the order '$\preceq$' implies that the matrix representation of $S|\Phi_{(n,k)}$ with respect to the basis $\left\{ g_{\lambda,\nu,\kappa} \,\middle|\, (\nu, \kappa) \preceq (n, k) \right\}$ is upper triangular with all diagonal entries being equal to λ and all off-diagonal entries being equal to zero or one. Keeping this picture in mind the following assertions are now obvious. If $(n, k) \in M$, then:

$$S(\Phi_{(n,k)}) \subset \Phi_{(n,k)}; \tag{4.4}$$

$$\sigma(S|\Phi_{(n,k)}) = \{\lambda\}; \tag{4.5}$$

$$\lambda \text{ is a pole of } S|\Phi_{(n,k)} \text{ of order } \sup\left\{ \kappa \,\middle|\, (\nu, \kappa) \preceq (n, k) \right\}. \tag{4.6}$$

Assertion (i) now follows proposition A.7.8 and the fact that the embedding $\Phi_{(\nu,\kappa)} \hookrightarrow H^2(F)$ is trivially continuous, while (ii) follows from the same proposition and (4.4). Assertion (iii) is implied by (4.5) and corollary A.7.11, and (iv) follows from (4.6) and proposition A.6.5. The last assertion is obvious.

QED

We are now able to prove the main result of this section.

4.2.3 Theorem *Let $\lambda : \mathbb{N} \to \mathbb{D}$ be a an injective function such that the set $\left\{ \lambda(n) \,\middle|\, n \in \mathbb{N} \right\}$ has a limit point in $\mathbb{D}$. Assume furthermore that there are functions*

$$\begin{aligned} m_g &: \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \\ m_a &: \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \\ p &: \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \end{aligned}$$

such that

$$m_g + p - 1 \leq m_a \leq m_g p. \tag{4.7}$$

Then there is a complete and barrelled space Φ such that

(i) Φ is densely and continuously embedded in $H^2(F)$;

(ii) $S|\Phi \in \mathcal{L}(\Phi)$;

(iii) $\sigma(S|\Phi) = \operatorname{clo} \left\{ \lambda(n) \,\middle|\, n \in \mathbb{N} \right\}$;

(iv) if $\lambda(n)$ is isolated in $\sigma(S|\Phi)$ then $\lambda(n)$ is a pole of order $p(n)$;

(v) each $\lambda(n)$ is an eigenvalue with geometric multiplicity $m_g(n)$ and algebraic multiplicity $m_a(n)$.

Proof Let $j : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection, and let pr_1 denote the projection of $\mathbb{N} \times \mathbb{N}$ onto the first factor, that is, $pr_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $pr_1(n, m) = n$. Put $j_1 := pr_1 \circ j$. We note that $j_1 : \mathbb{N} \to \mathbb{N}$ is surjective and that, for each $n \in \mathbb{N}$, $j_1^{-1}(n)$ contains infinitely many elements. We define the following sets:

$$N_n := \left\{ \nu \in \mathbb{N} \,\middle|\, j_1(n) \leq \nu \leq j_1(n) + m_g(n) \right\} \text{ for } n \in \mathbb{N};$$

$$K_n := \left\{ \kappa \in \mathbb{N} \,\middle|\, \kappa \leq p(n) \right\} \text{ for } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$ we choose a subset $M_n \subset N_n \times K_n$ with $\operatorname{card} M_n = m_a(n)$ such that the conditions (a), (b), and (c) of the previous lemma hold. This is possible by virtue of (4.7). With the data $\lambda(n)$, N_n, K_n, M_n we associate the spaces

$$\Phi_n := \Phi(\lambda(n), N_n, K_n, M_n) \text{ for } n \in \mathbb{N}, \tag{4.8}$$

constructed in the previous lemma. Finally, we let Φ be the locally convex direct sum (see [Sch, II.6.1]) of the spaces Φ_n:

$$\Phi := \bigoplus_{n\in\mathbb{N}} \Phi_n. \tag{4.9}$$

We shall now prove that Φ has the desired properties. First we show that Φ is dense in $H^2(F)$. To see this, note that

$$g_{\lambda(n),j_1(n),1} = (1-\lambda(n)z)^{-1}e_{j_1(n)} \in \Phi_n \text{ for every } n \in \mathbb{N}.$$

By lemma 4.2.1 (i) and (ii), the set

$$\left\{ g_{\lambda(n),j_1(n),1} \,\middle|\, n \in \mathbb{N} \right\} = \left\{ (1-\lambda(m)z)^{-1}e_k \,\middle|\, k \in \mathbb{N}, m \in j_1^{-1}(k) \right\}$$

is total in $H^2(F)$. Thus, Φ is dense in $H^2(F)$. In order to verify the remaining assertions we remark that the locally convex direct sum (4.9) is trivially an inductive limit as well. Let $\tilde{\Phi}_n$ denote the finite topological sum

$$\tilde{\Phi}_n := \bigoplus_{\nu\leq n} \Phi_\nu \text{ for } n \in \mathbb{N}.$$

Then Φ is the inductive limit of the spaces $\tilde{\Phi}_n$ with respect to the canonical embeddings $\tilde{\Phi}_n \hookrightarrow \tilde{\Phi}_m$ $(n \leq m)$:

$$\Phi = \underrightarrow{\mathrm{Ind}}\, \tilde{\Phi}_n.$$

The remaining part of assertion (i) now follows from an application of proposition A.7.8 to (i) of the previous lemma. The very same proposition applied to (ii) of the previous lemma yields assertion (ii). In order to prove (iii) we note that by corollary A.2.3,

$$\sigma(S|\tilde{\Phi}_n) = \left\{ \lambda(\nu) \,\middle|\, \nu \leq n \right\}.$$

This implies (iv) by virtue of corollary A.7.11. In proving the next assertion we assume that $\lambda(n_0)$ is isolated in $\sigma(S|\Phi)$. Let

$$E_1 := \Phi_{n_0},$$
$$E_2 := \bigoplus_{\substack{n\in\mathbb{N}\\ n\neq n_0}} \Phi_n.$$

Clearly, (E_1, E_2) reduces $S|\Phi$. Furthermore, $(S|E_1 - \lambda(n_0))$ is nilpotent of order $p(n_0)$ by assertion (iv) of the previous lemma. The same arguments we used to prove (iii) also imply that $\lambda(n_0) \notin \sigma(E_2)$. An application of theorem A.4.3 now shows that $\lambda(n_0)$ is a pole of order $p(n_0)$ of $S|\Phi$. The last assertion is obvious.

QED

This theorem shows that almost any set in $\mathbb{D}$, namely the closure of any sequence with a limit point in $\mathbb{D}$, can occur as the spectrum of the restriction of S to a dense subspace. Moreover, it is possible to turn the spectral points into eigenvalues of predetermined algebraic and geometric multiplicities. For isolated spectral points we may also choose the order of the corresponding poles of the resolvent. In particular, we have the following.

4.2.4 Corollary *Let* $\left\{ \lambda_n \in \mathbb{D} \,\middle|\, n \in \mathbb{N} \right\}$ *be a sequence converging to zero. Then there is a complete and barrelled, densely and continuously embedded subspace* Φ *of* $H^2(F)$ *such that*

$$S|\Phi = \sum_{n\in\mathbb{N}} \lambda_n P_n, \tag{4.10}$$

$$I = \sum_{n\in\mathbb{N}} P_n, \tag{4.11}$$

where P_n*,* $n \in \mathbb{N}$*, denotes a continuous projection and the sums converge in* $\mathcal{L}_s(\Phi)$*. Moreover, if* λ_n *is isolated in* $\sigma(U|\Phi)$ *then* P_n *is the spectral projection* $P_{\{\lambda_n\}}$ *associated with the spectral set* $\{\lambda_n\}$*.*

Proof Put $p = 1$ and choose m_g and m_a with $m_g = m_a$. Letting P_n, $n \in \mathbb{N}$, denote the projection onto the space Φ_n defined in (4.8) and taking Φ as in (4.9) the assertions of the corollary follow.

QED

In other words, the unilateral left-shift allows of a complete spectral decomposition on a suitably chosen subspace Φ. This decomposition is arbitrary to the extent that any sequence converging to zero may appear as the eigenvalue sequence in (4.10). We shall show in the next section that this result implies that the Frobenius-Perron operator of an exact dynamical system allows of an arbitrary spectral decomposition.

4.3 Spectral decomposition of exact systems

Let $(X, \mathcal{A}, \mu; T)$ be an exact dynamical system and let U be the associated Frobenius-Perron operator. We recall from Section 2.4 that in this case there is a unitary transformation V from $L^2(X, \mathcal{A}, \mu) \ominus \mathbb{C}$ onto $H^2(F)$ such that

$$U|L^2(X, \mathcal{A}, \mu) \ominus \mathbb{C} = V^{-1} S_F V.$$

Thus, all the results of the previous section also apply to the Frobenius-Perron operator of an exact system. In particular, the union of $\{1\}$ and the closure of any sequence having a limit point in $\mathbb{D}$ may occur as the spectrum of a suitable restriction of U. The resolvent of U may have poles of predetermined order at predetermined locations. Moreover, corollary 4.2.4 shows that every exact dynamical system admits a spectral decomposition.

Examples of exact systems are given by non-invertible Bernoulli systems (see Section 2.4) and weakly mixing systems for which there exists a test function space Φ such that the resolvent of $U|\Phi$ is meromorphic on the unit circle (see corollary 3.3.3). Concrete examples include:

- weakly mixing piecewise expanding interval transformations [HK] (see [Bow2] for sufficient conditions for weak mixing);
- weakly mixing piecewise expanding transformations in $\mathbb{R}^N$ [GB];
- one-sided subshifts of finite type with aperiodic transition matrix [PP1, Chapter 2, Remark 3].

4.4 Discussion

This chapter shows that for certain dynamical systems the spectral decompositions sought by the Brussels/Austin groups always exist. Indeed, we saw that the Frobenius-Perron operator of every exact dynamical system admits a spectral decomposition. We also saw that this spectral decomposition is to some extent arbitrary. Incidentally, we showed that there always exist test function spaces such that the resolvent of the restricted Frobenius-Perron operator has poles of predetermined order at predetermined locations. Since the poles of the resolvent correspond to the resonances of the system, resonances may occur at any location provided that the test function space is suitably chosen. In particular, one may always select a mode of description such that the collective decay rate of the system takes a predetermined value.

One might want to enquire as to the nature of the class of preparations producing these 'anomalous' resonances. In the case of the Rényi map test function spaces producing resonances different from $\{\beta^{-n}\}$ are likely to be ill-behaved. To construct a resonance corresponding to an eigenvalue with modulus greater β^{-1}, for example, one needs to include functions which are not once continuously differentiable[1]. Accepting β^{-1} to be the 'real' collective decay rate requires a commitment to the assumption that 'reality' is described by differentiable functions. Of course for the Rényi map, which is just a simple toy system, this 'reality' is the reality of computer experiments. Similar assumptions are made in the discussion of more sophisticated models of systems of the 'real' world. The decay of correlations in Axiom A systems or billiards, for example, is usually studied for well-behaved, smooth observables [Rue3, Rue4, Bum]. Ultimately, the 'right' choice of description is the one capable of reproducing observations obtained from experiments. If in a computer simulation of the Rényi map most of the decay rates are found to be larger than β^{-1} we need to take these anomalous decay rates seriously and seek an adequate description in terms of preparations which are not necessarily differentiable. In this case, candidates for test function spaces implementing this mode of description can be obtained using the construction outlined in this chapter.

It should be mentioned here that the smooth mode of description for the Rényi map is distinguished from other modes of description in that the $\mathcal{C}^\infty$-resonances coincide with the Ruelle resonances of the system (see Section 3.4). Let us recall that resonances, being poles of the power spectrum of the system, are defined on the probabilistic level, while Ruelle resonances are determined by a trajectory based object (the Ruelle ζ-function). This raises the question as to whether the 'anomalous' resonances of the Rényi map constructed in this chapter may be obtained from a trajectory based object in a similar way. This is unlikely, since our construction was based on the shift representation of the Frobenius-Perron operator of exact systems. Indeed, the Frobenius-Perron operators of all exact systems are unitarily equivalent suggesting that a coding of a system's dynamics finer than that provided by the Frobenius-Perron operator itself is required to retain information about the behaviour of the system on the trajectory level.

In this context one might wonder why the methods of the Brussels/Austin groups, which operate on the probabilistic level implemented by the Frobenius-Perron operator, yield the Ruelle resonances of the system. In the case of the Rényi map considered by Hasegawa and Saphir [HS1] and Antoniou and Tasaki [AT2], the authors choose a basis consisting of polynomials as a starting point (Legendre polynomials in [HS1] and monomials in [AT2]). In this case the procedures employed by the authors yield the spectrum of

[1] This follows from $\sigma(U|\mathcal{C}^1) = \left\{ \lambda \,\middle|\, |\lambda| \leq \beta^{-1} \right\} \cup \{1\}$.

the Frobenius-Perron operator restricted to the space of polynomials, which happens to coincide with the Ruelle resonances of the system (see Section 3.4). Other choices of a basis, for example one containing functions giving rise to anomalous resonances, would have produced different results.

Chapter 5

Discrete time subdynamics

This chapter and the one following are devoted to the earliest version of the Brussels formalism, also known as *subdynamics*, which may be seen to be a continuation of Boltzmann's original quest to reconcile the irreversible macroscopic laws governing the behaviour of matter in bulk with the basic time symmetric microscopic equations of motion. His search for the origin of the second law of thermodynamics led him to a characterisation of dynamical processes in many-body systems by a kinetic equation, which is now referred to as the Boltzmann equation. Since then numerous such equations have been derived: the Fokker-Planck equation for a Brownian particle, and the Vlasov and Balescu-Lenard equations for a plasma, to name but three. Their common feature is that they describe the Markovian dynamics of single particle distributions under certain physical conditions. The Boltzmann equation, for example, gives a correct description only for dilute gases. These conditions are usually formulated in terms of a limiting process for which an appropriately chosen scaling parameter of the system vanishes (see, for example, the work of Davies [Dav1, Dav2, Dav3], Lanford [Lan], and Spohn [Spo]).

The central concern of subdynamics is to dispense with this limiting procedure and provide instead a systematic method to derive kinetic equations which are valid over some finite range of values of the relevant scaling parameter. Let us recall (see Section 2.6) that the crucial element of this approach is the construction of an idempotent operator Π which commutes with the evolution operator of the system. The operator Π projects onto a subspace, the elements of which can be shown to obey a kinetic equation. For this reason, the range of Π is referred to as the 'kinetic subspace'. The projection Π thus determines a class of functions on the phase space of a system, the temporal evolution of which is governed by a kinetic equation and the original dynamics of the system.

Although this brief summary hardly does justice to the sophistication of the approach it does represent the key aspects of the theory as it stood in 1975. Further details may be found in the original articles listed in Section 2.6 and in the book by Balescu [Bal]. A concise and critical account of the theory is presented in [OB].

The technical details of subdynamics pose formidable mathematical problems, mostly due to the fact that unbounded non-selfadjoint operators need to be considered during the construction. These difficulties have as yet prevented a complete and mathematically rigorous treatment, although a number of authors have established the validity of certain parts of the formalism. The deep study of Lanz, Lugiato, and Ramella [LLR] should be mentioned here, as well as the papers of Kummer [Kumm], and Coveney and Penrose [CP].

In the present and the following chapter we shall investigate part of subdynamics from a rigorous point of view focussing on the arguments leading to the construction of the projection Π. To get a clear picture of the structure of the formalism we shall, contrary to the usual conventions of subdynamics, avoid elements of perturbation theory. In this chapter we shall study the discrete time version of subdynamics. This idea is due to Hasegawa and Saphir [HS1, HS2] who borrowed elements of the continuous time version to obtain generalised spectral decompositions for chaotic mappings. The discrete time version of the formalism is easier to handle than its continuous time counterpart, since the evolution operator of the system, that is, the Frobenius-Perron operator, is bounded. Yet it displays all the features of the continuous time version, which will be considered in the next chapter.

This chapter is organised as follows. We first derive the discrete time analogue of the generalised master equation for evolution operators on locally convex spaces. This leads to the Brussels decomposition of the resolvent of the evolution operator used by Hasegawa and Saphir. Next we formulate sufficient conditions which guarantee that this decomposition holds, which will then be used to prove theorem 5.2.2, where we recover the main features of the continuous time formalism. In Section 5.3 we discuss the discrete time analogues of the kinetic equations sought by the Brussels school. The next two sections are devoted to a characterisation of the class of operators fulfilling these conditions. In the final section we discuss which discrete time systems may be analysed using the framework of subdynamics.

Part of this chapter is based on material contained in the author's M.Phil dissertation [Ban][1], where the discrete time variant of subdynamics is analysed under the assumption that the evolution operator of the system is a continuous endomorphism of a Banach space. Sections 5.1 and 5.2 as well as the first part of Section 5.5 below generalise the corresponding results in [Ban] to evolution operators on locally convex spaces.

5.1 The discrete time master equation

One of the earliest attempts to generalise Boltzmann's kinetic equation to arbitrary systems was made by Pauli [Pau], who derived a master equation for the time evolution of the probability distribution of a quantum system by assuming that this was driven by the random steps in a Markov process, a hypothesis which is not in general consistent with the Liouville equation. Pauli's approach was later improved by van Hove [Hov1, Hov2], while Prigogine and Résibois [PR] arrived at an exact master equation for an arbitrary system. The underlying idea of the master equation approach is that the macroscopic properties of a system can be understood in terms of a 'reduced' description. This reduction amounts to selecting a 'relevant' and an 'irrelevant' part of the overall evolution of the system. A typical situation is that of a system coupled to a larger one, the reservoir. The overall dynamics are given by the compound system, that is, system and reservoir, and the relevant part may be chosen to be the dynamics of the system itself. In mathematical terms, this separation may be modelled by a projection operator. This idea goes back to Nakajima [Nak] and Zwanzig [Zwa1, Zwa2], who showed that earlier derivations of master equations could be simplified by using projection operators.

We shall now derive a discrete time analogue of the generalised master equation (GME) for the evolution operator of a discrete time dynamical system. For reasons which will become clear later in this chapter we shall use a very general setup: we assume that the evolution of the system is modelled by a continuous linear operator U acting on a sequentially complete and barrelled space E. Furthermore, we assume that $\mathcal{L}(E)$, the space of continuous linear operators on E, is equipped with an admissible $\mathfrak{S}$-topology

[1] This dissertation was approved for the degree of Master of Philosophy at the University of Cambridge. The main results are published in [BC].

turning $\mathcal{L}(E)$ into a locally convex algebra, and that the spectral radius $r(U)$ of U in this algebra is equal to 1. These conditions are certainly fulfilled if U is the Frobenius-Perron operator of a dynamical system acting on the Hilbert space of square-integrable functions over the phase-space of the system.

The temporal evolution of the system is governed by the difference equation of the dynamical semigroup $\left\{ U_n \,\middle|\, n \in \mathbb{N}_0 \right\}$,

$$U_{n+1} = UU_n \text{ for } n \in \mathbb{N}_0,$$

subject to the initial condition

$$U_0 := I.$$

By introducing into this difference equation a pair of projections P and Q with $P, Q \in \mathcal{L}(E)$ and $Q = I - P$, which model the separation into relevant and irrelevant parts, we obtain the following pair of equations:

$$PU_{n+1} = PUPU_n + PUQU_n; \tag{5.1}$$

$$QU_{n+1} = QUPU_n + QUQU_n. \tag{5.2}$$

The Brussels school refers to the P and Q subspaces as the 'vacuum' and the 'correlations', since in the original formalism P was meant to project onto the diagonal part of the density matrix of a quantum system.

In order to obtain a relation for the relevant component PU_n, which is the desired GME, we shall use $\mathcal{Z}$-transform techniques, which provide a simple but effective method to handle difference equations. Details may be found in Appendix B.

Applying a $\mathcal{Z}$-transform to equations (5.1) and (5.2) yields

$$z(P\tilde{U}(z) - P) = PUP\tilde{U}(z) + PUQ\tilde{U}(z) \tag{5.3}$$

$$z(Q\tilde{U}(z) - Q) = QUP\tilde{U}(z) + QUQ\tilde{U}(z), \tag{5.4}$$

where we have used the linearity of the $\mathcal{Z}$-transform and the shifting theorem. The symbol $\tilde{U}$ denotes the $\mathcal{Z}$-transform of $\{U_n\}$:

$$\tilde{U}(z) := \mathcal{Z}[U_n](z) = z[z - U]^{-1}.$$

If z belongs to the resolvent set $\varrho(QUQ)$ of QUQ we may solve for $Q\tilde{U}(z)$ in (5.4)

$$Q\tilde{U}(z) = z[z - QUQ]^{-1}Q + [z - QUQ]^{-1}QUP\tilde{U}(z). \tag{5.5}$$

Inserting this relation into (5.3) yields

$$\begin{aligned} z(P\tilde{U}(z) - P) = & \qquad (5.6) \\ PUP\tilde{U}(z) + zPUQ[z - QUQ]^{-1} + PUQ[z - QUQ]^{-1}QUP\tilde{U}(z). \end{aligned}$$

The Brussels school coined suggestive names for the continuous time analogues of the operators in (5.5) and (5.6), which we shall also adopt: the 'collision operator'

$$\tilde{\psi}(z) := PUQ[z - QUQ]^{-1}QUP;$$

the 'destruction operator'

$$\tilde{D}(z) := PUQ[z - QUQ]^{-1};$$

the 'creation operator'

$$\tilde{C}(z) := [z - QUQ]^{-1}QUP;$$

and the 'reduced resolvent'

$$\tilde{G}(z) := Q[z - QUQ]^{-1}Q.$$

For the sake of completeness we list their n-domain representations, that is their images under the inverse of the $\mathcal{Z}$-transform:

$$\begin{aligned} \psi_n &:= \begin{cases} PUQ(QUQ)^{n-1}QUP & n \geq 1 \\ 0 & n = 0 \end{cases}; \\ D_n &:= \begin{cases} PUQ(QUQ)^{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}; \\ C_n &:= \begin{cases} (QUQ)^{n-1}QUP & n \geq 1 \\ 0 & n = 0 \end{cases}; \\ G_n &:= \begin{cases} Q(QUQ)^{n-1}Q & n \geq 1 \\ 0 & n = 0 \end{cases}. \end{aligned}$$

The desired master equation may now be obtained from (5.6) by applying the inverse $\mathcal{Z}$-transform and using the convolution theorem:

$$PU_{n+1} = PUPU_n + D_{n+1}Q + \sum_{k=0}^{n} \psi_k PU_{n-k} \text{ for } n \in \mathbb{N}_0. \qquad (5.7)$$

This equation is an operator identity which, acting on an initial state $x_0 \in E$, provides a relation for the P-component of the iterates $x_n := U_n x_0$ of x_0:

$$Px_{n+1} = PUPx_n + D_{n+1}Qx_0 + \sum_{k=0}^{n} \psi_k Px_{n-k}. \tag{5.8}$$

This is a discrete time version of the generalised master equation. Like its continuous time counterpart it is an exact equation describing the rate of change of the reduced states Px_n. The third term on the right of this equation is known as the 'memory' term, since it relates the rate of change of Px_N at time N with the values of Px_n at earlier times. For this reason this equation is said to be non-Markovian. Kinetic equations, however, have the property that the future state of the system is completely determined by its present state, that is, they are Markovian. In order to pass from equation (5.8) to a Markovian equation, we shall follow the strategy of the Brussels school.

5.2 Subdynamics

The construction of the projection giving rise to independent subdynamics relies on a projective decomposition of the resolvent of the evolution operator, which we shall now derive.

We may *formally* solve for $P\tilde{U}(z)$ in equation (5.6),

$$z(P\tilde{U}(z) - P) = PUP\tilde{U}(z) + z\tilde{D}(z) + \tilde{\psi}(z)P\tilde{U}(z),$$

and obtain

$$P\tilde{U}(z) = z[z - PUP - \tilde{\psi}(z)]^{-1}[P + \tilde{D}(z)].$$

Adding this equation to equation (5.5), which provides an expression for $Q\tilde{U}(z)$,

$$Q\tilde{U}(z) = z\tilde{G}(z) + \tilde{C}(z)P\tilde{U}(z),$$

yields the following decomposition of $\tilde{U}(z)$:

$$\begin{aligned}
\tilde{U}(z) &= P\tilde{U}(z) + Q\tilde{U}(z) \\
&= [P + \tilde{C}(z)]P\tilde{U}(z) + z\tilde{G}(z) \\
&= z[P + \tilde{C}(z)][z - PUP - \tilde{\psi}(z)]^{-1}[P + \tilde{D}(z)] + z\tilde{G}(z).
\end{aligned} \tag{5.9}$$

For z outside the unit disk $\tilde{U}(z)$ is just $z[z-U]^{-1}$, so that the above equation is a decomposition of the resolvent of U, which is sometimes referred to as the 'Brussels decomposition' of the resolvent. A similar equation is at the heart of the continuous time formalism, with the infinitesimal generator L of the continuous semigroup of evolution operators in place of our U. In the continuous time version it is assumed that the functions $\tilde{C}$, $\tilde{D}$, and $\tilde{G}$ are holomorphic on a certain subset of the spectrum of L, and that the function $z \mapsto [z - PLP - \tilde{\psi}(z)]^{-1}$ has poles in this subset. The projections Π are then constructed from the residues at these poles. Before we formulate the discrete time analogues of these assumptions and proceed to the construction of the projections Π, we note that the decomposition above has to be interpreted as a relation between operator-valued functions on the complex plane. Letting $\mathcal{D}_L$ (respectively, $\mathcal{D}_R$) denote the set of complex numbers for which the left hand side (respectively, the right hand side) is defined we have

$$\mathcal{D}_L = \varrho(U),$$

$$\mathcal{D}_R = \varrho(QUQ) \cap \left\{ z \in \mathbb{C} \;\middle|\; [z - PUP - \tilde{\psi}(z)] \text{ is invertible in } \mathcal{L}(E) \right\},$$

so that equality holds in (5.9) for $z \in \mathcal{D}_L \cap \mathcal{D}_R$. Since this set may well be empty we shall make the following assumptions, which are the discrete time analogues of the hypothesis of the continuous time version mentioned above:

(A1) the operator-valued functions $\tilde{C}$, $\tilde{\psi}$, $\tilde{D}$, and $\tilde{G}$ are holomorphic on the complement of the open unit disk, that is, there is $r \in [0, 1)$ such that these functions are holomorphic on $\left\{ z \in \mathbb{C} \;\middle|\; |z| > r \right\}$;

(A2) the operator-valued function $z \mapsto [z - PUP - \tilde{\psi}(z)]^{-1}$ exists and is meromorphic on the complement of the open unit disk.

An immediate consequence of these assumptions is that the operator U is of a special form.

5.2.1 Proposition *Let E be sequentially complete and barrelled and let $U \in \mathcal{L}(E)$ with $r(U) = 1$. If (A1) and (A2) hold, then the resolvent of U is meromorphic on the complement of the open unit disk with at least one pole lying on the unit circle. In particular, $\sigma(U)$ may be decomposed into the union of a closed set σ, which lies inside a circle $|z| = r < 1$, and of a positive finite number of poles $\lambda_1, \cdots, \lambda_p$ with modulus less than or equal to one.*

Proof If (A1) and (A2) hold then the right hand side of equation (5.9) is meromorphic on the complement of the open unit disk. By hypothesis, $\tilde{U}$ is holomorphic on the complement of the closed unit disk. Thus, the right hand side defines a meromorphic extension of $\tilde{U}$ to the complement of the open unit disk. Since $r(U) = 1$, one point of the unit circle must be a singularity of the resolvent of U which, by what we have just proved, must be a pole.

QED

The existence of the projection operators sought by the Brussels school now follows from the Riesz projection theorem, which will be stated using the notation of the Brussels school.

5.2.2 Theorem *Let E be sequentially complete and barrelled and let $U \in \mathcal{L}_{\mathfrak{S}}(E)$ with $r(U) = 1$. If the assumptions (A1) and (A2) hold then there is $p \in \mathbb{N}$ such that there exist $p + 1$ continuous projections $\Pi^{(i)} \in \mathcal{L}(E)$, $0 \le i \le p$ with*

$$\sum_{i=0}^{p} \Pi^{(i)} = I, \tag{5.10}$$

$$\Pi^{(i)}\Pi^{(j)} = \delta_{ij}\Pi^{(i)} \text{ for } 0 \le i, j \le p, \tag{5.11}$$

$$U\Pi^{(i)} = \Pi^{(i)}U \text{ for } 0 \le i \le p. \tag{5.12}$$

Moreover, there are $p + 1$ commuting semigroups of continuous operators $\left\{ \Sigma_n^{(i)} \in \mathcal{L}(E) \,\middle|\, n \in \mathbb{N}_0 \right\}$, $0 \le i \le p$, that is,

$$\Sigma_n^{(i)}\Sigma_m^{(j)} = \Sigma_m^{(j)}\Sigma_n^{(i)} = \delta_{ij}\Sigma_{n+m}^{(i)} \text{ for } n, m \in \mathbb{N}_0 \text{ and } 0 \le i, j \le p, \tag{5.13}$$

such that

$$U_n = \Sigma_n^{(0)} + \Sigma_n^{(1)} + \Sigma_n^{(2)} + \cdots + \Sigma_n^{(p)} \text{ for } n \in \mathbb{N}_0. \tag{5.14}$$

Furthermore,

$$\Sigma_n^{(i)} = \lambda_i^n \Pi^{(i)} + \sum_{k=1}^{n} \binom{n}{k} \lambda_i^{n-k} N_i^k \text{ for } n \in \mathbb{N}_0 \text{ and } 1 \le i \le p, \tag{5.15}$$

where λ_i, $1 \le i \le p$, are complex numbers with modulus less than or equal to one, and $N_i \in \mathcal{L}(E)$, $1 \le i \le p$, are nilpotent operators. Finally, the semigroup $\{\Sigma_n^{(0)}\}$ satisfies

$$\limsup_{n\to\infty} \left|\left\langle \Sigma_n^{(0)}x, x' \right\rangle\right|^{1/n} < \min\left\{ |\lambda_i| \,\middle|\, 1 \le i \le p \right\} \text{ for } x \in E \text{ and } x' \in E'. \tag{5.16}$$

Proof Let σ and $\lambda_1, \cdots, \lambda_p$ have the same meaning as in the previous proposition. Define $\Pi^{(i)}$ to be the spectral projection associated with the spectral set $\{\lambda_i\}$ for $1 \leq i \leq p$ and let $\Pi^{(0)}$ be the spectral projection associated with the spectral set σ. The assertions (5.10), (5.11), and (5.12) now follow from the Riesz projection theorem (see theorem A.3.2). Putting

$$\Sigma_n^{(i)} := \Pi^{(i)} U^n \text{ for } n \in \mathbb{N}_0,\ 0 \leq i \leq p$$

immediately yields property (5.13) and the splitting of the evolution (5.14). In order to prove the next equation we define

$$N_i := \Pi^{(i)}(U - \lambda_i) \text{ for } 1 \leq i \leq p,$$

that is, N_i is the eigennilpotent of the eigenvalue λ_i. Thus, for $n \in \mathbb{N}_0$, we get

$$\begin{aligned} \Sigma_n^{(i)} &= \Pi^{(i)} U^n = \Pi^{(i)}(U - \lambda_i + \lambda_i)^n = \sum_{k=0}^{n} \binom{n}{k} N_i^k \lambda_i^{n-k} (\Pi^{(i)})^{n-k} \\ &= \lambda_i^n \Pi^{(i)} + \sum_{k=1}^{n} \binom{n}{k} \lambda_i^{n-k} N_i^k. \end{aligned}$$

The last relation follows from the fact that

$$\sigma(\Pi^{(0)} U) = \sigma \cup \{0\}$$

and the spectral radius formula.

QED

This theorem covers the basic features of subdynamics as defined by the Brussels school. The temporal evolution of the system may be separated into independently evolving parts by virtue of the idempotents $\Pi^{(i)}$ which, for $1 \leq i \leq p$, are the projections onto the algebraic eigenspace of the eigenvalue λ_i. The temporal evolution in the subspace $\Pi^{(i)}E$ is governed by the semigroup $\{\Sigma_n^{(i)}\}$. The operators $\Sigma_n^{(i)}$, $1 \leq i \leq p$, are the discrete time analogues of the so-called 'asymptotic evolution operators'. This terminology originates from the fact that the contribution of the 'remainder operator' $\Sigma_n^{(0)}$ to the evolution of the whole system becomes negligible for large n by (5.16), so that the groups $\{\Sigma_n^{(i)}\}$, $1 \leq i \leq p$, describe the dominant long time behaviour of the system. In the next section we shall show how these operators may be used to provide a discrete time analogue of kinetic equations.

To prove the theorem above we had to make certain hypotheses concerning the operators occurring in the decomposition of the resolvent, which are difficult to verify in practice. In order to get more manageable conditions for the existence of the theory we shall now formulate another set of hypotheses.

The assumption (A1) is equivalent to the following assumption:

(B1) the spectral radius of QUQ is strictly less than one: $r(QUQ) < 1$.

This is easily seen by taking into account that the resolvent set of QUQ is the natural domain of analyticity of the resolvent of QUQ, and that the function $\tilde{G}$ is holomorphic on the complement of the unit disk if and only if the resolvent of QUQ is holomorphic there. A physical interpretation of (B1) can be obtained from the spectral radius formula. If $r(QUQ) < 1$ then there is $r < 1$ such that

$$|\langle (QUQ)^n x, x' \rangle| = O(r^n) \text{ for every } x \in E \text{ and } x' \in E'.$$

This means that the correlations in the irrelevant part of the evolution decay exponentially fast.

In order to get sufficient conditions for (A2) we introduce the following hypothesis:

(B2) PU is a finite rank operator.

This assumption is no doubt very restrictive; it is clearly not satisfied in those situations for which the theory was originally designed, for example, in the case of a classical many-particle system, where P is meant to project onto the space of single-particle distributions. However, it appears to be very difficult to justify (A2) without making some compactness assumption concerning the operator $PUP + \tilde{\psi}(z)$. In the continuous time scenario similar assumptions have been made by Coveney and Penrose [CP], who assumed that P itself is a finite rank operator, or by Grecos, Guo and Guo [GGG] and Obcemea and Brändas [OB], who assumed that $PUP + \tilde{\psi}(z)$ has a 'spectral representation' as a sum of finite rank operators.

We shall show in Section 5.5 that (B1) and (B2) imply (A2), so that (B1) and (B2) are sufficient for the existence of subdynamics. We shall also see that (B1) and (B2) determine a special class of operators, and that the Frobenius-Perron operators of certain dynamical systems are of this type.

5.3 Discrete time kinetic equations

The projections $\Pi^{(i)}$ constructed in the last section may be used to pass from the discrete time master equation derived in section 5.1 to the analogue of a kinetic equation. The starting point is the $\mathcal{Z}$-transformed master equation (5.7):

$$zP(\tilde{U}(z) - I) = PUP\tilde{U}(z) + \tilde{D}(z) + \tilde{\psi}(z)\tilde{U}(z).$$

Multiplying this equation by z^{n-1} and using $\tilde{U}(z) = z[z-U]^{-1}$ we obtain

$$z^n P(z[z-U]^{-1} - I) = z^n PUP[z-U]^{-1} + z^{n-1}\tilde{D}(z) + z^n\tilde{\psi}(z)[z-U]^{-1}. \quad (5.17)$$

If the assumptions (A1) and (A2) hold, then $\tilde{D}$ and $\tilde{\psi}$ are holomorphic on the complement of the open unit disk, and, by proposition 5.2.1, the resolvent of U is meromorphic on the complement of the open unit disk with poles at $\lambda_1, \ldots, \lambda_p$.

First we shall study the case where all the poles are *simple*. Then the resolvent of U has a Laurent expansion at λ_i of the form

$$[z-U]^{-1} = (z-\lambda_i)^{-1}\Pi^{(i)} + W_i(z), \quad (5.18)$$

valid in a small punctured disk entered at λ_i. Here $W_i(z)$ is an $\mathcal{L}(E)$-valued function holomorphic at λ_i. Integrating equation (5.17) around a small positively oriented circle, centred at λ_i and lying inside the region where (5.18) is valid, we obtain

$$P\lambda_i^{n+1}\Pi^{(i)} = PUP\lambda_i^n\Pi^{(i)} + \tilde{\psi}(\lambda_i)\lambda_i^n\Pi^{(i)} \text{ for } n \in \mathbb{N}_0. \quad (5.19)$$

Note that the contribution of $z^{n-1}\tilde{D}(z)$ vanishes, since, by (A1), the function $\tilde{D}$ is holomorphic at λ_i. Since the poles are simple, the associated eigennilpotents vanish, so that

$$\lambda_i^n\Pi^{(i)} = \Pi^{(i)}U^n = \Sigma_n^{(i)} \text{ for } n \in \mathbb{N}_0$$

by theorem 5.2.2. We may thus write equation (5.19) in the following way

$$P\Sigma_{n+1}^{(i)} = [PUP + \tilde{\psi}(\lambda_i)]\Sigma_n^{(i)} \text{ for } n \in \mathbb{N}_0,$$

or

$$P\Pi^{(i)}U^{n+1} = [PUP + \tilde{\psi}(\lambda_i)]P\Pi^{(i)}U^n \text{ for } n \in \mathbb{N}_0.$$

This operator difference equation is the discrete time analogue of the kinetic equation sought by the Brussels school. Acting on an initial state $x_0 \in E$ it provides a relation for the rate of change of the P-components of the iterates $x_n := U_n x_0$ in the $\Pi^{(i)}E$ subspace:

$$P\Pi^{(i)}x_{n+1} = [PUP + \tilde{\psi}(\lambda_i)]P\Pi^{(i)}x_n \text{ for } n \in \mathbb{N}_0.$$

If the initial state x_0 belongs to the subspace $\Pi^{(i)}E$, then the above equation reads

$$Px_{n+1} = [PUP + \tilde{\psi}(\lambda_i)]Px_n \text{ for } n \in \mathbb{N}_0.$$

The projections $\Pi^{(i)}$ thus select a class of initial states with the property that its relevant part, that is, the P-component, satisfies a 'Markovian' rate equation. This means the rate of change of x_N at time N is completely determined by its present state. For this reason, the Brussels school refers to the range of the projection $\Pi^{(i)}$ as the 'kinetic subspace'. It should be noted that if U is the Frobenius-Perron operator of a dynamical system then kinetic subspaces associated with spectral points not lying on the unit circle cannot be probability densities, as they are not strictly non-negative. The corresponding 'kinetic equation' may, however, be interpreted as describing the relaxation of deviations from the equilibrium distribution.

The rate equations derived above are the discrete time analogues of differential equations which are of first order in time. It turns out that the kinetic equation takes a different form if the poles are not simple. As an example we consider the case where λ_i is a pole of order 2. Then the resolvent of U has an expansion of the form

$$[z - U]^{-1} = (z - \lambda_i)^{-2}N_i + (z - \lambda_i)^{-1}\Pi^{(i)} + \tilde{W}_i(z), \tag{5.20}$$

valid in a small punctured disk centred at λ_i. Here N_i is the eigennilpotent of the eigenvalue λ_i and $\tilde{W}_i(z)$ is a function holomorphic at λ_i. Integrating equation (5.17) around a small circle centred at λ_i as before yields the following relation, where we suppress the index i to avoid cluttered notation:

$$P[(n+1)\lambda^n N + \lambda^{n+1}\Pi] = [PUP + \tilde{\psi}(\lambda)][n\lambda^{n-1}N + \lambda^n\Pi] + \lambda^n\tilde{\psi}'(\lambda)N,$$

where $\tilde{\psi}'(\lambda)$ denotes the derivative of $\tilde{\psi}$ at λ. Note that, by theorem 5.2.2, we have

$$n\lambda^{n-1}N + \lambda^n\Pi = \Pi U^n,$$

so that the above equation reads

$$P\Pi U^{n+1} = [PUP + \tilde{\psi}(\lambda)]P\Pi U^n + \lambda^n \tilde{\psi}'(\lambda)N. \tag{5.21}$$

This equation may be rewritten as a difference equation involving higher iterates. In order to do this recall that

$$N = \Pi(U - \lambda).$$

Letting $n = 0$ in equation (5.21) and using the above expression for N yields

$$[I - \tilde{\psi}'(\lambda)]P\Pi U = [PUP + \tilde{\psi}(\lambda) - \lambda\tilde{\psi}'(\lambda)]\Pi, \tag{5.22}$$

which provides a relation for $P\Pi U$ for given $P\Pi$. A straightforward calculation shows that a solution $\{P\Pi U^n\}$ of equation (5.21) also satisfies the following equation

$$P\Pi U^{n+2} - [PUP + \tilde{\psi}(\lambda) + \lambda]P\Pi U^{n+1} + \lambda[PUP + \tilde{\psi}(\lambda)]P\Pi U^n = 0 \text{ for } n \in \mathbb{N}_0,$$

provided that the relation (5.22) holds. This equation is a 'second order' difference equation for $P\Pi U^n$. In general, higher order poles will give rise to higher order rate equations. All the kinetic equations listed in the introduction to this chapter are, however, of first order in time. The assumption that the poles of the resolvent are simple turns out to be an important ingredient when deriving the analogues of the physically relevant first order equations. In Section 5.5 we shall give conditions which guarantee that the poles of the resolvent of the evolution operator are simple.

5.4 A holomorphic Fredholm alternative

In this section we shall state and prove a simple characterisation of the inverse of certain holomorphic operator-valued functions, which will be instrumental for the investigation of the invertibility of $PUP + \tilde{\psi}(z)$. The material presented here is a straightforward generalisation of some results of Ribarič and Vidav [RV], which were originally stated for Banach spaces, to the locally convex case.

Let E be locally convex and let $T : E \to E$ be a finite rank operator, that is $\dim \mathcal{R}(T) < \infty$. Suppose that $x_1, \ldots, x_m$ is a basis of $\mathcal{R}(T)$ and that

$x'_1, \ldots, x'_m$ are elements of E' with $\left\langle x_i, x'_j \right\rangle = \delta_{ij}$. Then T has a representation of the form

$$Tx = \sum_{i=1}^{m} \langle Tx, x'_i \rangle \, x_i \text{ for } x \in E. \tag{5.23}$$

The *determinant* of $I - T$ is defined to be the determinant of the matrix $M_{ij} = (\delta_{ij} - \langle Tx_j, x'_i \rangle)$. It is not difficult to see that the determinant of $I - T$, which will be denoted by $\det(I - T)$, is well defined (the reasoning is essentially the same as in [Kat, III.§4.3]). Moreover, the determinant tells us whether $I - T$ is invertible or not.

5.4.1 Lemma *Let E and T be as above. Then the inverse $(I - T)^{-1}$ exists in $\mathcal{L}(E)$ if and only if $\det(I - T) \neq 0$. In this case, $S = I - (I - T)^{-1}$ has the same range as T and*

$$\begin{pmatrix} \langle Sx, x'_1 \rangle \\ \vdots \\ \langle Sx, x'_m \rangle \end{pmatrix} = -M^{-1} \begin{pmatrix} \langle Tx, x'_1 \rangle \\ \vdots \\ \langle Tx, x'_m \rangle \end{pmatrix}, \tag{5.24}$$

where M^{-1} is the inverse of the matrix $M_{ij} := (\delta_{ij} - \langle Tx_j, x'_i \rangle)$.

Proof The proof is essentially the same as [RV, lemma I].

QED

We note that the inverse of $(I - T)$, provided that it exists, can be explicitly calculated using equations (5.24) and (5.23). Before we state the main result of this section let us recall that a function $\mathcal{T} : D \to \mathcal{L}_s(E)_s$ with D a region of $\mathbb{C}$ is holomorphic if for every $x \in E$ and every $x' \in E'$ the scalar-valued functions $z \mapsto \langle \mathcal{T}(z)x, x' \rangle$ are holomorphic on D.

5.4.2 Proposition (Holomorphic Fredholm Alternative) *Let E be locally convex and let $\mathcal{T} : D \to \mathcal{L}_s(E)_s$ be a holomorphic function on the region D of the complex plane such that the ranges of all $\mathcal{T}(z)$, $z \in D$ belong to a finite-dimensional subspace of E. Then the following alternative holds. Either*

(i) $(I - \mathcal{T}(z))^{-1}$ exists for no $z \in D$ and $\lambda = 1$ is an eigenvalue for every operator $\mathcal{T}(z)$, $z \in D$;

or

(ii) $(I - \mathcal{T}(z))^{-1}$ *exists everywhere in* D *except possibly for a discrete set of points. In this case, the function* $z \mapsto (I - \mathcal{T}(z))^{-1}$ *is meromorphic from* D *into* $\mathcal{L}_s(E)_s$*, with the residues at the poles being finite rank operators.*

Proof Let $x_1, \ldots, x_m$ be a basis for $\mathcal{R}(\mathcal{T})$ and let $x'_1, \ldots, x'_m$ be elements of E' with $\left\langle x_i, x'_j \right\rangle = \delta_{ij}$. Furthermore, let $\mathcal{M}(z)$, $z \in D$, be the matrix $\mathcal{M}(z)_{ij} := (\delta_{ij} - \langle \mathcal{T}(z)x_j, x'_i \rangle)$. Put $\omega(z) := \det(I - \mathcal{T}(z)) = \det \mathcal{M}(z)$ and observe that, by hypothesis, ω is holomorphic on the region D.

Suppose now that $\omega = 0$ on D. Then by the previous lemma the operator $(I - \mathcal{T}(z))$ is not invertible for any $z \in D$. Thus for every $z \in D$ the number 1 belongs to the spectrum of $\mathcal{T}(z)$, which since $\mathcal{T}(z)$ is compact must be an eigenvalue of $\mathcal{T}(z)$.

If $\omega \neq 0$ on D, then $(I - \mathcal{T}(z))$ is invertible except for a set $\tilde{D}$ with no point of accumulation in D by virtue of the previous lemma. Now let $\mathcal{S}(z) := I - (I - \mathcal{T}(z))^{-1}$, $z \in D \setminus \tilde{D}$, and observe that

$$\begin{pmatrix} \langle \mathcal{S}x, x'_1 \rangle \\ \vdots \\ \langle \mathcal{S}x, x'_m \rangle \end{pmatrix} = -\mathcal{M}^{-1} \begin{pmatrix} \langle \mathcal{T}x, x'_1 \rangle \\ \vdots \\ \langle \mathcal{T}x, x'_m \rangle \end{pmatrix}.$$

Since $z \mapsto (\mathcal{M}(z))^{-1}$ is meromorphic on D, the remaining assertions follow.

QED

The term 'holomorphic Fredholm alternative' stems from the fact that, under the assumptions of the proposition, $(I - \mathcal{T}(z))$ is a Fredholm operator for every $z \in D$.

5.5 Quasi-finite rank operators and an ergodic theorem

In this section we shall examine the scope of the assumptions (B1) and (B2). These assumptions determine a special class of operators to be defined below. We shall also supply sufficient conditions for poles to be simple, which will be used to prove an operator ergodic theorem. Incidentally, this theorem provides an alternative characterisation of the Brussels projections Π.

5.5.1 Definition Let E be locally convex. An operator $U \in \mathcal{L}(E)$ is called *quasi-finite rank* if there is a projection $Q \in \mathcal{L}(E)$ such that

(i) $(I-Q)U$ is a finite rank operator

and

(ii) $r(QUQ) < 1$.

Clearly, conditions (B1) and (B2) imply that U is a quasi-finite rank operator. In order to relate $\{(A1),(A2)\}$ and $\{(B1),(B2)\}$ we need to study the function

$$z \mapsto [z - PUP - \tilde{\psi}(z)]^{-1},$$

where

$$\tilde{\psi}(z) = PUQ[z - QUQ]^{-1}QUP.$$

Using the results of the previous section we may now prove the following.

5.5.2 Proposition *Let E be sequentially complete and barrelled. Suppose that (B2) holds, that is, PU is a finite rank operator. Then $z \mapsto [z - PUP - \tilde{\psi}(z)]^{-1}$ is meromorphic on the annulus $A := \left\{ z \in \mathbb{C} \,\middle|\, |z| > r(QUQ) \right\}$, with the residues at the poles being finite rank operators.*

Proof Since the resolvent of QUQ is holomorphic on A the function $z \mapsto z^{-1}[PUP - \tilde{\psi}(z)]$ is holomorphic on A. Furthermore, all the values of this function are operators whose ranges belong to $\mathcal{R}(PU)$. Thus, the assertions follow immediately from proposition 5.4.2, provided that we can show that the inverse exists at one point in A. In order to prove this let $x_1, \dots, x_m$ be a basis of $\mathcal{R}(PU)$ and let $x'_1, \dots, x'_m$ belong to E' with $\left\langle x_i, x'_j \right\rangle = \delta_{ij}$. Let $\boldsymbol{PUP}$ and $\tilde{\boldsymbol{\psi}}(z)$, $z \in A$, be the $m \times m$ matrices defined by

$$\boldsymbol{PUP}_{ij} := \langle PUPx_j, x'_i \rangle$$

$$\tilde{\boldsymbol{\psi}}(z)_{ij} := \left\langle \tilde{\psi}(z)x_j, x'_i \right\rangle \text{ for } z \in A.$$

Choose $r > r(QUQ)$ and observe that, by the spectral radius formula, there is $K > 0$ such that

$$\left| \left\langle (QUQ)^n QUPx_j, (PUQ)'x'_i \right\rangle \right| \le Kr^n \text{ for } n \in \mathbb{N}_0.$$

Since $\tilde{\psi}(z)$ has a $\mathcal{L}_s(E)$-convergent Laurent expansion on A, given by

$$\tilde{\psi}(z) = \sum_{n=0}^{\infty} z^{-(n+1)} PUQ(QUQ)^n QUP,$$

we have

$$\left|\tilde{\boldsymbol{\psi}}(z)_{ij}\right| \leq K \sum_{n=0}^{\infty} |z|^{-(n+1)} r^n = K \frac{1}{|z| - r} \text{ for } |z| > r \text{ and } 1 \leq i, j \leq m. \tag{5.25}$$

Letting $\|\boldsymbol{M}\|$ denote the matrix-norm of the matrix $\boldsymbol{M}$ given by

$$\|\boldsymbol{M}\| := \sum_{i,j} |\boldsymbol{M}_{ij}|,$$

we see that

$$|z|^{-1} \left\|\boldsymbol{PUP} + \tilde{\boldsymbol{\psi}}(z)\right\| \to 0 \text{ as } |z| \to \infty,$$

by virtue of equation (5.25). This implies that the matrix $\boldsymbol{I} - z^{-1}[\boldsymbol{PUP} + \tilde{\boldsymbol{\psi}}(z)]$ is invertible for some z_0 with $|z_0|$ large enough. Thus, by lemma 5.4.1, the operator $I - z_0^{-1}[PUP + \tilde{\psi}(z_0)]$ is invertible in $\mathcal{L}(E)$.

QED

As an immediate consequence of this proposition we get the following to corollaries.

5.5.3 Corollary *(B1) and (B2) together imply (A2).*

5.5.4 Corollary *Let E be sequentially complete and barrelled. Then U is a quasi-finite rank operator if and only if the resolvent of U is meromorphic on the complement of the open unit disk, with the residues at the poles being finite rank operators.*

Proof The 'only if' clause is the content of the previous proposition, and the 'if' clause follows from theorem 5.2.2 by choosing Q to be $\Pi^{(0)}$.

QED

The last two corollaries show that the assumption of rapid decay of correlations in the Q-subspace (B1) and a certain compactness assumption (B2) are sufficient to justify the steps leading to the construction of the projection Π. Incidentally, these assumptions single out a special class of evolution operators, namely the class of quasi-finite rank operators characterised by corollary 5.5.4.

An apparently weaker set of assumptions guaranteeing the existence of Π is discussed in [Ban, BC], where the discrete time version of subdynamics is analysed for evolution operators on Banach spaces. There, our assumption (B2) is weakened to

(B2') PU is a compact operator.

Operators for which there exists a projection P such that (B1) and (B2') hold are referred to as 'quasicompact' operators in [Ban, BC]. It turns out that the class of quasicompact operators and the class of quasi-finite rank operators coincide for $U \in \mathcal{L}(E)$ with E being a Banach space. This follows from the characterisation of quasi-finite rank operators in corollary 5.5.4 of the present thesis and the characterisation of quasicompact operators in theorem 1 of [BC]. Thus, the set of assumptions $\{$(B1),(B2)$\}$ is equivalent to the set of assumptions $\{$(B1),(B2')$\}$ for evolution operators acting on Banach spaces. In this respect, the analysis presented in previous sections of this chapter yields an extension of the results in [Ban, BC] to evolution operators on locally convex spaces.

We shall now return to the problem of determining when poles of the resolvent of the evolution operator are simple. The following proposition provides sufficient conditions.

5.5.5 Proposition *Let E be sequentially complete and barrelled, and let $U \in \mathcal{L}(E)$. Suppose that $n^{-1}U^n \to 0$ in $\mathcal{L}_s(E)_s$ as $n \to \infty$. Then:*

(i) the spectrum of U is a subset of the closed unit disk;

(ii) every eigenvalue λ of U on the unit circle is simple, that is,

$$\mathcal{N}((\lambda - U)) = \mathcal{N}((\lambda - U)^2);$$

in particular, every pole of U on the unit circle is simple.

Proof Since $n^{-1}U^n \to 0$ in $\mathcal{L}_s(E)_s$, we have

$$|\langle U^n x, x' \rangle| \leq nK \text{ for } x \in E \text{ and } x' \in E',$$

where K is some positive constant depending on x and x'. Thus

$$\limsup_{n\to\infty} |\langle U^n x, x' \rangle|^{1/n} \leq 1 \text{ for every } x \in E,\ x' \in E'.$$

This implies that $r(U) \leq 1$ by virtue of the spectral radius formula and assertion (i) is proved.

In proving the second assertion we observe that if λ is an eigenvalue of U on the unit circle then 1 is an eigenvalue of $\hat{U} := \lambda^{-1}U$ and $n^{-1}\hat{U} \to 0$ in $\mathcal{L}_s(E)_s$. Thus to prove (ii) it suffices to consider the case where 1 is an eigenvalue of U. Suppose now to the contrary that 1 is not a simple eigenvalue of U. Then there exists an $x_0 \in E$ such that $(U-I)x_0 \neq 0$ but $(U-I)^2 x_0 = 0$. Since

$$U^n = (U - I + I)^n = \sum_{k=0}^{n} \binom{n}{k} (U-I)^k$$

we have

$$n^{-1}U^n x_0 = n^{-1}x_0 + (U-I)x_0 \text{ for } n \in \mathbb{N},$$

so that

$$n^{-1} \langle U^n x_0, x' \rangle = n^{-1} \langle x_0, x' \rangle + \langle (U-I)x_0, x' \rangle \text{ for } n \in \mathbb{N} \text{ and } x' \in E'.$$

Letting $n \to \infty$ in the equation above implies that $\langle (U-I)x_0, x' \rangle = 0$ for every $x' \in E'$. Thus $(U-I)x_0 = 0$, which is the desired contradiction.

QED

Using this result in connection with theorem 5.2.2 we obtain the following representation of the evolution operators.

5.5.6 Corollary *Let E be sequentially complete and barrelled and let U be a quasi-finite rank operator. If $n^{-1}U^n \to 0$ in $\mathcal{L}_s(E)_s$ as $n \to \infty$ then U has a representation of the form*

$$U^n = \sum_{|\lambda|=1} \lambda^n \Pi_\lambda + \left(\Sigma^{(0)}\right)^n = \sum_{i=1}^{p} \lambda_i^n \Pi^{(i)} + \left(\Sigma^{(0)}\right)^n \quad \textit{for } n \in \mathbb{N}_0,$$

and the following assertions hold:

- *(i) for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ the operator Π_λ is the spectral projection associated with the (possibly empty) spectral set $\{\lambda\} \cap \sigma(U)$ of U;*
- *(ii) $r(\Sigma^{(0)}) < 1$.*

The previous results may now be used to prove an operator ergodic theorem for quasi-finite rank operators, which also provides an alternative characterisation of the projections $\Pi^{(i)}$. Let $A_n(U)$ denote the sequence of the arithmetic means of the operator U:

$$A_n(U) := n^{-1} \sum_{k=0}^{n-1} U^k \text{ for } n \in \mathbb{N}.$$

The theorem may now be stated as follows.

5.5.7 Theorem *Let E be sequentially complete and barrelled and let U be a quasi-finite rank operator. Then the following statements are equivalent:*

(i) $n^{-1}U^n \to 0$ *in* $\mathcal{L}_s(E)_s$ *as* $n \to \infty$*;*

(ii) $A_n(\lambda^{-1}U) \to \Pi_\lambda$ *in* $\mathcal{L}_b(E)$ *as* $n \to \infty$ *for every* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$.

Here Π_λ *is defined as in corollary 5.5.6.*

Proof First we show that (i) follows from (ii). Since

$$n^{-1}U^n = A_{n+1}(U) - A_n(U) + n^{-1}A_{n+1}(U) \text{ for } n \in \mathbb{N},$$

the assumption $A_n(U) \to \Pi_1$ in $\mathcal{L}_b(E)$ implies that $n^{-1}U^n \to 0$ in $\mathcal{L}_b(E)$, and hence also in $\mathcal{L}_s(E)_s$.

In order to prove that (i) implies (ii), we note that by the previous corollary we may write

$$U^k = \sum_{i=1}^{p} \lambda_i^k \Pi^{(i)} + V^k \text{ for } k \in \mathbb{N}_0,$$

where $V := \Sigma^{(0)}$. Thus, for any complex λ with $|\lambda| = 1$ we have

$$n^{-1} \sum_{k=0}^{n-1} \lambda^{-k} U^k = \sum_{i=1}^{p} n^{-1} \sum_{k=0}^{n-1} (\lambda^{-1}\lambda_i)^k \Pi^{(i)} + n^{-1} \sum_{k=0}^{n-1} \lambda^{-k} V^k.$$

The second term on the right hand side of this equation converges to zero in $\mathcal{L}_b(E)$ as $n \to \infty$. To see this, let q be a continuous seminorm on $\mathcal{L}_b(E)$. Since $r(V) < 1$ there is $r < 1$ and $K > 0$ such that

$$q(V^k) \leq K r^k \text{ for } k \in \mathbb{N}_0$$

by the spectral radius formula. Thus

$$n^{-1}q\left(\sum_{k=0}^{n-1}\lambda^{-k}V^k\right) \leq n^{-1}\sum_{k=0}^{n-1}q(V^k) \leq n^{-1}\frac{1}{1-r},$$

which converges to zero as n approaches infinity. To investigate the first term we note that for i with $1 \leq i \leq p$ we have

$$\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}(\lambda^{-1}\lambda_i)^k = \begin{cases} 0 & \text{for } \lambda \neq \lambda_i \\ 1 & \text{for } \lambda = \lambda_i \end{cases} .$$

Thus, the first term converges to $\Pi^{(i)}$ if $\lambda = \lambda_i$ for $1 \leq i \leq p$. Otherwise it converges to zero.

QED

5.6 Are there systems with quasi-finite rank evolution operators?

We showed that the discrete time version of the resolvent based Brussels formalism holds provided that the assumptions (A1) and (A2) are satisfied. It turned out that these assumptions imply that the resolvent of the evolution operator of the system is meromorphic on the complement of the open unit disk, which enabled us to prove the existence of the projections $\Pi^{(i)}$ and the asymptotic evolution operators $\Sigma^{(i)}$. We also showed that these operators could be used to derive the discrete time analogue of a kinetic equation, which is the main concern of the continuous time formalism. Replacing the standing assumptions (A1) and (A2) by a stronger set of assumptions (B1) and (B2), we were able to determine a class of evolution operators for which the strategy of the Brussels school is viable.

At this point we need to ask whether there are any examples of discrete time systems whose evolution operators U belong to this class. To address this question let us suppose that $U : L^2(X, \mathcal{A}, \mu) \to L^2(X, \mathcal{A}, \mu)$ is the Frobenius-Perron operator of the dynamical system $(X, \mathcal{A}, \mu; T)$, which may be either invertible or non-invertible. If the system is non-invertible then the spectrum of U on the original Hilbert space L^2 is the closed unit disk, so that U cannot be a quasi-finite rank operator. Similarly, if the system is invertible and weakly mixing then U has continuous spectrum, and once again cannot be a quasi-finite rank operator. However, we saw in Chapters 3 and 4 that the resolvent of the Frobenius-Perron operator of certain non-invertible systems is a meromorphic function with finite rank residues on the complement of the open unit disk, and hence a quasi-finite rank operator, provided that U is considered as an operator on certain subspaces of the original Hilbert space, the so-called test function spaces introduced in Chapter 3. We were also able to show that, if the system is exact, such a test function space always exists. The resolvent based Brussels formalism presented in this chapter is thus applicable to exact systems provided that the space E, the state space of the probabilistic evolution, is chosen to be any of the test function spaces Φ constructed in Chapter 4.

The situation is less satisfactory for invertible systems. We saw in Chapter 3 that ergodic systems for which there exists a test function space Φ such that the resolvent of $U|\Phi$ is meromorphic on the unit circle are usually non-invertible (see proposition 3.3.5). This implies that invertible ergodic systems cannot be treated within the framework presented here, even if one allows the possibility of restricting the domain of the Frobenius-Perron operator of the system. In this context an interesting result by Evans and Coveney [EC] should be mentioned. The authors analyse a strongly mixing invertible system, the so-called 'pastry-cook's transformation' introduced by Penrose and Coveney [PC], in the framework presented here by considering a Banach space of functions satisfying a certain smoothness condition. This space is obtained by completing a dense subspace of the original Hilbert space with respect to a certain norm. As a consequence of this construction the authors are able to show that the resolvent of evolution operator considered as an endomorphism on the new Banach space is meromorphic on the unit circle. The new Banach space, however, cannot be viewed as a subspace of the original Hilbert space, since the injectivity of the embedding is lost during the completion. The new space thus contains functions which were not in the original Hilbert space and which cannot be interpreted as probability distributions. In this respect it is possible to justify the analyticity requirements of the resolvent based formalism even for an invertible system by partly enlarging the original Hilbert space.

Chapter 6

Continuous time subdynamics

The analysis of dynamical systems presented in the previous chapters was limited to systems with a discrete time parameter, that is, to $\mathbb{Z}$ or $\mathbb{N}$ actions. In this scenario the temporal evolution of the system could be described by a single continuous linear transformation U acting on a certain space and its iterates $\{U^n\}$. The spectral properties of U could then be used to describe the asymptotic behaviour of the dynamical system. At this point it is natural to ask which of the results presented in the previous chapter carry over to the physically more relevant systems with a continuous time parameter. In this case the temporal evolution can be modelled by a one-parameter semigroup $\{U_t\}_{t\in\mathbb{R}^+}$ of continuous linear transformations, where the parameter t representing the time variable belongs to the set of non-negative real numbers $\mathbb{R}_0^+$. This semigroup is uniquely determined by a single transformation L, the so-called infinitesimal generator. Analysing the spectral properties of L then yields information about the temporal behaviour of the system just as in the discrete time case. The possibility of describing the continuous time system by the single transformation L, however, is bought at the expense of losing the continuity of L. This technicality is one of the main difficulties in deriving the analogues of the discrete time results of the previous chapter. Nevertheless, most of the results of the discrete time case can be extended to the continuous time case. The following table may serve as a guideline for the correspondence between the discrete and continuous time setups:

discrete time	continuous time
discrete semigroup $\{U^n\}_{n\in\mathbb{N}_0}$	continuous semigroup $\{\exp(tL)\}_{t\in\mathbb{R}_0^+}$
generator U	infinitesimal generator L
$\mathcal{Z}$-transform	Laplace-transform
difference equation	differential equation

The first section is devoted to a brief introduction to the analytical theory of semigroups in Banach spaces. In the following two sections we shall discuss the derivation of the generalised master equation (GME), and the subdynamics approach aiming at the 'Markovisation' of the GME using the projection Π. To avoid unnecessary repetition, we shall focus on the differences in the mathematical structure of the equations and objects, as the physical interpretation is to a large extent the same as in the discrete time case. In the last section we calculate some of the objects of the theory using a specific example of a dynamical system and address a problem of Coveney and Penrose.

6.1 Semigroups and their infinitesimal generators

This section serves as a brief introduction to the theory of semigroups in Banach spaces so far as it is needed in the remainder of this chapter. We shall present the definitions of the basic objects and state the most important results including the Hille-Yosida theorem. Broadly speaking, the analytical theory of semigroups, extensively developed in the fifties and sixties, is an extension of the theory of ordinary differential equations to infinite-dimensional spaces. The standard references are the monographs by Hille and Phillips [HP] and Yosida [Yos].

Subsequently, we shall assume that $(E, \|.\|)$ is a Banach space with norm $\|.\|$. The domain of definition of a linear operator T acting on E will be denoted by $\mathcal{D}(T)$. We shall adopt the Banach space terminology for the topology of $\mathcal{L}_b(E)$ (respectively, $\mathcal{L}_s(E)$; respectively, $\mathcal{L}_s(E)_s$) which we shall refer to as the uniform (respectively, strong; respectively, weak) operator topology. The norm generating the topology of $\mathcal{L}_b(E)$, that is, the operator norm, will also be denoted by $\|.\|$, since confusion is unlikely to arise.

6.1.1 Definition A family $\left\{ U_t \in \mathcal{L}(E) \,\middle|\, t \in \mathbb{R}_0^+ \right\}$ is called a C_0 *semigroup of operators* if the following three conditions are satisfied:

(i) $U_t U_s = U_{t+s}$ for any $t, s \in \mathbb{R}_0^+$;

(ii) $U_0 = I$;

(iii) $\lim_{t \to t_0} U_t x = U_{t_0} x$ for each $t \in \mathbb{R}_0^+$ and each $x \in E$.

If there are constants M and β such that

$$\|U_t\| \leq M e^{\beta t} \text{ for any } x \in E \text{ and any } t \in \mathbb{R}_0^+$$

then the C_0 semigroup $\{U_t\}$ is said to belong to the class $\mathcal{G}(M, \beta)$.

Condition (i) means that $\{U_t\}$ is a semigroup, and condition (iii) states that the function $t \mapsto U_t$ is strongly continuous, that is, continuous in the strong operator topology. The latter condition is sufficiently general as most of the semigroups encountered in theoretical physics are of this type, for example, the quantum mechanical unitary groups generated by self-adjoint Hamiltonians. We note that $\{U_t\}$ belongs to $\mathcal{G}(M, \beta)$ if and only if the set $\left\{ e^{-\beta t} U_t \,\middle|\, t \in \mathbb{R}_0^+ \right\}$ is uniformly bounded. Since the uniformly bounded sets coincide with the weakly bounded sets, we have $\{U_t\} \in \mathcal{G}(M, \beta)$ if and only if for every $x \in E$ and every $x' \in E'$ there is a constant K such that

$$|\langle U_t x, x' \rangle| \leq K e^{\beta t} \text{ for every } t \in \mathbb{R}_0^+.$$

Suppose now that $\{U_t\}$ is a C_0 semigroup. Then the *infinitesimal generator* L of $\{U_t\}$ is defined as

$$Lx := \lim_{t \downarrow 0} t^{-1}(U_t x - x),$$

and the domain $\mathcal{D}(L)$ of L consists of all $x \in E$ for which this limit exists.

It turns out that $\mathcal{D}(L)$ is dense in E and that L is a closed operator. Moreover, for any $x \in \mathcal{D}(L)$ the function $t \mapsto U_t x$ is continuously differentiable and satisfies the differential equation

$$\frac{d}{dt} U_t x = L U_t x = U_t L x.$$

The interested reader should consult Chapter IX of [Yos] for the short proofs of these assertions. The above construction shows that we may associate a closed and densely defined linear transformation with any given C_0 semigroup. Conversely, we may ask the question as to whether a given closed and densely defined linear transformation is the infinitesimal generator of a C_0 semigroup. This problem is settled by the Hille-Yosida theorem. Before we state this important result we recall (see [Tay1, §5]) that the *resolvent set* $\varrho(L)$ of a closed operator L acting on a Banach space is the set of all complex numbers z for which $z - L$ has an everywhere defined continuous inverse:

$$\varrho(L) := \left\{\, z \in \mathbb{C} \,\middle|\, (z-L)^{-1} \in \mathcal{L}(E) \,\right\}.$$

The holomorphic function $\varrho(L) \ni z \mapsto (z-L)^{-1}$ is called the *resolvent* of L.

6.1.2 Theorem (Hille-Yosida) *A linear transformation L is the infinitesimal generator of a C_0 semigroup $\{U_t\}$ of class $\mathcal{G}(M,\beta)$ if and only if the following two conditions hold:*

(i) L is closed and $\mathcal{D}(L)$ is dense in E;

(ii) $\varrho(L)$ contains the half plane $\left\{\, z \in \mathbb{C} \,\middle|\, \Re z > \beta \,\right\}$ and

$$\left\|(z-L)^{-n}\right\| \leq M(z-\beta)^{-n} \text{ for every real } z > \beta \text{ and every } n \in \mathbb{N}.$$

If this is the case, $\{U_t\}$ and the resolvent of L are related as follows:

(a) for each $x \in E$ and each z with $\Re z > \beta$ we have

$$(z-L)^{-1}x = \int_0^\infty e^{-zt} U_t x \, dt;$$

(b) let $\gamma > \max(0,\beta)$. Then for each $x \in \mathcal{D}(L)$ and each $t \in \mathbb{R}^+$ we have

$$U_t x = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma - iR}^{\gamma + iR} e^{zt} (z-L)^{-1} x \, dz,$$

the limit existing uniformly with respect to t in any interval $(\epsilon, 1/\epsilon)$, $\epsilon > 0$.

Proof See [Yos, IX.7 theorem and IX.4 corollary 1] for the case where $\{U_t\} \in \mathcal{G}(M,0)$. The general case, that is $\{U_t\} \in \mathcal{G}(M,\beta)$, follows by applying [HP, 12.2.2 theorem] to the special case above. For (a), see [Yos, IX.4 theorem 1] and for (b), see [HP, 11.6.1 theorem].

QED

Formula (a) may be rephrased by saying that the resolvent of the infinitesimal generator is the (strong) Laplace-transform of the semigroup it generates, and formula (b) is just the inversion formula for the Laplace-transform. We note that $x \in \mathcal{D}(L)$ in (b) cannot be replaced by $x \in E$ in general, and that the contour of integration needs to be chosen in the right half plane in general, even if $\{U_t\} \in \mathcal{G}(M, \beta)$ with $\beta < 0$. Formula (b) also implies that a C_0 semigroup $\{U_t\}$ is uniquely determined by its infinitesimal generator L, and we write

$$\exp(tL) := U_t$$

to denote the unique semigroup generated by L.

6.2 Deriving the continuous time generalised master equation

The continuous time generalised master equation can be derived along the same lines as its discrete time counterpart using Laplace-transforms instead of $\mathcal{Z}$-transforms. In fact, this is the approach originally taken by Nakajima [Nak], Zwanzig [Zwa1, Zwa2], and Prigogine and Résibois [PR]. It is, however, quite instructive to treat the derivation of the GME as a problem concerning ordinary differential equations in infinite-dimensional spaces, since the conditions which the objects occurring in the GME have to fulfill become immediately apparent.

We assume that the temporal evolution of the system is governed by a C_0 semigroup $\{U_t\}$ of class $\mathcal{G}(1, 0)$, that is, $\{U_t\}$ is a semigroup of contractions. This is certainly the case if $\{U_t\}$ is the Frobenius-Perron operator of a continuous time dynamical system (see proposition 2.3.6). Denoting the infinitesimal generator of this semigroup by L we have

$$\frac{d}{dt} U_t x_0 = L U_t x_0 \text{ for } x_0 \in \mathcal{D}(L),$$

so that $x_t := U_t x_0$, $t \in \mathbb{R}_0^+$, is a continuously differentiable solution of the equation

$$\frac{d}{dt} x_t = L x_t. \tag{6.1}$$

In statistical mechanics this equation is known as the Liouville-von Neumann equation, which, by virtue of the Hille-Yosida theorem, may also be used as the starting point for the derivation of the GME. In this context, L is called the Liouville operator of the system.

As in the discrete time case the separation into relevant and irrelevant parts will be modelled by a pair of projections P and Q belonging to $\mathcal{L}(E)$ with $P + Q = I$. Contrary to the discrete time case the operator L is no longer continuous and its domain of definition does not coincide with all of E. To proceed we assume that:

(M1) $Px \in \mathcal{D}(L)$ and $Qx \in \mathcal{D}(L)$ for every $x \in \mathcal{D}(L)$;

(M2) PLQ and QLP extend to everywhere defined continuous operators.

The first assumption guarantees that the operators PLP, PLQ, QLP, and QLQ have non-trivial domains of definition; in fact, they contain $\mathcal{D}(L)$ and are thus dense in E. The second assumption can be interpreted as a regularity condition for the transition between the relevant and the irrelevant part of the evolution.

Inserting P and Q in equation (6.1) we obtain the pair of differential equations

$$\frac{d}{dt}Px_t = PLx_t = PLPx_t + PLQx_t \tag{6.2}$$

$$\frac{d}{dt}Qx_t = QLx_t = QLPx_t + QLQx_t. \tag{6.3}$$

The second equation is an inhomogeneous differential equation for Qx_t. To solve it we assume that

(M3) QLQ is the infinitesimal generator of a C_0 semigroup.

Since the function $\mathbb{R}_0^+ \ni t \mapsto QLPx_t$ is continuous, which follows from $QLP \in \mathcal{L}(E)$, and since QLQ generates a C_0 semigroup, we may invoke a theorem of Phillips (see [Kat, IX. theorem 1.19]) to obtain a solution of equation (6.3):

$$Qx_t = \exp(tQLQ)Qx_0 + \int_0^t \exp((t-s)QLQ)QLPx_s\, ds. \tag{6.4}$$

Inserting this expression into (6.2) yields the GME

$$\frac{d}{dt}Px_t = PLPx_t + PLQ\exp(tQLQ)Qx_0 + \int_0^t PLQ\exp((t-s)QLQ)QLPx_s\, ds,$$

where we used assumption (M2) to put PLQ under the integral. We summarise this result in the following proposition.

6.2.1 Proposition *Suppose that the continuous projections P and Q with $P+Q=I$ are chosen such that:*

(M1) $P(\mathcal{D}(L)) \subset \mathcal{D}(L)$ *and* $Q(\mathcal{D}(L)) \subset \mathcal{D}(L)$*;*

(M2) $PLQ \in \mathcal{L}(E)$ *and* $QLP \in \mathcal{L}(E)$*;*

(M3) QLQ *is the infinitesimal generator of a* C_0 *semigroup.*

Then a solution of the equation

$$\frac{d}{dt}x_t = Lx_t$$

with $x_0 \in \mathcal{D}(L)$ *is also a solution of the continuous time GME*

$$\frac{d}{dt}Px_t = PLPx_t + D(t)Qx_0 + \int_0^t \psi(t-s)x_s\,ds. \tag{6.5}$$

Here, $D(t)$ *and* $\psi(t)$ *are defined as*

$$\begin{array}{rcll} D(t) & := & PLQ\exp(tQLQ) & \text{for } t \in \mathbb{R}_0^+, \\ \psi(t) & := & PLQ\exp(tQLQ)QLP & \text{for } t \in \mathbb{R}_0^+ \end{array}$$

with $t \mapsto D(t)$ *and* $t \mapsto \psi(t)$ *being* $\mathcal{L}(E)$*-valued strongly continuous functions.*

This proposition summarises the main idea of the master equation approach to statistical mechanics. The integro-differential equation (6.5) for the relevant part Px_t of the evolution is 'exact' in the sense that it holds for all solutions x_t of the Liouville equation (6.1) with suitable initial conditions, that is, $x_0 \in \mathcal{D}(L)$. As in the discrete time case, this equation is non-local in time, or non-Markovian, due to the presence of the 'memory term', that is, the integral involving the time-domain collision operator ψ.

6.3 Subdynamics and kinetic equations

To derive the projective decomposition of the resolvent of L we assume that the assumptions (M1),(M2), and (M3) of proposition 6.2.1 hold. Since $\mathcal{R}((z-L)^{-1}) = \mathcal{D}(L)$ for $z \in \varrho(L)$ the following identity holds for every $z \in \varrho(L)$:

$$z(z-L)^{-1} - I = z(z-L)^{-1} - (z-L)(z-L)^{-1} = L(z-L)^{-1}.$$

Inserting P and Q into the above relation yields

$$Pz[z-L]^{-1} - P = PLP[z-L]^{-1} + PLQ[z-L]^{-1},$$

$$Qz[z-L]^{-1} - Q = QLP[z-L]^{-1} + QLQ[z-L]^{-1}.$$

This pair of equations has the same structure as the corresponding pair of the discrete time version (see equations (5.3) and (5.4)). The same algebraic manipulations may then be used to obtain the decomposition

$$[z-L]^{-1} = [P+\tilde{C}(z)][z-PLP-\tilde{\psi}(z)]^{-1}[P+\tilde{D}(z)] + \tilde{G}(z), \qquad (6.6)$$

where the continuous time analogues of the creation, destruction, and collision operators are defined as

$$\begin{array}{rcl} \tilde{C}(z) & := & [z-QLQ]^{-1}QLP, \\ \tilde{\psi}(z) & := & PLQ[z-QLQ]^{-1}QLP, \\ \tilde{D}(z) & := & PLQ[z-QLQ]^{-1}, \\ \tilde{G}(z) & := & Q[z-QLQ]^{-1}Q, \end{array}$$

for $z \in \varrho(QLQ)$. Owing to the continuity of PLQ and QLP the functions listed above are $\mathcal{L}(E)$-valued and holomorphic on $\varrho(QLQ)$. Moreover, these functions are the Laplace-transforms of the $\mathcal{L}(E)$-valued strongly continuous functions defined for $t \in \mathbb{R}_0^+$:

$$\begin{array}{rcl} C(t) & := & \exp(tQLQ)QLP; \\ \psi(t) & := & PLQ\exp(tQLQ)QLP; \\ D(t) & := & PLQ\exp(tQLQ); \\ G(t) & := & Q\exp(tQLQ)Q. \end{array}$$

As in the discrete time case, the decomposition (6.6) has to be understood as a relation between $\mathcal{L}(E)$-valued functions on the complex plane with equality holding for $z \in \mathcal{D}_L \cap \mathcal{D}_R$, where

$$\mathcal{D}_L := \varrho(L),$$

$$\mathcal{D}_R := \varrho(QLQ) \cap \left\{ z \in \mathbb{C} \,\middle|\, [z-PLP-\tilde{\psi}(z)]^{-1} \in \mathcal{L}(E) \right\}.$$

To construct the projections Π the Brussels school makes the following assumptions (see, for example, [PGH, GGG, OB]):

(C1) $\tilde{C}$, $\tilde{D}$, $\tilde{\psi}$, and $\tilde{G}$ are holomorphic at zero;

(C2) the function $z \mapsto [z - PLP - \tilde{\psi}(z)]^{-1}$ is meromorphic at zero.

These assumptions are slightly weaker than their discrete time counterparts presented in the last chapter. We shall briefly return to this difference later in this section.

If (C1) and (C2) hold then, by virtue of the decomposition (6.6), the resolvent of L must be meromorphic at zero. The functional calculus for closed operators on a Banach space (see [Tay1, §5.6]) may then be used to construct a projection operator Π as the residue of $[z - L]^{-1}$ at each pole.

6.3.1 Theorem *If (C1) and (C2) hold and if λ is a pole of order m of the resolvent of L, then $\Re\lambda \leq 0$ and there is a projection $\Pi \in \mathcal{L}(E)$ given by*

$$\Pi := \frac{1}{2\pi i} \oint_{|z-\lambda|=\epsilon} (z - L)^{-1}\, dz \tag{6.7}$$

where $\epsilon > 0$ is chosen small enough so that the contour in (6.7) belongs to $\varrho(L)$ and encloses λ but no other singularity of the resolvent of L.

This projection satisfies

$$\Pi L x = L \Pi x \text{ for } x \in \mathcal{D}(L), \tag{6.8}$$

$$\Pi U_t x = U_t \Pi x \text{ for } x \in E \text{ and } t \in \mathbb{R}_0^+. \tag{6.9}$$

In particular, the function $t \mapsto \Sigma_t := \Pi U_t$ is a strongly continuous semigroup of operators having a representation of the form

$$\Sigma_t = e^{\lambda t}\Pi + \sum_{k=1}^{m-1} \frac{t^k}{k!} N^k \text{ for } t \in \mathbb{R}_0^+, \tag{6.10}$$

where N is nilpotent of order m.

Proof Since L is the infinitesimal generator of a C_0 semigroup of class $\mathcal{G}(1,0)$ the open right half plane belongs to $\varrho(L)$ by virtue of the Hille-Yosida theorem, whence it follows that $\Re\lambda \leq 0$. The assertion that Π is a continuous projection satisfying (6.8) follows immediately from the functional calculus for closed operators. In proving the next relation we note that (6.9) is trivially satisfied for $t = 0$. For $t > 0$ we use the formula (see theorem 6.1.2)

$$U_t x = \frac{1}{2\pi i} \int_{\gamma-\infty}^{\gamma+\infty} e^{zt}(z - L)^{-1} x\, dz \tag{6.11}$$

which holds for $x \in \mathcal{D}(L)$. Since Π is continuous and commutes with the resolvent of L the assertion is true for every $x \in \mathcal{D}(L)$. Since U_{t_0} is continuous for every $t_0 \in \mathbb{R}_0^+$ and $\mathcal{D}(L)$ is dense in E, the relation must hold for every $x \in E$. To prove (6.10) we note that, as a consequence of the functional calculus, the Laurent expansion of $\Pi[z-L]^{-1}$ at λ is given by

$$\Pi[z-L]^{-1} = (z-\lambda)^{-1}\Pi + \sum_{k=1}^{m-1} (z-\lambda)^{-(k+1)} N^k, \tag{6.12}$$

valid in a small punctured disk centred at λ. In the above equation N denotes the eigennilpotent associated with λ,

$$N := (\lambda - L)\Pi.$$

Equations (6.11) and (6.12) together imply that for $x \in \mathcal{D}(L)$

$$\begin{aligned}\Pi U_t x = \; & \frac{1}{2\pi i}\int_{\gamma-\infty}^{\gamma+\infty} e^{zt}(z-\lambda)^{-1}\, dz \cdot \Pi x + \\ & \sum_{k=1}^{m-1} \frac{1}{2\pi i}\int_{\gamma-\infty}^{\gamma+\infty} e^{zt}(z-\lambda)^{-(k+1)}\, dz \cdot N^k x.\end{aligned}$$

Evaluating the integrals we see that (6.10) holds for $x \in \mathcal{D}(L)$ and, by virtue of the continuity of Π and N, also for any $x \in E$.

QED

The construction of Π above as the Riesz projection associated with the spectral set $\{\lambda\}$ of L differs from the procedure used by the Brussels school in the early 1970s, where the projection is obtained by considering a Laurent expansion of $[z - PLP - \tilde{\psi}(z)]^{-1}$ at zero (see [PGH, Bal, GGG]). In the case of a simple pole the equivalence of the two constructions can be established as follows. Suppose that $A_{-1} \in \mathcal{L}(E)$ is the residue of the function $z \mapsto [z - PLP - \tilde{\psi}(z)]^{-1}$ at λ.[1] Then provided that λ is a simple pole of L we obtain

$$\Pi = [P + \tilde{C}(\lambda)]A_{-1}[P + \tilde{D}(\lambda)], \tag{6.13}$$

$$\Sigma_t = [P + \tilde{C}(\lambda)]e^{\lambda t}A_{-1}[P + \tilde{D}(\lambda)] \tag{6.14}$$

[1] In general, there is no simple relation between A_{-1} and the operators PLP and $\tilde{\psi}(z)$. The formula $A_{-1} = [I - \tilde{\psi}'(\lambda)]^{-1}$ given in [OB, p. 398] is true for $\dim P(E) = 1$ in the case of a simple pole only, and false, in general, if $\dim P(E) > 1$.

by comparing coefficients in the Laurent expansion of the decomposition (6.6) at λ and using assumption (C1). The above equations, however, are just the formulæ given by the Brussels school [Bal, GGG]. To conclude this discussion of the equivalence of the two constructions we remark that if $\{U_t\}$ is a C_0 group of unitary operators, which is the scenario the Brussels group is primarily concerned with, then every pole of L must be simple. For then, by virtue of Stone's theorem [Yos, IX.9 corollary 2], the operator iL is selfadjoint which, by [Kat, V §3.5], implies that every pole of the resolvent of iL is simple.

The projection Π can of course be constructed at any pole of the resolvent of L. Denoting these poles by $\lambda_1, \dots, \lambda_p$ we arrive at a set of projections $\{\Pi^{(1)}, \dots, \Pi^{(p)}\}$, all of which have the properties stated in the theorem above. If $\Pi^{(0)}$ is defined as

$$\Pi^{(0)} := I - \sum_{i=1}^{p} \Pi^{(i)},$$

then $\{\Pi^{(0)}, \dots, \Pi^{(p)}\}$ is a set of continuous projections satisfying

$$\sum_{i=0}^{p} \Pi^{(i)} = I, \tag{6.15}$$

$$\Pi^{(i)}\Pi^{(j)} = \delta_{ij}\Pi^{(i)} \text{ for } 0 \le i, j \le p. \tag{6.16}$$

Furthermore, the functions $\mathbb{R}_0^+ \ni t \mapsto \Sigma_t^{(i)} := \Pi^{(i)}U_t$, $0 \le i \le p$ are strongly continuous commuting semigroups of operators. We have thus recovered the results of the discrete time case (see theorem 5.2.2) with one exception: the asymptotic bound on the behaviour of the semigroup $\Sigma_t^{(0)}$. This is a difficult problem, which we shall not pursue any further. We simply remark that an exponential bound similar to the discrete time case can be obtained provided that the contour in the inversion formula

$$U_t x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} \{\tilde{R}(z) + \tilde{G}(z)\} x \, dz,$$

where

$$\tilde{R}(z) := [P + \tilde{C}(z)][z - PLP - \tilde{\psi}(z)]^{-1}[P + \tilde{D}(z)],$$

can be displaced to the left across the imaginary axis in a neighbourhood of infinity. This requires that $\tilde{R}$ and $\tilde{G}$ be holomorphic in a 'strip'

$$\left\{ z \in \mathbb{C} \,\middle|\, |\Re z| < \epsilon \text{ and } |\Im z| \ge K \right\},$$

with $\epsilon > 0$ and $K \geq 0$, and satisfy certain growth conditions in this strip. The continuous time analogues of conditions (A1) and (A2), which read

(C1') the operator-valued functions $\tilde{C}$, $\tilde{\psi}$, $\tilde{D}$, and $\tilde{G}$ are holomorphic on the complement of the open left half plane;

(C2') the operator-valued function $z \mapsto [z - PLP - \tilde{\psi}(z)]^{-1}$ exists and is meromorphic on the complement of the open left half plane,

are not sufficient in general. The reason is that the operator-valued function $z \mapsto [z - PLP - \tilde{\psi}(z)]^{-1}$ might have infinitely many poles. In this case, the procedure of 'picking up' the contributions of the poles and deforming the contour in a neighbourhood of infinity requires an additional justification.

However, even without an asymptotic bound on the remainder operator $\Sigma_t^{(0)}$ the 'Markovisation' of the GME envisaged by the Brussels school can be achieved. The procedure is essentially the same as in the discrete time case. We assume that the hypotheses (M1), (M2) and (M3) of proposition 6.2.1 are satisfied and that

(M4) $PLP \in \mathcal{L}(E)$

holds. In this case we may apply a Laplace-transform on both sides of the GME (6.5) and obtain ($x_0 \in \mathcal{D}(L)$)

$$P[z-L]^{-1}x_0 - Px_0 = PLP[z-L]^{-1}x_0 + \tilde{D}(z)Qx_0 + \tilde{\psi}(z)[z-L]^{-1}x_0, \quad (6.17)$$

where we used condition (M4) to conclude that $PLP[z-L]^{-1}x_0$ is the Laplace-transform of $PLPU_tx_0$.

Suppose now that the assumptions (C1) and (C2) hold and that λ is a simple pole of the resolvent of L. Multiplying both sides of equation (6.17) by e^{zt} and integrating the resulting equation around a small positively oriented circle, centred at λ, lying in the region where the Laurent expansion

$$[z-L]^{-1} = (z-\lambda)^{-1}\Pi + W(z)$$

is valid, we obtain

$$P\Pi\lambda e^{\lambda t}x_0 = PLP\Pi e^{\lambda t}x_0 + \tilde{\psi}(\lambda)\Pi e^{\lambda t}x_0. \quad (6.18)$$

By theorem 6.3.1 we have

$$\Pi U_t x_0 = e^{\lambda t}\Pi x_0,$$

so that equation (6.18) may be written

$$\frac{d}{dt}P\Pi U_t x_0 = [PLP + \tilde{\psi}(\lambda)]P\Pi U_t x_0,$$

or

$$\frac{d}{dt}P\Pi x_t = [PLP + \tilde{\psi}(\lambda)]P\Pi x_t. \tag{6.19}$$

We have arrived at a differential equation for the relevant part of the Π-component of the total evolution. If x_t belongs to the kinetic subspace, which by theorem 6.3.1 is the case if and only if $x_0 \in \Pi(E)$, then equation (6.19) reads

$$\frac{d}{dt}Px_t = [PLP + \tilde{\psi}(\lambda)]Px_t,$$

which is the kinetic equation sought by the Brussels school.

6.4 A problem of Coveney and Penrose

In a recent article, Coveney and Penrose [CP] have analysed certain aspects of the subdynamics approach and have given conditions under which the P-P component of the total evolution U_t can be split into two parts $W(t)$ and $\hat{W}(t)$, with $W(t)$ exhibiting exponential decay and $\hat{W}(t)$ decaying faster than $W(t)$. Their conditions are formulated in the time-dependent language, as opposed to the stationary language which is concerned with the holomorphic structure of the Laplace-transformed objects of the theory (see the discussion in the previous section). The purpose of this section is to give an example of a dynamical system which satisfies these conditions.

To appreciate their result we take a closer look at the GME

$$\frac{d}{dt}PU_t x_0 = PLPU_t x_0 + D(t)Qx_0 + \int_0^t \psi(t-s)U_s x_0\, ds \text{ with } x_0 \in \mathcal{D}(L).$$

If x_0 belongs to the range of P, that is, if $Px_0 = x_0$, then the equation takes the form

$$\frac{d}{dt}V(t)x_0 = PLPV(t)x_0 + \int_0^t \psi(t-s)V(s)x_0\, ds, \tag{6.20}$$

where we have made the definition

$$V(t) := PU_tP \text{ for } t \in \mathbb{R}_0^+.$$

Equation (6.20) is a *closed* integro-differential equation for the relevant part $PU_tx_0 = Px_t$ of the total evolution. In quantum statistical mechanics, where P is meant to project on the diagonal part of density matrices, the assumption $Px_0 = x_0$, or equivalently $Qx_0 = 0$, can be interpreted as an initial phase randomisation.

Taking the Laplace-transform of both sides of equation (6.20) we obtain

$$z\tilde{V}(z)x_0 - Px_0 = PLP\tilde{V}(z)x_0 + \tilde{\psi}(z)\tilde{V}(z)x_0,$$

where the tildes denote the Laplace transforms.

This yields

$$P[z-L]^{-1}Px_0 = \tilde{V}(z)x_0 = P[z - PLP - \tilde{\psi}(z)]^{-1}Px_0. \tag{6.21}$$

Taking the inverse Laplace-transform of this equation we obtain a solution of equation (6.20) ($\gamma > 0$):

$$V(t)x_0 = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt}P[z - PLP - \tilde{\psi}(z)]^{-1}Px_0\,dz \text{ for } t \in \mathbb{R}^+. \tag{6.22}$$

Coveney and Penrose studied the asymptotic behaviour of $V(t)$ under the assumption that the state space E of the system is a Hilbert space and that $\{U_t\}$ is a C_0 group of unitary operators. In addition, they assumed that P is a finite rank operator, that is $\dim P(E) < \infty$. The central result of their paper shows that if the time-domain collision operator ψ is bounded by a decaying exponential, that is if ψ satisfies

$$\|\psi(t)\| \leq Ke^{-\beta t} \text{ for } t \in \mathbb{R}^+,$$

with $K > 0$ and $\beta > 0$, then $V(t)$ splits into two parts $W(t)$ and $\hat{W}(t)$, with $W(t)$ exhibiting exponential decay and $\hat{W}(t)$ decaying faster than $W(t)$. Incidentally, they also show that the condition (C2) is satisfied in this case. In this context, Coveney and Penrose raised the question as to whether there exist dynamical systems for which the time-domain collision operator is bounded by a decaying exponential. We shall now answer this question in the affirmative by giving an example of an invertible dynamical system for which a rank 1 projection P can be chosen such that the collision operator ψ decays exponentially.

Before we discuss this example, we observe that if L is selfadjoint and P is a finite rank operator with

$$P(E) \subset \mathcal{D}(L),$$

then $P(\mathcal{D}(L)) \subset \mathcal{D}(L)$, $Q(\mathcal{D}(L)) \subset \mathcal{D}(L)$ and $LP \in \mathcal{L}(E)$, $PL \in \mathcal{L}(E)$. Moreover, since for $x \in \mathcal{D}(L)$

$$QLQx = Lx - (PLPx + PLQx + QLPx), \tag{6.23}$$

the operator QLQ is of the form 'L + finite rank perturbation', which by a known result [Kat, IX. theorem 2.1] implies that QLQ is the infinitesimal generator of a C_0 semigroup. Thus the assumptions (M1), (M2), and (M3) of proposition 6.2.1 are satisfied, so that the manipulations leading to equation (6.21) are justified.

Suppose now that the evolution of the system is given by a C_0 group of unitary transformations $\{U_t\}$ acting on a Hilbert space E with the following property:

(P) there is a reducing subspace $H \subset E$ of $\{U_t\}$, that is, H is invariant under $\{U_t\}$

$$U_t H \subset H \text{ for } t \in \mathbb{R}$$

such that the restriction of $\{U_t\}$ to H has homogeneous Lebesgue spectrum on the real line, that is, there is a unitary transformation V from H onto $L^2(\mathbb{R}; F)$ such that

$$U_t|H = V^{-1} e^{itw} V \text{ for every } t \in \mathbb{R}.$$

Here, $L^2(\mathbb{R}; F) = L^2(F)$ (F a separable Hilbert space) is the space of all weakly measurable functions $f : \mathbb{R} \to F$ for which

$$\int_{\mathbb{R}} \|f(x)\|_F^2 \, dm(x)$$

remains finite; $L^2(F)$ is a Hilbert space with inner product

$$(f, g)_{L^2(F)} = \int_{\mathbb{R}} (f(x), g(x))_F \, dm(x) \text{ for } f,\, g \in L^2(F).$$

For $t \in \mathbb{R}$, the action of the operator e^{itw} on $f \in L^2(F)$ is given by

$$(e^{itw} f)(x) := e^{itx} f(x).$$

Condition (P) is certainly satisfied if $\{U_t\}$ is the group of Frobenius-Perron operators associated with a K-flow $(X, \mathcal{A}, \mu; \{T_t\})$. In this case, H is the orthocomplement of the constant functions

$$H = L^2(X, \mathcal{A}, \mu) \ominus \mathbb{C}$$

and F is infinite-dimensional (see Section 2.4).

We shall define P to be the orthogonal projection onto the subspace spanned by a certain element of the reducing subspace H of $\{U_t\}$. We may thus, with no loss of generality, assume that $\{U_t\}$ is the group $\{e^{itw}\}$.

Now let h be a normalised element of F and define $f_0 \in L^2(F)$ by

$$f_0(x) := \sqrt{\frac{2r^3}{\pi}} \frac{1}{(x+ir)^2} h \text{ with } r > 0.$$

Since $\|f_0\|_{L^2(F)} = 1$, the formula

$$Pf := (f, f_0)_{L^2(F)} f_0$$

defines the orthogonal projection onto the subspace spanned by f_0. As P is a rank 1 projection, the operators $V(t)$, PLP, and $\psi(t)$ must be of the form

$$\begin{aligned} V(t) &= \alpha(t)P \\ PLP &= \beta P \\ \psi(t) &= \gamma(t)P \end{aligned}$$

where

$$\alpha(t) := (U_t f_0, f_0)_{L^2(F)} \tag{6.24}$$

$$\beta := (Lf_0, f_0)_{L^2(F)} \tag{6.25}$$

and $\gamma(t)$ is the function to be determined. The unknown $\gamma(t)$ may be calculated from a perturbation expansion of $\exp(tQLQ)$, which, however, turns out to be very cumbersome. It is much easier to proceed using Laplace-transforms, which we denote by tildes. To do this, we note that

$$\begin{aligned} \tilde{V}(z) &= \tilde{\alpha}(z)P \\ \tilde{\psi}(z) &= \tilde{\gamma}(z)P, \end{aligned}$$

where

$$\tilde{\alpha}(z) := \left([z-L]^{-1} f_0, f_0\right)_{L^2(F)}.$$

In order to determine $\tilde{\gamma}(z)$ from $\tilde{\alpha}(z)$ and β we recall the identity (6.21):

$$\tilde{V}(z) = P[z - PLP - \tilde{\psi}(z)]^{-1}P.$$

The term on the right hand side is of the form

$$P[z - a(z)P]^{-1}P,$$

with $a(z) = \beta + \tilde{\gamma}(z)$. Since

$$(I - aP)(I - \frac{a}{a-1}P) = (I - \frac{a}{a-1}P)(I - aP) = I,$$

which follows from $P^2 = P$, we have

$$(I - aP)^{-1} = (I - \frac{a}{a-1}P)$$

and thus

$$(z - aP)^{-1} = z^{-1}(I - \frac{a}{a-z}P), \tag{6.26}$$

whence

$$P(z - aP)^{-1}P = \frac{1}{z-a}P.$$

The last equation and equation (6.21) imply that

$$\tilde{\alpha}(z) = (z - \beta - \tilde{\gamma}(z))^{-1}$$

which yields

$$\tilde{\gamma}(z) = z - \beta - \tilde{\alpha}(z)^{-1}. \tag{6.27}$$

Having obtained an equation for $\tilde{\gamma}(z)$ we only need to calculate $\tilde{\alpha}(z)$ and β. To do this we observe that the infinitesimal generator of the group $\{e^{itw}\}$ is the operator iw defined by

$$(iwf)(x) = ixf(x)$$

with domain

$$\mathcal{D}(iw) = \left\{ f \in L^2(F) \,\middle|\, iwf \in L^2(F) \right\}.$$

Thus, $\tilde{\alpha}(z)$ and β are given by

$$\tilde{\alpha}(z) = \frac{2r^3}{\pi} \int_{-\infty}^{\infty} \frac{1}{(z - ix)(x + ir)^2(x - ir)^2}\, dx = \frac{z + 2r}{(z + r)^2} \text{ for } \Re z > 0$$

$$\beta = \frac{2r^3}{\pi} \int_{-\infty}^{\infty} \frac{ix}{(x + ir)^2(x - ir)^2}\, dx = 0.$$

Using equation (6.27) we get an expression for the collision operator

$$\tilde{\psi}(z) = -\frac{r^2}{z + 2r}P \text{ for } \Re z > 0 \tag{6.28}$$

which, on taking the inverse Laplace-transform, yields the time-domain collision operator

$$\psi(t) = -r^2 e^{-2rt} P \text{ for } t \in \mathbb{R}^+.$$

Since $r > 0$, the collision operator is indeed bounded by a decreasing exponential.

We note that $V(t)$, which can be obtained from the inverse Laplace-transform of $\alpha(z)$

$$V(t) = (1 + rt)e^{-rt} P \text{ for } t \in \mathbb{R}^+,$$

is a solution of the GME (6.20). In this case, the remainder term $\hat{W}(t)$ given by Coveney and Penrose vanishes and $V(t)$ itself is the asymptotic operator $W(t)$.

We saw that for every dynamical system whose evolution operator satisfies condition (P), and in particular for every K-flow, it is possible to choose a rank 1 projection giving rise to an exponentially decaying collision operator. This example also shows that the condition (C2) may hold without condition (C1) being satisfied. To see this, we note that the essential spectrum of L is the entire imaginary axis. Since QLQ can be obtained from L by an L-degenerate perturbation (see equation (6.23)), the essential spectrum of QLQ must contain the imaginary axis (see [Kat, IV.5.35]). This means that the resolvent of QLQ cannot be holomorphic at 0, which implies that the function $z \mapsto \tilde{G}(z) = Q[z - QLQ]^{-1}Q$ cannot be holomorphic at 0. Thus condition (C1) is not satisfied.

On the other hand, the collision operator $\tilde{\psi}(z) = PLQ[z - QLQ]^{-1}QLP$, which is *a priori* defined for $z \in \varrho(QLQ)$ only, may well have a larger domain of analyticity (a trivial case occurring when P and L commute, for then $\tilde{\psi}(z) = 0$ for every $z \in \mathbb{C}$). In our example, the collision operator is holomorphic in the entire complex plane with the exception of the point $-2r$. This statement should be understood as saying that the collision operator $\tilde{\psi}(z)$ can be analytically continued from the right half plane across the imaginary axis to the left half plane, with the analytic continuation being given by equation (6.28). The function $z \mapsto [z - PLP - \tilde{\psi}(z)]^{-1}$ which, using (6.26), is given by

$$[z - PLP - \tilde{\psi}(z)]^{-1} = z^{-1}(I - \frac{r^2}{(z+r)^2} P),$$

clearly also admits an analytic continuation into the left half plane with the resulting extension being meromorphic on the entire complex plane. Thus, the condition (C2) is satisfied.

6.5 Discussion

In this and the previous chapter we explored the mathematical structure of the resolvent-based version of the Brussels formalism. We showed that the standing assumptions of the Brussels school, that is, the assumptions (C1) and (C2) ((A1) and (A2) in the discrete time scenario), lead to a special class of evolution operators, namely those with meromorphic resolvents. Using this fact we were able to invoke the Riesz projection theorem, which immediately furnished the desired projection Π, thus avoiding the rather involved construction presented in the original articles of the Brussels school. The spectral characterisation of Π also provided a simple way to obtain the kinetic equations sought by the Brussels school. In the discrete time scenario the assumption of rapid decay of correlations in the Q-subspace (A1) in combination with the assumption (A2) guaranteed that the kinetic subspaces are associated with the dominant long-time modes of the system. The continuous time counterpart of these assumptions appear to be too weak to establish the validity of the corresponding continuous time result.

It should be mentioned here that phase space functions which belong to the range of a projection Π associated with a spectral point not lying on the imaginary axis (on the unit circle for discrete time systems) cannot be strictly non-negative and, hence, cannot be probability densities. In this case, the corresponding 'kinetic equation' should be interpreted as describing the relaxation of deviations from the equilibrium distribution.

We also saw that in the original Hilbert space the analyticity requirements of the theory cannot be met by non-invertible systems in general. The same is true for invertible systems which are at least weakly mixing. For certain non-invertible discrete time systems, however, these requirements can be fulfilled provided that the domain of the evolution operator is restricted to a suitable test function space. This suggests that in concrete applications of discrete time subdynamics a suitable choice of a mode of description is essential. The situation is different for invertible discrete time systems. Here we saw that, at least for ergodic systems, restricting the domain does not give rise to evolution operators with meromorphic resolvents (see proposition 3.3.5). In this case structures other than test function spaces need to be considered if one wishes to justify the standing assumptions of the theory. The construction of Evans and Coveney [EC] discussed in Section 5.6 may provide a promising alternative, as may the so-called 'spectral deformation techniques' discussed, for example, by Courbage [Cou1, Cou2, Cou4], Obcemea and Brändas [OB], and Kumičák and Brändas [KB1, KB2].

On the other hand, no alternative structures are required if one is only interested in the P-P component of the full evolution. Indeed, we saw that the hypotheses of the Coveney-Penrose result [CP] discussed in Section 6.4 can be satisfied by the evolution operators of K-systems on the original Hilbert space provided that the projection P is suitably chosen. It should be clear that the result of Coveney and Penrose does not entail the existence of the full projection Π as they only consider the P-P part of the full resolvent. In this respect, one still has to abandon the original Hilbert space for a different structure if one wishes to justify the construction of Π.

Chapter 7

In conclusion

In exploring the temporal behaviour of dynamical systems we were guided by the idea that this behaviour can be studied quite profitably from a probabilistic point of view and that the fine structure of this behaviour can be analysed by applying the methods of spectral theory to the evolution operators effecting the probabilistic description of these systems. This idea proved to be an important ingredient in our analysis and justification of the two versions of the Brussels formalism referred to as subdynamics and the spectral decomposition algorithm.

A crucial element of our investigations of discrete time dynamical systems in Chapters 3–5 was the notion of a test function space for the evolution operator of the system. In Chapter 3 we used the spectral theory for operators on locally convex spaces presented in Appendix A to argue that test function spaces implement a mode of description for the dynamical system, since choosing a test function space amounts to preselecting a class of probability distributions of the system, the collective or typical temporal behaviour of which can be obtained from the spectrum of the restricted evolution operator. To illustrate possible modes of description for a concrete system we analysed the spectral properties of the Frobenius-Perron operator of the Rényi map on a sequence of test function spaces comprising functions with a certain degree of smoothness. It turned out that for these test function spaces the isolated points of the spectra coincide and are given by the Ruelle resonances of the system. At the end of Chapter 3 we presented a definition of the notion of a spectral decomposition used by the Brussels/Austin groups and argued that spectral decompositions may be interpreted as a physical representation of the system in so far as they incorporate the resonances occurring at the selected mode of description. We also showed that the Frobenius-Perron operator of the Rényi map admits a complete spectral decomposition on certain spaces of entire functions.

In Chapter 4 we embarked upon a further investigation of spectral decompositions and gave a constructive proof that the Frobenius-Perron operator of every exact system admits a complete spectral decomposition. It turned out that the spectral decomposition is not unique. In fact, we saw that every sequence converging to zero can occur as the spectrum of the Frobenius-Perron operator provided that its domain is restricted to a suitable test function space. As a consequence every such sequence gives rise to a different spectral decomposition. In particular, the Rényi map (or any other exact system for which the Ruelle ζ-function can be shown to be meromorphic in a certain region) may have resonances which are not Ruelle resonances. These anomalous resonances are not likely to be related to a trajectory based object.

Spectral methods also played an important role in the next two chapters, where we dealt with the mathematical structure of subdynamics. In Chapter 5 we analysed the discrete time version focussing on the conditions which guarantee the existence of the projection operator Π. These conditions furnished a spectral characterisation of Π which we used to derive the discrete time analogues of the kinetic equations sought by the Brussels school. It turned out that in order to derive the analogues of the physically relevant first order equations the poles of the resolvent have to be simple. We supplied sufficient conditions for this to be the case and obtained an alternative characterisation of the projection Π as an ergodic average. Finally, we addressed the question as to whether there are any systems which meet the conditions guaranteeing the existence of Π. We saw that these conditions are not satisfied for a large class of dynamical systems[1] if the Frobenius-Perron operator is considered on the original Hilbert space. However, as a consequence of the results in Chapters 3 and 4, certain non-invertible systems (including all exact systems) meet these conditions provided that the domain of the Frobenius-Perron operator is restricted to a suitable test function space. We also showed that for invertible systems which are at least ergodic this procedure cannot be used to justify these conditions.

In Chapter 6 we focussed on the continuous time version of subdynamics and obtained most of the analogues of the discrete time scenario, includ-

[1] This class comprises all non-invertible systems and at least all weakly mixing invertible systems.

ing the spectral characterisation of Π and a short derivation of the kinetic equations sought by the Brussels school. Finally, we addressed a question of Coveney and Penrose regarding the existence of a continuous time dynamical system for which the time domain collision operator is norm-bounded by a decreasing exponential. We showed that every K-flow gives rise to a collision operator with this property provided that the P-part of the evolution is suitably chosen.

The results of the present thesis show that spectral methods provide a convenient tool to analyse the fine structure of the probabilistic evolution of non-invertible dynamical systems. The mathematical model based on the notion of a test function space supplies a wide range of possible modes of description, the right choice being determined by which aspects of the dynamics are to be described. At least in the case of non-invertible systems this model may be used to justify the procedures of subdynamics and the generalised spectral decomposition technique. The situation is less satisfactory for invertible systems which, no doubt, play an important role in the enterprise of the Brussels school. Nevertheless, we hope that the research presented here will turn out to be useful for the development of a mathematical model justifying both the invertible and the non-invertible case.

Appendices

Appendix A

Operator spectral theory

It will be convenient to introduce here a notion, namely that of the latent roots of a matrix — latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf.
James Joseph Sylvester

The spectral theory of operators has its roots in the theory of matrices and the theory of integral equations, and, as such, goes back at least to the second half of the last century. Its present form grew out of a considerable abstraction of these ideas yielding a theory which is based on a fertile combination of function-theoretic and algebraic concepts. At the heart of this theory is the notion of the *spectrum*, which generalises Sylvester's latent roots of a matrix and the singular values of an integral equation. Let us recall that a complex number λ is said to belong to the spectrum of a continuous operator T of a Banach space, if $\lambda - T$ fails to have a continuous inverse. With this definition the spectrum of T is a non-empty compact subset of the complex plane. An extension of the above results to more general topological vector spaces relies heavily on a suitable concept of spectrum. If the very same definition is used for a continuous operator on a locally convex space, the spectrum may be empty or unbounded, and might not be closed (see [Mae] for examples). To our knowledge, the first concept of spectrum to provide a satisfactory analogy of the results in Banach spaces is due to Williamson [Wil] (unpublished). Not much later, Waelbroeck employed a similar idea to develop a spectral theory for locally convex algebras, which incorporates the main features of the Banach space theory [Wae1, Wae2]. Parts of his theory were generalised by Neubauer [Neu], Allan [All], Moore [Moo], and König [Kön].

In this appendix we shall be concerned with the spectral theory of operators on a locally convex space. The collection of all operators on a locally convex space carries the structure of a locally convex algebra in a natural way, so that some part of operator spectral theory is covered by the spectral theory of locally convex algebras developed by Waelbroeck. Operators, however, have some internal structure adding certain features to their spectral theory. Our central aim will be to derive the analogues of some well-known theorems of the spectral theory of Banach space operators, which, to our knowledge, have not yet appeared in the literature in this generality. After a brief summary of the main results of Allan's spectral theory, which, to the author's taste, appears as the most transparent extension of the Banach space theory, we shall focus on the derivation of the Riesz projection theorem and give a characterisation of poles of the resolvent. Next we discuss the relation between the spectrum of an operator and the spectrum of its adjoint, which will then be used to generalise a result of Grabiner concerning the spectra of intertwined operators. In the final section we consider the spectral theory of operators on inductive and projective limits in terms of their restrictions to the generating spaces.

A.0.1 Notation In the following, E,F,G shall denote Hausdorff locally convex topological vector spaces over the complex field unless otherwise stated. We write (E,τ) or (E,P), where P is a directed system of seminorms generating the topology τ of E, if we want to emphasise the topology on E. The topological dual of (E,τ) will be denoted by E'. For a functional $x'\in E'$ we write $\langle x,x'\rangle := x'(x)$ for the evaluation of x' at x. The collection of continuous linear transformations from E into F shall be denoted by $\mathcal{L}(E,F)$, and we use $\mathcal{L}(E)$ for $\mathcal{L}(E,E)$. Note that a linear transformation $T:(E,P)\to(F,Q)$ is continuous if and only if for every $q\in Q$ there is a $p\in P$ and a $K>0$ such that $q(Tx)\le Kp(x)$ for every $x\in E$. See [Sch, Köt1, Köt2] for the easy proof as well as for the relevant background in the theory of locally convex topological vector spaces. Finally, $\overline{\mathbb{C}}$ will denote the one-point compactification of $\mathbb{C}$ with its usual topology.

A.1 Allan's spectral theory

Basic definitions

A.1.1 Definition A *locally convex algebra* is an associative algebra A with unit over the complex field, equipped with a topology τ such that

(i) (A, τ) is a Hausdorff locally convex topological vector space;

(ii) multiplication is separately continuous, that is, for any element a_0 of A, the maps $a \mapsto a_0 a$ and $a \mapsto a a_0$ of A into itself, are continuous.

Locally convex algebras arise naturally from operator algebras. First we need to equip $\mathcal{L}(E)$, or more generally $\mathcal{L}(E, F)$, with a locally convex topology. Given locally convex spaces (E, P) and (F, Q), let $\mathfrak{S}$ be a family of bounded subsets of (E, P) which is directed under set theoretic inclusion, and which covers E, that is, $\bigcup_{B \in \mathfrak{S}} B = E$. Then the topology of uniform convergence on elements of $\mathfrak{S}$, the $\mathfrak{S}$-*topology* for short, is a Hausdorff locally convex topology on $\mathcal{L}(E, F)$ (see [Sch, III.3.1]). The resulting topological vector space is denoted by $\mathcal{L}_{\mathfrak{S}}(E, F)$. Analytically this topology is given by the directed system of seminorms

$$T \mapsto \sup_{x \in B} q(Tx),$$

where B runs through $\mathfrak{S}$ and q through Q.

The following $\mathfrak{S}$-topologies will be of special importance:

(i) The topology of *simple* convergence or *s-topology*: $\mathfrak{S}$ the family of all finite subsets of E;

(ii) the topology of *bounded* convergence or *b-topology*: $\mathfrak{S}$ the family of all bounded subsets of E.

The s-topology is the weakest, and the b-topology the strongest $\mathfrak{S}$-topology. $\mathcal{L}(E)$ equipped with the s-topology or b-topology will be denoted by $\mathcal{L}_s(E)$ or $\mathcal{L}_b(E)$, respectively.

A.1.2 Definition Let E_1 and E_2 be locally convex, and let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be directed and covering families of bounded sets of E_1 and E_2 respectively. The pair $(\mathfrak{S}_1, \mathfrak{S}_2)$ is called *admissible for* (E_1, E_2) if for every $B_1 \in \mathfrak{S}_1$ and every $T \in \mathcal{L}(E_1, E_2)$ there is $B_2 \in \mathfrak{S}_2$ such that $TB_1 \subset B_2$. We call $\mathfrak{S}$ *admissible for* E, or simply *admissible* if no confusion is possible, if $(\mathfrak{S}, \mathfrak{S})$ is admissible for (E, E).

A.1.3 Proposition *If E is locally convex and $\mathfrak{S}$ is admissible for E, then $\mathcal{L}_{\mathfrak{S}}(E)$ is a locally convex algebra. In particular, $\mathcal{L}_s(E)$ and $\mathcal{L}_b(E)$ are locally convex algebras.*

Proof Taking multiplication to be the composition of transformations on E, $\mathcal{L}(E)$ becomes an associative algebra with the identity acting as the unit. Thus, we only need to show that multiplication is separately continuous. Now let $T_0 \in \mathcal{L}(E)$. If $p \in P$ and $B \in \mathfrak{S}$, then there is $q \in P$ and $K > 0$ such that $p(Tx) \leq Kq(x)$ by virtue of the continuity of T_0. Hence

$$\sup_{x \in B} p(T_0 Tx) \leq K \sup_{x \in B} q(Tx) \text{ for every } T \in \mathcal{L}(E),$$

which implies that $T \mapsto T_0 T$ is continuous in $\mathcal{L}_{\mathfrak{S}}(E)$. On the other hand, if $p \in P$ and $B \in \mathfrak{S}$, then there is $\tilde{B} \in \mathfrak{S}$ such that $T_0 B \subset \tilde{B}$, since $\mathfrak{S}$ is admissible by hypothesis. Thus,

$$\sup_{x \in B} p(TT_0 x) \leq \sup_{x \in \tilde{B}} p(Tx) \text{ for every } T \in \mathcal{L}(E),$$

which implies that $T \mapsto TT_0$ is continuous in $\mathcal{L}_{\mathfrak{S}}(E)$. Finally, the b-topology is admissible, since the continuous image of a bounded set is bounded, while the s-topology is trivially admissible.

QED

If E is a normed space then the s-topology corresponds to the strong operator topology, and the b-topology to the uniform operator topology. The analogue of the weak operator topology enters this framework as the weak topology of $\mathcal{L}_s(E)$ (see [Sch, II.5] for the definition of the weak topology). We write $\mathcal{L}_s(E)_s$ to denote $\mathcal{L}(E)$ equipped with this topology, and note that a net $\{T_\alpha\}$ converges to zero in $\mathcal{L}_s(E)_s$ if and only if $\langle T_\alpha x, x' \rangle$ converges to zero for any $x \in E$ and any $x' \in E'$. Thus, if E is a normed space, the weak topology of $\mathcal{L}_s(E)$ is just the familiar weak operator topology.

Allan's concept of spectrum is based on the notion of a 'regular element', which is defined as follows.

A.1.4 Definition Let A be a locally convex algebra. An element $a \in A$ is called *regular*, if there is a $k > 0$ such that $\{(a/k)^n \mid n \in \mathbb{N}\}$ is a bounded subset of A. The set of regular elements of A will be denoted by A_0.

We are now able to introduce two concepts of spectrum for an element of a locally convex algebra.

A.1.5 Definition Let A be a locally convex algebra with unit e. For $a \in A$ let $\sigma'_A(a)$ and $\sigma_A(a)$ denote the following subsets of the extended complex plane $\overline{\mathbb{C}}$:

(i) $\sigma'_A(a) := \{\lambda \in \mathbb{C} \mid (\lambda e - a) \text{ has no inverse in } A\}$;

(ii) (a) for $\lambda \neq \infty$, $\lambda \in \sigma_A(a)$ if and only if $(\lambda e - a)$ has no inverse belonging to A_0;

(b) $\infty \in \sigma_A(a)$ if and only if $a \notin A_0$.

We call $\sigma'_A(a)$ the *classical* or *Banach spectrum of a in A* and $\sigma_A(a)$ the *spectrum of a in A*. The complements of these sets in $\overline{\mathbb{C}}$, denoted by $\varrho'_A(a)$ and $\varrho_A(a)$ respectively, are called the *classical* or *Banach resolvent set of a* and the *resolvent set of a*, respectively.

If A is a Banach algebra the spectrum and the Banach spectrum of an element of the algebra obviously coincide. Since the spectrum of an element of a locally convex algebra (A, τ) depends on the topology τ, the spectrum of an operator $T \in \mathcal{L}_{\mathfrak{S}}(E)$ depends on $\mathfrak{S}$. For certain spaces E, however, this is not the case.

A.1.6 Proposition *Let E be locally convex, let $\mathfrak{S}$ be admissible, and let $T \in \mathcal{L}(E)$. If E is quasi-complete or barrelled, then*

$$\sigma_{\mathcal{L}_{\mathfrak{S}}(E)}(T) = \sigma_{\mathcal{L}_s(E)}(T).$$

Proof By hypothesis, a subset of $\mathcal{L}_{\mathfrak{S}}(E)$ is bounded if and only if it is bounded in $\mathcal{L}_s(E)$ (see [Sch, III.3.4 corollary and III.4.2]). Thus, an element of $\mathcal{L}_{\mathfrak{S}}(E)$ is regular if and only if it is regular in $\mathcal{L}_s(E)$.

QED

Main results

A satisfactory analogue of the Banach space theory requires some assumption of completeness. Allan's theory is based on the notion of pseudo-completeness [All, 2.5]. If E is *sequentially complete and barrelled*, then $\mathcal{L}_{\mathfrak{S}}(E)$ is a pseudo-complete locally convex algebra for any admissible $\mathfrak{S}$. This follows from the fact that in this case $\mathcal{L}_{\mathfrak{S}}(E)$ is sequentially complete (see [Sch, III.4.4]), which implies that $\mathcal{L}_{\mathfrak{S}}(E)$ is pseudo-complete [All, 2.6]. The theorems to be presented in the remainder of this section, which form the core of Allan's spectral theory of pseudo-complete locally convex algebras, will be formulated for the algebra $\mathcal{L}_{\mathfrak{S}}(E)$ with sequentially complete and barrelled E. In this case, the spectrum of $T \in \mathcal{L}_{\mathfrak{S}}(E)$ does not depend on the choice of $\mathfrak{S}$, and we shall simply speak of *the* spectrum of T.

A.1.7 Definition Let E be locally convex and let $T \in \mathcal{L}(E)$. Then the *resolvent* of T is the function $R(\lambda)$ defined by

$$R(\lambda) := (\lambda - T)^{-1},$$

whenever this inverse exists in $\mathcal{L}(E)$. Note that the domain of definition contains $\varrho'(T) \cap \mathbb{C}$.

We note that the resolvent satisfies the so-called *resolvent equation*:

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu) \text{ for } \lambda, \mu \in \varrho'(T) \cap \mathbb{C}. \tag{A.1}$$

A.1.8 Theorem (Allan) *Let E be non-zero sequentially complete and barrelled, and let $T \in \mathcal{L}(E)$. Then:*

(i) R is holomorphic at λ if and only if $\lambda \in \varrho(T)$;

(ii) $\sigma(T)$ is closed in $\overline{\mathbb{C}}$, hence compact in $\overline{\mathbb{C}}$;

(iii) $\sigma(T)$ is compact in $\mathbb{C}$ if and only if T is regular; in this case, the resolvent is holomorphic at ∞ and its Laurent expansion is given by the Neumann series,

$$R(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n,$$

valid for $|\lambda| > M$, with M suitably large;

(iv) $\sigma(T)$ is not empty.

Proof See [All, 3.8] for (i), (ii), and (iii).

(iv) Allan shows that the assumption $\sigma(T) = \emptyset$ implies that $R(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n = 0$. It follows that the first term of the series vanishes, that is, $I = 0$. This contradicts the assumption $E \neq \{0\}$.

QED

The last theorem fully justifies the definition of σ as it allows us to recover the important properties of the spectrum of operators on Banach spaces. The term *holomorphic* occurring in (i), as applied to a function defined on a subset of $\overline{\mathbb{C}}$ taking its values in a locally convex space should be understood as in [GWS, III.III].

As usual, the *spectral radius* $r_{\mathcal{L}(E)}$ of an operator $T \in \mathcal{L}(E)$ will be defined by

$$r_{\mathcal{L}(E)}(T) = \sup \left\{ |\lambda| \,\middle|\, \lambda \in \sigma_{\mathcal{L}(E)}(T) \right\}.$$

The quantities

$$\begin{aligned}
\beta_{\mathcal{L}(E)}(T) &:= \inf \left\{ k > 0 \,\middle|\, \{(T/k)^n\}_{n\in\mathbb{N}} \text{ is bounded in } \mathcal{L}(E) \right\},\\
\beta'_{\mathcal{L}(E)}(T) &:= \sup \left\{ \limsup_{n\to\infty} |\langle T^n x, x'\rangle|^{1/n} \,\middle|\, x \in E, x' \in E' \right\},\\
\beta''_{\mathcal{L}(E)}(T) &:= \sup \left\{ \limsup_{n\to\infty} p(T^n)^{1/n} \,\middle|\, p \in P \right\},
\end{aligned}$$

where P is a directed system of seminorms generating the topology of $\mathcal{L}(E)$, provide an analogue of Gelfand's spectral radius formula.

A.1.9 Theorem (Allan) *Let E be non-zero sequentially complete and barrelled. Let $T \in \mathcal{L}(E)$ and $\lambda \in \varrho(T) \cap \mathbb{C}$. Then:*

(i) $\beta(T) = \beta'(T) = \beta''(T) = r(T)$;

(ii) $r((\lambda - T)^{-1})^{-1} = \operatorname{dist}(\lambda, \sigma(T)) := \inf \left\{ |\lambda - \lambda'| \,\middle|\, \lambda' \in \sigma(T) \right\}$.

Proof See [All, 2.18 and 3.12] for (i). Assertion (ii) can be easily deduced from Allan's spectral mapping theorem [All, 6.11].

QED

The last element of Allan's theory we require is his holomorphic functional calculus. The various functional calculi provide powerful devices to reduce certain problems concerning algebras of operators to problems concerning algebras of scalar-valued functions. The algebra of scalar-valued functions used by Allan is defined as follows. Let E be sequentially complete and barrelled and let $T \in \mathcal{L}(E)$. Using $\mathcal{H}(T)$ to denote the set of all complex-valued functions which are holomorphic on $\sigma(T)$ we define an equivalence relation '$\sim$' on $\mathcal{H}(T)$ by: $f \sim g$ if and only if $f(\lambda) = g(\lambda)$ for all λ in some neighbourhood of $\sigma(T)$. The quotient of $\mathcal{H}(T)$ by this equivalence relation, denoted by $H(T)$, is an algebra when the algebraic operations are understood pointwise for representatives on suitable neighbourhoods of $\sigma(T)$.

A.1.10 Theorem (Allan's holomorphic functional calculus) *Let E be non-zero sequentially complete and barrelled, and let $T \in \mathcal{L}(E)$ with $\varrho(T) \neq \emptyset$. Then there is an algebra-homomorphism $f \mapsto f(T)$ of $H(T)$ into $\mathcal{L}(E)_0$ given by:*

(i) if T is regular then

$$f(T) = \frac{1}{2\pi i} \oint_{+\partial D} f(\lambda) R(\lambda)\, d\lambda,$$

where $+\partial D$ is the positively oriented boundary of a Cauchy domain D for f (see [All, 5.2] for the definition of a Cauchy domain);

(ii) if T is not regular then

$$f(T) = f(\infty) I + \frac{1}{2\pi i} \oint_{+\partial D} f(\lambda) R(\lambda)\, d\lambda$$

with D as before.

For $\mu \in \varrho(T) \cap \mathbb{C}$ the functions $f_1(\lambda) = 1$, $f_2(\lambda) = (\mu - \lambda)^{-1}$ belong to $H(T)$ and

$$f_1(T) = I \tag{A.2}$$

$$f_2(T) = R(\mu) \tag{A.3}$$

Proof All statements but (A.3) follow from [All, 3.8]. To prove (A.3) we write the resolvent equation (A.1) in the form

$$(\mu - \lambda)^{-1} R(\lambda) = (\mu - \lambda)^{-1} R(\mu) + R(\lambda) R(\mu). \tag{A.4}$$

If T is regular, then (i) and (A.4) yield

$$f_2(T) = \frac{1}{2\pi i} \oint_{+\partial D} (\mu - \lambda)^{-1}\, d\lambda \cdot R(\mu) + \frac{1}{2\pi i} \oint_{+\partial D} R(\lambda)\, d\lambda \cdot R(\mu). \tag{A.5}$$

The first integral on the right-hand side of this equation is zero, while the second is I by (A.2).

If T is not regular then (ii) and (A.4) again yield equation (A.5), since $f_2(\infty) = 0$. In this case the first integral is 1 and the second integral vanishes by (A.2).

QED

A.1.11 Remark
One may put the two cases we distinguished in theorem A.1.10 on an equal footing, at the cost of losing some functions in $H(T)$ for regular T. If T is regular, and $f \in H(T)$ is holomorphic at ∞, then one may show that

$$f(T) = f(\infty)I + \frac{1}{2\pi i} \oint_{+\partial D} f(\lambda)R(\lambda)\, d\lambda,$$

where D is a Cauchy domain for $f \in H(T)$ — just as in Taylor's functional calculus for closed operators on a Banach space (see [Tay1, p. 293]).

We note that, if T is not regular, then $H(T)$ does not contain non-constant polynomials, so that the theorem above cannot be used to conclude that $f(T) = T$ for $f(\lambda) := \lambda$. Nevertheless, we have the following proposition.

A.1.12 Proposition *Let E and T be as in theorem A.1.10. If $f \in H(T)$ has a zero of order at least one at ∞, or if f vanishes identically on a neighbourhood of ∞, then $g(\lambda) := \lambda f(\lambda) \in H(T)$ and $g(T) = Tf(T)$.*

Proof Under the assumptions made on f, we obviously have $g \in H(T)$. Let $\mu \in \varrho(T) \cap \mathbb{C}$ and define $\tilde{g}(\lambda) := (\mu - \lambda)f(\lambda)$. Again $\tilde{g} \in H(T)$ and $f(\lambda) = (\mu - \lambda)^{-1}\tilde{g}(\lambda)$. Then $f(T) = R(\mu)\tilde{g}(T)$ by the functional calculus. Since $g(\lambda) = -\tilde{g}(\lambda) + \mu f(\lambda)$, we get $g(T) = -(\mu - T)f(T) + \mu f(T) = Tf(T)$.

QED

A.2 Reducing subspaces

So far, we have only been concerned with the algebraic properties of $\mathcal{L}(E)$, and no use was made of the fact that every element of $\mathcal{L}(E)$ is a continuous linear transformation of E into itself. We shall now embark on a finer analysis of the spectral properties of the elements of $\mathcal{L}(E)$ using the fact that they have some 'internal' structure. Since these elements — the continuous linear transformations on E — are known as *operators*, their spectral analysis is referred to as *operator spectral theory*. The results presented in this section are mostly extensions of some well-known theorems of the theory of operators on Banach spaces to sequentially complete and barrelled spaces, and as such, rely heavily on the machinery introduced in the previous section.

A.2.1 Notation Given a linear transformation $T : E \to F$, we shall use

$$\mathcal{N}(T) := \{x \in E \mid Tx = 0\}$$

for the *kernel* of T, and

$$\mathcal{R}(T) := T(E) = \{Tx \mid x \in E\}$$

for the *range* of T. If $G \subset E$ is a subspace, or more generally a subset of E, we write $T|G$ to denote the *restriction* of T to G:

$$T|G(x) := T(x) \text{ for } x \in G.$$

We remind the reader that a linear transformation $P : E \to E$ on a vector space E is called a *projection* if $P^2 = P$. A projection gives rise to a splitting of E into an algebraic sum $E = E_1 \oplus_{\text{alg}} E_2$ with $E_1 := \mathcal{R}(P)$ and $E_2 := \mathcal{R}(I - P)$. Conversely, every splitting of E into an algebraic direct sum $E = E_1 \oplus_{\text{alg}} E_2$ determines projections P_1 and P_2, which obey $P_1 + P_2 = I$ and $\mathcal{R}(P_i) = E_i$ for $i = 1, 2$. The projections are both continuous if and only if the sum is topological, in which case it will be written $E = E_1 \oplus_{\text{top}} E_2$ (see [Sch, I.2.2]).

Recall that a linear transformation T acting on a vector space E is said to be *reduced by the pair of subspaces* (E_1, E_2), if $E = E_1 \oplus_{\text{alg}} E_2$ and T leaves E_1 and E_2 invariant, that is, $TE_1 \subset E_1$ and $TE_2 \subset E_2$. Equivalently, T is reduced by (E_1, E_2) if $TP_i = P_iT$, where P_i denotes the projection onto E_i. If an operator is reduced by a pair of subspaces, the operator may be analysed by studying its restrictions to these subspaces, which often reduces the complexity of the problem. As a first step, we shall investigate the relation between the spectrum of the operator and that of its restrictions to a pair of subspaces reducing the operator.

A.2.2 Proposition *Let E be locally convex and let $\mathfrak{S}$ generate an admissible $\mathfrak{S}$-topology on $\mathcal{L}(E)$. Assume that $E = E_1 \oplus_{\text{top}} E_2$ and that the linear transformation $T : E \to E$ is reduced by (E_1, E_2). Let $T_i : E_i \to E_i$ denote the restrictions of T to E_i. Then:*

(i) T is bijective if and only if T_1 and T_2 are bijective;

(ii) T is continuous if and only if T_1 and T_2 are continuous;

(iii) $\mathfrak{S}_i := \{P_i(B) \mid B \in \mathfrak{S}\}$ *is admissible for* E_i*; moreover,* T *is regular in* $\mathcal{L}_{\mathfrak{S}}(E)$ *if and only if* T_1 *and* T_2 *are regular in* $\mathcal{L}_{\mathfrak{S}_1}(E_1)$ *and* $\mathcal{L}_{\mathfrak{S}_2}(E_2)$*, respectively.*

Proof Let Φ denote the topological isomorphism induced by the topological direct sum $E = E_1 \oplus_{\text{top}} E_2$:

$$\Phi : E_1 \times E_2 \to E,$$

$$\Phi(x_1, x_2) = x_1 + x_2.$$

It is not difficult to verify that

$$T = \Phi \circ (T_1 \times T_2) \circ \Phi^{-1}, \tag{A.6}$$

where $T_1 \times T_2 : E_1 \times E_2 \to E_1 \times E_2$ is given by $(T_1 \times T_2)(x_1, x_2) = (T_1 x_1, T_2 x_2)$.

(i) Since Φ is an isomorphism, T is bijective if and only if $T_1 \times T_2$ is bijective, which holds if and only if T_1 and T_2 are bijective.

(ii) Similarly, since Φ is a topological isomorphism, T is continuous if and only if $T_1 \times T_2$ is continuous, which in turn is continuous if and only if T_1 and T_2 are continuous by the definition of the product-topology.

(iii) To avoid cluttered notation we shall use the following abbreviations:

$$\mathcal{L} := \mathcal{L}(E), \tag{A.7}$$

$$\mathcal{L}_i := \mathcal{L}_{\mathfrak{S}_i}(E_i). \tag{A.8}$$

As P_i is continuous, $\mathfrak{S}_i$ is a system of bounded sets in E_i, which is easily seen to be directed and covering in E_i as well. In order to show that $\mathfrak{S}_1$ is admissible we define for every $S \in \mathcal{L}_1$ a continuous extension $\tilde{S}$ to E by

$$\tilde{S} := \Phi \circ (S \times I) \circ \Phi^{-1}.$$

Let $B \in \mathfrak{S}$ and $S \in \mathcal{L}_1$. Since $\mathfrak{S}$ is admissible there is $\tilde{B} \in \mathfrak{S}$ such that $\tilde{S}(B) \subset \tilde{B}$. Now $SP_1(B) = \tilde{S}P_1(B) = P_1\tilde{S}(B) \subset P_1(\tilde{B})$, that is, $\mathfrak{S}_1$ is admissible for E_i. Similarly for $\mathfrak{S}_2$.

To prove the 'if' clause we assume that T_1 and T_2 are regular in $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Choose $k > \max\{\beta_{\mathcal{L}_1}(T_1), \beta_{\mathcal{L}_2}(T_2)\}$. We shall show that for every $B \in \mathfrak{S}$ the set $\bigcup_{n\in\mathbb{N}}(T/k)^n(B)$ is bounded in E, whence it follows that T is regular. For $x \in E$ equation (A.6) yields

$$(T/k)^n x = \Phi((T_1/k)^n P_1 x \times (T_2/k)^n P_2 x). \tag{A.9}$$

For $B \in \mathfrak{S}$ let B_i denote the set

$$B_i := \bigcup_{n\in\mathbb{N}} (T_i/k)^n P_i(B).$$

Now

$$\bigcup_{n\in\mathbb{N}} (T/k)^n(B) = \Phi(\bigcup_{n\in\mathbb{N}} \{(T_1/k)^n P_1(B) \times (T_2/k)^n P_2(B)\}) \subset \Phi(B_1 \times B_2),$$

with the equality following from equation (A.9). Since B_i is bounded in E_i by our choice of k, the set $B_1 \times B_2$ is bounded in $E_1 \times E_2$ by the definition of the product-topology. Thus, $\Phi(B_1 \times B_2)$ is bounded in E by virtue of the continuity of Φ. This implies that $\bigcup_{n\in\mathbb{N}}(T/k)^n(B)$ is bounded in E for every $B \in \mathfrak{S}$.

In proving the 'only if' clause let I_i denote the canonical embedding of E_i into E. Then $T_i = TI_i$, and $(T_i/k)^n = (T/k)^n I_i$ for any $k > 0$. Let $k > \beta_{\mathcal{L}}(T)$ and $B \in \mathfrak{S}$. Then

$$\bigcup_{n\in\mathbb{N}} (T_i/k)^n P_i(B) = \bigcup_{n\in\mathbb{N}} (T/k)^n I_i P_i(B).$$

Since $I_iP_i = P_i$ is continuous and $\mathfrak{S}$ is admissible, the set $I_iP_i(B)$ belongs to $\mathfrak{S}$. By our choice of k the right-hand side of the above equation is a bounded subset of E lying entirely in E_i. Hence, it is a bounded subset of E_i, and therefore $\{(T_i/k)^n \mid n \in \mathbb{N}\}$ is bounded in $\mathcal{L}_i$. This shows that T_1 and T_2 are regular.

QED

A.2.3 Corollary *Under the hypotheses of the proposition, we have*

$$\sigma'_{\mathcal{L}}(T) = \sigma'_{\mathcal{L}_1}(T_1) \cup \sigma'_{\mathcal{L}_2}(T_2),$$

$$\sigma_{\mathcal{L}}(T) = \sigma_{\mathcal{L}_1}(T_1) \cup \sigma_{\mathcal{L}_2}(T_2),$$

with the abbreviations introduced in (A.7) and (A.8).

In this context we note that if E is sequentially complete and barrelled and if $E = E_1 \oplus_{\text{top}} E_2$, then E_1 and E_2 are sequentially complete and barrelled with respect to the topology induced by E. Thus, if $T \in \mathcal{L}(E)$ is reduced by (E_1, E_2) with $E = E_1 \oplus_{\text{top}} E_2$, then the main results of Allan's spectral theory also hold for the operators $T|E_1$ and $T|E_2$ provided that E is sequentially complete and barrelled.

A.3 The Riesz projection theorem

As a first application of the functional calculus we shall now prove an extension of the well-known Riesz projection theorem, which asserts that operators have reducing subspaces provided that their spectra can be split into two parts of a suitable nature.

A.3.1 Definition A subset σ of $\sigma(T)$, T an operator on a locally convex space, is called a *spectral set of* T, if it is open and closed in $\sigma(T)$, or equivalently if both σ and $\sigma(T) \setminus \sigma$ are closed in $\overline{\mathbb{C}}$. Two spectral sets σ_1 and σ_2 are called *complementary* if $\sigma_1 \cap \sigma_2 = \emptyset$ and $\sigma_1 \cup \sigma_2 = \sigma(T)$.

The Riesz projection theorem can be stated as follows.

A.3.2 Theorem (Riesz projection theorem) *Let E be non-zero, sequentially complete and barrelled, and let σ_1 and σ_2 be complementary spectral sets of $T \in \mathcal{L}(E)$. Then there are two continuous projections P_1 and P_2 such that*

(i) $P_1 + P_2 = I$;

(ii) $P_i P_j = \delta_{ij} P_j$;

(iii) $P_i T = T P_i$.

Moreover, if T_i denotes the restriction of T to $P_i(E)$, then:

(iv) $\infty \notin \sigma_i$ *if and only if* T_i *is regular;*

(v) $\sigma_i = \sigma(T_i)$;

(vi) P_i *is zero if and only if* σ_i *is empty.*

Proof We shall with no loss of generality assume that $\varrho(T)$ is not empty. For otherwise $\sigma(T) = \overline{\mathbb{C}}$ and either σ_1 or σ_2 must be empty. Assume that σ_2 is empty. Setting $P_1 = 1$ and $P_2 = 0$ the assertions of the theorem follow.

Since σ_1 and σ_2 are closed in $\overline{\mathbb{C}}$, there are two non-intersecting open sets Δ_1 and Δ_2, such that $\sigma_i \subset \Delta_i$. Define functions f_1 and f_2 on $\Delta_1 \cup \Delta_2$ by

$$f_1(\lambda) := \begin{cases} 1 & \text{for } \lambda \in \Delta_1 \\ 0 & \text{for } \lambda \in \Delta_2 \end{cases}, \quad f_2(\lambda) := \begin{cases} 0 & \text{for } \lambda \in \Delta_1 \\ 1 & \text{for } \lambda \in \Delta_2 \end{cases}.$$

Obviously, f_1 and f_2 belong to $H(T)$ and satisfy the relations

$$f_1 + f_2 = 1,$$

$$f_i f_j = \delta_{ij} f_j.$$

Setting $P_i := f_i(T)$, the assertions (i), (ii) and (iii) are an immediate consequence of the functional calculus (theorem A.1.10).

In proving (iv) we shall deal with the 'only if' clause first. Define $g_i(\lambda) := \lambda f_i(\lambda)$. Since $\infty \notin \sigma_i$, f_i vanishes on a neighbourhood of ∞. By proposition A.1.12, $g_i(T) = TP_i$, and since $g_i \in H(T)$, we also know that $g_i(T)$ is regular by theorem A.1.10. The regularity of T_i now follows from the regularity of TP_i. To see this let $k > \beta(TP_i)$. Then for any $x \in P_i(E)$, we have $(T_i/k)^n x = (TP_i/k)^n x$, and consequently $\bigcup_n (T_i/k)^n x$ is a bounded set of E lying in $P_i(E)$. Hence, $\{(T_i/k)^n \mid n \in \mathbb{N}\}$ is bounded in $\mathcal{L}_s(E)$, which means that T_i is regular in $\mathcal{L}(E_i)$.

For the proof of the 'if' clause we set $i = 1$, that is, we assume that $\infty \in \sigma_1$. Then $\infty \in \sigma(T)$ and T is not regular. However, $\infty \notin \sigma_2$ and therefore T_2 is regular by the 'only if' part. Now proposition A.2.2 implies that T_1 is not regular. Similarly, for $i = 2$.

(v) Let $\mu \in \mathbb{C} \setminus \sigma_i$ and define $g_i(\lambda) = (\mu - \lambda)^{-1} f_i(\lambda)$. Then $g_i \in H(T)$ and $g_i f_i = g_i$. Consequently $(\mu - T)g_i(T) = P_i$ and $g_i(T)P_i = g_i(T)$. Since $g_i(T)$ is regular it follows that $\mu - T$ restricted to $P(E_i)$ has a regular inverse on $P_i(E)$. Hence, $\sigma(T_i) \cap \mathbb{C} \subset \sigma_i$, which together with (iv) yields $\sigma(T_i) \subset \sigma_i$. Now, $\sigma(T_1) \cap \sigma(T_2)$ is empty, but since $\sigma(T_1) \cup \sigma(T_2) = \sigma(T)$ by corollary A.2.3, we must have $\sigma(T_1) = \sigma_1$ and $\sigma(T_2) = \sigma_2$.

For (vi) we assume with no loss of generality $i = 1$. If $P_1 = 0$, then $P_2 = I$ and $\sigma_2 = \sigma(T_2) = \sigma(T)$. Hence $\sigma_1 = \sigma(T) \setminus \sigma_2$ is empty. If conversely $P_1 \neq 0$, then $P(E_1) \neq 0$ and it follows from theorem A.1.8 that $\sigma(T_1)$ is not empty. Hence, σ_1 is not empty.

QED

The projections P_i constructed in the theorem are called the *Riesz* or *spectral projections* associated with the spectral sets σ_i; they will be denoted by P_{σ_i}. Suitably rephrased, the Riesz projection theorem states that an operator T with a non-trivial spectral set, that is, a spectral set which is neither empty nor all of $\sigma(T)$, has non-trivial reducing subspaces. This follows immediately from the statements (i), (iii), and (vi) above. In the next section we shall investigate the case where the spectral set consists of a single point only.

A.4 Isolated singularities of the resolvent

A particularly interesting case of the Riesz projection theorem occurs when an operator T has a spectral set consisting of a single point $\lambda_0 \in \mathbb{C}$ only. This point is necessarily an isolated point of the spectrum and, consequently, also a singularity of the resolvent. Before we give necessary and sufficient conditions for λ_0 to be a pole and describe the range of the associated Riesz projection, we introduce some concepts familiar from the Banach space theory.

A.4.1 Terminology Let E be a vector space over $\mathbb{C}$ and let $T : E \to E$ be a linear transformation. We remind the reader that $\lambda \in \mathbb{C}$ is an *eigenvalue* of T if $\mathcal{N}(\lambda - T)$ is non-zero. If this is the case, a non-zero element of $\mathcal{N}(\lambda - T)$ is called an *eigenvector*. Non-zero elements of $\cup_{n=0}^{\infty}\mathcal{N}((\lambda - T)^n)$ are called *algebraic eigenvectors* of T. The space $\mathcal{N}(\lambda - T)$ (respectively, $\cup_{n=0}^{\infty}\mathcal{N}((\lambda - T)^n)$) is usually referred to as the *eigenspace* (respectively, *algebraic eigenspace*); its dimension is known as the *geometric multiplicity* (respectively, *algebraic multiplicity*) of the eigenvalue λ and will be denoted by $m_{\text{geo}}(\lambda; T)$ (respectively, $m_{\text{alg}}(\lambda; T)$).

Let us return to out original task. Suppose that $\lambda_0 \in \mathbb{C}$ is an isolated point of the spectrum of an operator T on a sequentially complete and barrelled space E. We may then expand the resolvent $R(\lambda)$ into a Laurent series valid in a small punctured disk centred at λ_0:

$$R(\lambda) = \sum_{n=-\infty}^{\infty} (\lambda - \lambda_0)^n A_n,$$

where the coefficient operators $A_n \in \mathcal{L}(E)$ are given by

$$A_n = \frac{1}{2\pi i} \oint_{+\partial D_\epsilon} (\lambda - \lambda_0)^{-n-1} R(\lambda)\, d\lambda.$$

Here D_ϵ denotes a closed disk of radius ϵ centred at λ_0 containing no other singularities of R other than λ_0. The functional calculus immediately provides us with some important relations amongst these coefficient operators.

Let $0 < 2r < \epsilon$ and define functions f_n as follows:

$$f_n(\lambda) := \begin{cases} 0 & \text{for } |\lambda - \lambda_0| < r \\ (\lambda - \lambda_0)^{-n-1} & \text{for } |\lambda - \lambda_0| > 2r \end{cases} \quad \text{if } n \geq 0;$$

$$f_n(\lambda) := \begin{cases} (\lambda-\lambda_0)^{-n-1} & \text{for } |\lambda-\lambda_0| < r \\ 0 & \text{for } |\lambda-\lambda_0| > 2r \end{cases} \quad \text{if } n < 0.$$

Obviously, $f_n \in H(T)$ and

$$\begin{aligned} A_n &= -f_n(T) \quad \text{for } n \geq 0, \\ A_n &= f_n(T) \quad \text{for } n < 0 \end{aligned}$$

by theorem A.1.10. In particular, the operator-valued residue A_{-1} is the Riesz projection $P_{\{\lambda_0\}}$ associated with the spectral set $\{\lambda_0\}$. Let us observe that

$$\begin{aligned} (\lambda-\lambda_0)f_{n+1}(\lambda) &= f_n(\lambda) && \text{for } n \geq 0, \\ (\lambda-\lambda_0)f_{-n}(\lambda) &= f_{-(n+1)}(\lambda) && \text{for } n > 0, \\ (\lambda-\lambda_0)f_0(\lambda) &= 1 - f_{-1}(\lambda). \end{aligned}$$

Under the functional calculus these relations translate into the formulæ

$$(T-\lambda_0)A_{n+1} = A_n \quad \text{for } n \geq 0, \tag{A.10}$$
$$(T-\lambda_0)A_{-n} = A_{-(n+1)} \quad \text{for } n > 0, \tag{A.11}$$
$$(T-\lambda_0)A_0 = P_{\{\lambda_0\}} - I; \tag{A.12}$$

here we have also used proposition A.1.12. Equations (A.10) and (A.11) may be rewritten in the following form:

$$(T-\lambda_0)^n A_n = A_0 \quad \text{for } n \geq 0, \tag{A.13}$$
$$(T-\lambda_0)^n P_{\{\lambda_0\}} = A_{-(n+1)} \quad \text{for } n > 0. \tag{A.14}$$

A.4.2 Definition Let m be a positive integer. An isolated singularity λ_0 of the resolvent of T is called a *pole of order* m if $A_{-m} \neq 0$ and $A_{-n} = 0$ for $n > m$. Here, A_n, $n \in \mathbb{Z}$ are the coefficients of the Laurent expansion of the resolvent at λ_0. If this is the case we shall sometimes say that λ_0 is a 'pole of T'. The order of the pole will be denoted $\nu_p(\lambda_0; T)$.

By (A.14), the isolated singularity λ_0 is a pole of order m if and only if

$$(T-\lambda_0)^{m-1}P_{\{\lambda_0\}} \neq 0 \text{ and } (T-\lambda_0)^n P_{\{\lambda_0\}} = 0 \text{ for } n \geq m. \tag{A.15}$$

Now let (E_1, E_2), with $E_1 := \mathcal{R}(P_{\{\lambda_0\}})$ and $E_2 := \mathcal{N}(P_{\{\lambda_0\}})$, be the pair of reducing subspaces given by the Riesz projection theorem and let T_i denote the restriction of T to E_i. Let us make the following observations: firstly, we know from assertion (v) of the Riesz projection theorem that λ_0 must belong to the resolvent set of T_2; secondly, if λ_0 is a pole of order m, condition (A.15) implies that $T_1 - \lambda_0$ is *nilpotent of order* m. This means that $(T_1-\lambda_0)^{m-1} \neq 0$ and $(T_1 - \lambda_0)^m = 0$.

It turns out that these two observations characterise poles of the resolvent in the following sense.

A.4.3 Theorem *Let E be non-zero sequentially complete and barrelled. Suppose that $\lambda_0 \in \mathbb{C}$ is an isolated singularity of the resolvent of $T \in \mathcal{L}(E)$, and let m be a positive integer. Then λ_0 is a pole of order m if and only if E splits into a topological sum $E = E_1 \oplus_{\mathrm{top}} E_2$ such that:*

(i) (E_1, E_2) *reduces* T*;*

(ii) $\lambda_0 - T_1$ *is nilpotent of order* m*;*

(iii) $\lambda_0 \in \varrho(T_2)$,

with T_i denoting the restriction of T to E_i.

If this is the case, λ_0 is an eigenvalue of T and

$$E_1 = \mathcal{R}(P_{\{\lambda_0\}}) = \mathcal{N}((\lambda_0 - T)^n), \tag{A.16}$$

$$E_2 = \mathcal{N}(P_{\{\lambda_0\}}) = \mathcal{R}((\lambda_0 - T)^n) \tag{A.17}$$

for $n \geq m$.

Proof We shall prove the 'only if' clause first, that is, we assume that λ_0 is a pole of order m. We put $E_1 := \mathcal{R}(P_{\{\lambda_0\}})$ and $E_2 = \mathcal{N}(P_{\{\lambda_0\}})$. Then, as we already noted, (i) and (iii) are immediate consequences of the Riesz projection theorem, while (ii) follows from (A.15). We now turn to the proof of (A.16) and (A.17). It is convenient to introduce the following abbreviations:

$$\begin{aligned} S &:= T - \lambda_0; && \text{(A.18)}\\ \mathcal{N}_n &:= \mathcal{N}(S^n); && \text{(A.19)}\\ \mathcal{R}_n &:= \mathcal{R}(S^n). && \text{(A.20)} \end{aligned}$$

In proving (A.16) we note that the 'pole condition' (A.15) immediately implies that $E_1 \subset \mathcal{N}_n$ if $n \geq m$. For the opposite inclusion let $n \geq 1$ and $x \in \mathcal{N}_n$. Then, (A.12) and (A.13) yield

$$0 = A_{n-1}S^n x = S^n A_{n-1} x = S A_0 x = P_{\{\lambda_0\}} x - x,$$

so that $x = P_{\{\lambda_0\}}x \in E_1$. Thus $\mathcal{N}_n \subset E_1$ for $n \geq 1$ and (A.16) is proved.

Before we proceed to verify (A.17) we show that

$$\mathcal{R}_n \cap \mathcal{N}_n = \{0\} \text{ for } n \geq m. \tag{A.21}$$

To see this, let $x \in \mathcal{R}_n \cap \mathcal{N}_n$. Then $x = S^n y$ for some $y \in E$ and $S^n x = 0$, and therefore $y \in \mathcal{N}_{2n}$. Since $\mathcal{N}_{2n} = \mathcal{N}_n$ by (A.16), it follows that $x = 0$, and (A.21) holds.

In order to prove (A.17), recall that $\mathcal{N}(P_{\{\lambda_0\}}) = \mathcal{R}(I - P_{\{\lambda_0\}})$. Hence, $E_2 \subset \mathcal{R}_n$ follows from $S^n A_{n-1} = P_{\{\lambda_0\}} - I$. Now suppose, conversely, that $x \in \mathcal{R}_n$ with $n \geq m$, and write $x = x_1 + x_2$, where $x_1 \in E_1$ and $x_2 \in E_2$. Then, since $E_2 \subset \mathcal{R}_n$, we have $x_1 = x - x_2 \in \mathcal{R}_n$. But $x_1 \in E_1 = \mathcal{N}_n$ by (A.16), and so $x_1 \in \mathcal{R}_n \cap \mathcal{N}_n = \{0\}$ by (A.21). This shows that $x = x_2 \in E_2$ and, therefore, $\mathcal{R}_n \subset E_2$ for $n \geq m$. This finishes the proof of (A.17).

In order to show that λ_0 is an eigenvalue, suppose to the contrary that $\mathcal{N}(T - \lambda_0) = \mathcal{N}_1 = \{0\}$. It is not difficult to see that this implies $\mathcal{N}_n = \{0\}$ for $n \geq 1$. But then, by (A.16), we must have $P_{\{\lambda_0\}} = 0$, which contradicts the Riesz projection theorem.

To prove the 'if' clause we assume that λ_0 is an isolated point of $\sigma(T)$ and that conditions (i), (ii), and (iii) hold. It suffices to show that 0 is a pole of order m of the resolvent of $S = T - \lambda_0$. First we note that 0 is an isolated point of $\sigma(S)$ and that S is reduced by (E_1, E_2). Letting $S_i := S|_{E_i}$, conditions (ii) and (iii) read:

(ii′) S_1 is nilpotent of order m;

(iii′) $0 \in \varrho(S_2)$.

Since 0 is an isolated singularity of the resolvent of S we have the Laurent expansion

$$(\lambda - S)^{-1} = \sum_{n=-\infty}^{\infty} \lambda^n C_n,$$

valid in a small punctured disk centred at 0. As S is reduced by (E_1, E_2), the operator-valued coefficients C_n are also reduced by (E_1, E_2); this follows from remark A.1.11 (iii). With $C_n^{(i)}$ denoting the restriction of C_n to E_i it is easily verified that

$$(\lambda - S_i)^{-1} = \sum_{n=-\infty}^{\infty} \lambda^n C_n^{(i)}$$

is the Laurent expansion of the resolvent of S_i. Now, by (ii′), S_1 is nilpotent. Therefore, its spectral radius is zero and the Neumann expansion (see theorem A.1.8)

$$(\lambda - S_1)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} S_1^{n-1}$$

holds for all $\lambda \neq 0$. The uniqueness of the Laurent expansion for $(\lambda - S_1)^{-1}$ implies that $C_{-n}^{(1)} = S_1^{n-1}$ for $n \geq 1$. Furthermore, condition (iii′) implies that $(\lambda - S_2)^{-1}$ is holomorphic at 0, and consequently $C_{-n}^{(2)} = 0$ for $n \geq 1$. If P_i denotes the projection of E onto E_i determined by $E = E_1 \oplus_{\mathrm{top}} E_2$, then for $n \geq 1$

$$C_{-n} = C_{-n}P_1 + C_{-n}P_2 = C_{-n}^{(1)}P_1 + C_{-n}^{(2)}P_2 = S_1^{n-1}P_1 + 0.$$

Since S_1 is nilpotent of order m, it follows that $C_{-m} \neq 0$ and $C_{-n} = 0$ for $n > m$. Thus, $(\lambda - S)^{-1}$ has a pole of order m at 0.

QED

A.4.4 Corollary *Let E be sequentially complete and barrelled, and let $\lambda_0 \in \mathbb{C} \setminus \{0\}$ be a pole of order m of $T \in \mathcal{L}(E)$. Then T has a decomposition of the form*

$$T = \lambda_0 P_{\{\lambda_0\}} + N_{\{\lambda_0\}} + V,$$

where $N_{\{\lambda_0\}} \in \mathcal{L}(E)$, the so-called eigennilpotent, *is nilpotent of order m and $V \in \mathcal{L}(E)$ with $\lambda_0 \notin \sigma(V)$.*

Proof Put $V := (1 - P_{\{\lambda_0\}})T$ and $N_{\{\lambda_0\}} := (T - \lambda_0)P_{\{\lambda_0\}}$. Now, $N_{\{\lambda_0\}}$ is nilpotent of order m by virtue of (A.11); the rest follows from the Riesz projection theorem.

QED

This theorem asserts that a pole of the resolvent is an eigenvalue, and that the spectral projection associated with the pole projects onto the algebraic eigenspace. Additionally, the 'kernel chain'

$$\{0\} = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots$$

and the 'range chain'

$$E = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \mathcal{R}_2 \supset \cdots,$$

with $\mathcal{N}_n$ and $\mathcal{R}_n$ as in (A.19) and (A.20), become stationary in the sense that $\mathcal{N}_m = \mathcal{N}_n$ and $\mathcal{R}_m = \mathcal{R}_n$ for $n \geq m$. The smallest non-negative integer for which this is true is called the *ascent* (in the case of the kernel chain), or the *descent* (in the case of the range chain), of the operator. A closer look at the proof of our theorem reveals that the order of the pole is equal to the ascent and the descent of $\lambda_0 - T$.

If E is a Banach space, a sharper version of the theorem exists (see, for example, [Heu, §50]). In this case, λ_0 is a pole of order m if and only if the ascent and the descent of $\lambda_0 - T$ are equal to m. This result was first established by Caradus in his doctoral dissertation [Car], and later generalised to closed operators on a Banach space by Taylor [Tay2] (see also the paper by Lay [Lay], from which the idea of the proof of the 'if' clause was taken).

In locally convex spaces the chain conditions are necessary, but fail to be sufficient in general. A counter-example may be constructed as follows. Maeda gives an example of an operator $T : E \to E$ on a Fréchet space E with $\sigma(T) = \{\infty\}$ [Mae, Example 2]. Thus, T and T^{-1} are both bijective, and $\sigma(T^{-1}) = \{0\}$. Consider the operator $\hat{T}$ defined by

$$\hat{T} : E \oplus_{\text{top}} \mathbb{C} \to E \oplus_{\text{top}} \mathbb{C}$$

$$\hat{T}(x, y) := (T^{-1}x, 0).$$

Since T^{-1} is bijective, $\mathcal{N}(\hat{T}^n) = \mathbb{C}$ for $n \geq 1$ and $\mathcal{R}(\hat{T}^n) = E$ for $n \geq 1$. Thus, the ascent and the descent of $\hat{T}$ are equal to 1. However, 0 is an essential singularity of the resolvent of $\hat{T}$. To see this, note that by corollary A.2.3 we have $\sigma(\hat{T}) = \{0\}$. Hence, the Neumann series

$$(\lambda - \hat{T})^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} \hat{T}^n$$

is valid for $\lambda \neq 0$. Since $\hat{T}^n \neq 0$ for $n \geq 0$, the resolvent of $\hat{T}$ has an essential singularity at 0.

A.5 The adjoint

Given locally convex spaces E and F, and a continuous linear transformation $T : E \to F$, we may construct a linear transformation T' from F' to E' as follows. For every $y' \in F'$, the equation

$$\langle x, x' \rangle := \langle Tx, y' \rangle \text{ with } x \in E \tag{A.22}$$

defines a unique linear functional x' on E. Since T and y' are continuous, x' is continuous. Setting

$$T'y' := x'$$

then defines an evidently linear transformation T' from F' to E'. This transformation is called the *adjoint of* T. As an immediate consequence of the definition we get the following.

A.5.1 Lemma *Let E, F and G be locally convex.*

(i) If $T \in \mathcal{L}(E, F)$ and $S \in \mathcal{L}(F, G)$, then

$$(ST)' = T'S'.$$

(ii) If T and T^{-1} belong to $\mathcal{L}(E)$, then

$$(T^{-1})' = (T')^{-1}.$$

Proof

(i) follows from

$$\langle x, (ST)'x' \rangle = \langle TSx, x' \rangle = \langle Tx, S'x' \rangle = \langle x, T'S'x' \rangle ,$$

where $x \in E$ and $x' \in G'$.

(ii) By (i), the relations $TT^{-1} = I$, $T^{-1}T = I$ imply the relations $(T^{-1})'T' = I'$, $T'(T^{-1})' = I'$. Since I' is the identity on E', (ii) follows.

QED

A closer inspection of the above definition of the adjoint of an operator $T : E \to F$ reveals that the assumption that T be continuous can be considerably weakened. It suffices, for example, to assume that T is continuous as a transformation from E into F with the weak topology, that is, even $T \in \mathcal{L}(E, F_s)$ guarantees that (A.22) defines a linear transformation taking F' into E'. We shall see later that under certain assumptions on E a 'continuous' T' always stems from a T continuous with respect to the original topologies.

Before we raise the question as to when the adjoint is continuous, we need to specify a topology on the dual of a locally convex space. For $\mathfrak{S}$ a directed and covering family of bounded subsets of E, we set

$$E'_{\mathfrak{S}} := \mathcal{L}_{\mathfrak{S}}(E, \mathbb{C}),$$

that is, we equip E' with the topology of uniform convergence on elements of $\mathfrak{S}$.

A.5.2 Proposition *Let E and F be locally convex, and let $(\mathfrak{S}, \mathfrak{T})$ be admissible for (E, F). If $T : E \to F$ is continuous, then $T' : F'_{\mathfrak{T}} \to E'_{\mathfrak{S}}$ is continuous.*

Proof Let $B \in \mathfrak{S}$. Since $(\mathfrak{S}, \mathfrak{T})$ is admissible for (E, F), there is $\tilde{B} \in \mathfrak{T}$ such that $T(B) \subset \tilde{B}$. For $y' \in F'$ we have

$$\sup_{x \in B} |\langle x, T'y' \rangle| = \sup_{x \in B} |\langle Tx, y' \rangle| \leq \sup_{x \in \tilde{B}} |\langle x, y' \rangle| \,,$$

that is, the adjoint T' is continuous with respect to the indicated topologies.

QED

The above proposition allows us to repeat the construction of the adjoint for T'. If $(\mathfrak{S}, \mathfrak{T})$ is admissible for (E, F), then $T'' := (T')'$ is a linear transformation from $(E'_{\mathfrak{S}})'$ into $(F'_{\mathfrak{T}})'$, which — if $(\mathfrak{T}', \mathfrak{S}')$ is admissible for $(F'_{\mathfrak{T}}, E'_{\mathfrak{S}})$ — is continuous from $(E'_{\mathfrak{S}})'_{\mathfrak{S}'}$ into $(F'_{\mathfrak{T}})'_{\mathfrak{T}'}$. We shall employ this procedure to establish the following properties of adjoints and the operation of taking adjoints in barrelled spaces.

A.5.3 Proposition *Let E be barrelled and let $\mathfrak{S}$ be admissible for E. Then:*

(i) the canonical embedding

$$J : E \to (E'_{\mathfrak{S}})'_b,$$

$$\langle x', Jx \rangle := \langle x, x' \rangle$$

where $x \in E$ and $x' \in E'$, is a topological monomorphism, that is, J is injective, continuous, and open;

(ii) $T \in \mathcal{L}(E)$ if and only if $T' \in \mathcal{L}(E'_{\mathfrak{S}})$;

(iii) the linear transformation

$$' : \mathcal{L}_{\mathfrak{S}}(E) \to \mathcal{L}_b(E'_{\mathfrak{S}}),$$

$$T \mapsto T'$$

is a topological monomorphism.

Proof First we note that E barrelled implies that the seminorms

$$E \ni x \mapsto p_{B'}(x) := \sup_{x' \in B'} |\langle x, x' \rangle| \,,$$

where B' runs through all bounded subsets of $E'_{\mathfrak{S}}$, generate the topology of E (see [Sch, IV.5.2]).

To prove (i) we first need to show that J is well-defined, that is, that $x \in E$ implies $Jx \in (E'_{\mathfrak{S}})'$. Let $x \in E$. Since $\mathfrak{S}$ is covering, there is a $B \in \mathfrak{S}$ such that $x \in S$. But then

$$|\langle x', Jx \rangle| = |\langle x, x' \rangle| \leq \sup_{y \in B} |\langle y, x' \rangle| \,,$$

which implies that Jx is a continuous linear functional on $E'_{\mathfrak{S}}$. Secondly, J is injective, since $Jx = 0$ means that $\langle x, x' \rangle = 0$ for every $x' \in E'$. The Hahn-Banach theorem then shows that $x = 0$. Now, if B' is a bounded subset of $E'_{\mathfrak{S}}$, then, by the above characterisation of the topology on E,

$$\sup_{x' \in B'} |\langle x', Jx \rangle| = \sup_{x' \in B'} |\langle x, x' \rangle| = p_{B'}(x)$$

shows that a net $\{x_\alpha\}$ converges to zero in E if and only if the net $\{Jx_\alpha\}$ converges to zero in $(E'_{\mathfrak{S}})'_b$.

(ii) The 'only if' clause was already proved in proposition A.5.2. Suppose conversely that $T' \in \mathcal{L}(E'_{\mathfrak{S}})$. Then, again by proposition A.5.2, T'' belongs to $\mathcal{L}((E'_{\mathfrak{S}})'_b)$. But T and T'' are related through the canonical embedding J:

$$T''J = JT.$$

To see this, let $x \in E$ and $x' \in E'$; then

$$\langle x', T''Jx \rangle = \langle T'x', Jx \rangle = \langle x, T'x' \rangle = \langle Tx, x' \rangle = \langle x', JTx \rangle \,.$$

Since, by (i), the canonical embedding is a topological monomorphism, T must be continuous.

(iii) Recall that the seminorms

$$E' \ni x' \mapsto q_B(x') := \sup_{x \in B} |\langle x, x' \rangle| \,,$$

where B runs through $\mathfrak{S}$, generate the topology on $E'_{\mathfrak{S}}$.

Let $T \in \mathcal{L}(E)$, $B \in \mathfrak{S}$ and let B' be a bounded subset of $E'_{\mathfrak{S}}$. Since

$$\sup_{x\in B'} q_B(T'x') = \sup_{x'\in B', x\in B} |\langle x, T'x'\rangle| = \sup_{x\in B, x'\in B'} |\langle Tx, x'\rangle| = \sup_{x\in B} p_{B'}(Tx),$$

a net $\{T_\alpha\}$ converges to zero in $\mathcal{L}(E)$ if and only if the net $\{T'_\alpha\}$ converges to zero in $\mathcal{L}(E'_{\mathfrak{S}})$.

QED

We are now able to prove the main result of this section characterising the relation between the spectrum of an operator and that of its adjoint.

A.5.4 Theorem *Let E be barrelled and let $\mathfrak{S}$ be admissible for E. Let, furthermore, $\mathfrak{S}'$ be admissible for $E'_{\mathfrak{S}}$. Then:*

(i) $\sigma'_{\mathcal{L}(E)}(T) = \sigma'_{\mathcal{L}(E'_{\mathfrak{S}})}(T')$;

(ii) $\sigma_{\mathcal{L}_{\mathfrak{S}}(E)}(T) = \sigma_{\mathcal{L}_{\mathfrak{S}'}(E'_{\mathfrak{S}})}(T')$.

Proof

(i) Let $\lambda \in \mathbb{C}$. We need to show that $(\lambda - T)^{-1} \in \mathcal{L}(E)$ if and only if $(\lambda - T')^{-1} \in \mathcal{L}(E'_{\mathfrak{S}})$. This, however, follows immediately from lemma A.5.1 (ii) and proposition A.5.2 (ii).

(ii) Suppose, for the moment, that $\mathfrak{S}'$ is the collection of all bounded subsets of $E'_{\mathfrak{S}}$. Let $\lambda \in \mathbb{C}$. By proposition A.5.2 (iii), T (respectively, $(\lambda - T)^{-1}$) is regular as an element of $\mathcal{L}_{\mathfrak{S}}(E)$ if and only if T' (respectively, $(\lambda - T')^{-1}$) is regular as an element of $\mathcal{L}_b(E'_{\mathfrak{S}})$, and the assertion follows for the b-topology on $\mathcal{L}(E'_{\mathfrak{S}})$. However, since E is barrelled, $E'_{\mathfrak{S}}$ is quasi-complete (see [Sch, III.4.4 corollary]), and hence, by proposition A.1.6 (i), the spectrum of T' in $\mathcal{L}_b(E'_{\mathfrak{S}})$ coincides with the spectrum of T' in $\mathcal{L}_{\mathfrak{S}'}(E'_{\mathfrak{S}})$.

QED

A.5.5 Remark

König [Kön] proves the theorem above under slightly different assumptions and methods.

A.6 The spectrum of intertwined operators

Let E and F be sequentially complete and barrelled. Suppose there are transformations $S \in \mathcal{L}(F)$, $Q \in \mathcal{L}(F, E)$, and $T \in \mathcal{L}(E)$ such that $TQ = QS$. We shall say that Q *intertwines* T *and* S. A typical situation giving rise to intertwined operators is the following. Suppose that F is a subspace of E carrying a topology stronger than the one induced by E, and suppose that the restriction of a $T \in \mathcal{L}(E)$ induces a continuous operator from F to F. Then the canonical embedding of F into E intertwines T and its restriction to F. In this section we shall study how the existence of intertwining transformations affects the spectra of the intertwined operators.

A.6.1 Lemma *Let E and F be non-zero sequentially complete and barrelled, and let $S \in \mathcal{L}(F)$ and $T \in \mathcal{L}(E)$ have non-empty resolvent sets. Assume that there is $Q \in \mathcal{L}(F, E)$ such that $TQ = QS$. If f is holomorphic on an open set Δ containing $\sigma(S) \cup \sigma(T)$, then*

$$f(T)Q = Qf(S).$$

Proof For $\lambda \in \varrho(S) \cap \varrho(T) \cap \mathbb{C}$, we have $(\lambda - T)^{-1}Q = Q(\lambda - S)^{-1}$. Thus, the assertion follows for the case where either both S and T are regular or both S and T are not regular. If one of the operators is not regular, then f is holomorphic at ∞ and the assertion follows from remark A.1.11.

QED

The main result of this section may now be stated.

A.6.2 Theorem (Intertwining theorem) *Let E and F be non-zero sequentially complete and barrelled, and let $S \in \mathcal{L}(F)$, $Q \in \mathcal{L}(F, E)$, and $T \in \mathcal{L}(E)$ such that $TQ = QS$.*

(i) If Q is injective then:

(a) each non-empty spectral set of S meets $\sigma(T)$;

(b) each component of $\sigma(S)$ meets $\sigma(T)$.

(ii) If Q has dense range then:

(a) each non-empty spectral set of T meets $\sigma(S)$;

(b) each component of $\sigma(T)$ meets $\sigma(S)$.

Proof The theorem is trivial if either $\varrho(S)$ or $\varrho(T)$ is empty. We shall therefore assume that $\varrho(S)$ and $\varrho(T)$ are not empty.

(i) In proving (a) let τ be a non-empty spectral set of $\sigma(S)$, and assume to the contrary that $\sigma(T) \cap \tau = \emptyset$. Then τ and $\sigma(T) \cup (\sigma(S) \setminus \tau)$ are disjoint closed sets. Thus, there are two non-intersecting open sets Δ_1 and Δ_2 such that $\tau \subset \Delta_1$ and $\sigma(T) \cup (\sigma(S) \setminus \tau) \subset \Delta_2$. Define a function f by

$$f(\lambda) := \begin{cases} 1 & \text{for } \lambda \in \Delta_1 \\ 0 & \text{for } \lambda \in \Delta_2 \end{cases} .$$

The function f is holomorphic on $\sigma(S) \cup \sigma(T)$ and by hypothesis and lemma A.6.1

$$f(T)Q = Qf(S). \tag{A.23}$$

Now, $f(T)$ vanishes identically while $f(S)$ is the *non-zero* spectral projection associated with τ. Thus, by (A.23),

$$Qf(S) = 0.$$

But since Q is injective, $f(S)$ must be zero, which is the desired contradiction.

To prove (b) let κ be a component of $\sigma(S)$ and observe that $\sigma(S)$ is a compact metric space by theorem A.1.8 (ii). Thus, there is a nested sequence $\{\kappa_n\}$ of closed and open subsets κ_n of $\sigma(S)$ for which $\kappa = \bigcap_n \kappa_n$ (see [Rud2, Appendix A, theorem A7]). By (a), each $\kappa_n \cap \sigma(T)$ is non-empty. Thus, the intersection of the nested sequence $\{\kappa_n \cap \sigma(T)\}$ of closed subsets of the compact metric space $\sigma(T)$ is not empty. Hence,

$$\kappa \cap \sigma(T) = \bigcap_n (\kappa_n \cap \sigma(T)) \neq \emptyset.$$

(ii) Suppose that Q has dense range. An application of the Hahn-Banach theorem shows that the adjoint transformation $Q' : E' \to F'$ is injective. By proposition A.5.2 we have $T' \in \mathcal{L}(E'_s)$, $Q \in \mathcal{L}(E'_s, F'_s)$, and $S' \in \mathcal{L}(F'_s)$; the adjoint intertwining relation reads

$$S'Q' = Q'T'$$

by lemma A.5.1 (i). Hence, by (i), each non-empty spectral set of T' and each component of $\sigma(T')$ meets $\sigma(S')$. This finishes the proof, since $\sigma(T') = \sigma(T)$ and $\sigma(S') = \sigma(S)$ by virtue of theorem A.5.4.

QED

A.6.3 Corollary *Let E and F be non-zero sequentially complete and barrelled, and let $S \in \mathcal{L}(F)$, $Q \in \mathcal{L}(F, E)$, and $T \in \mathcal{L}(E)$ such that $TQ = QS$. If Q is injective with dense range, and if $\sigma(S)$ and $\sigma(T)$ are both totally disconnected, then*

$$\sigma(S) = \sigma(T).$$

Proof By hypothesis, each singleton of $\sigma(T)$ or $\sigma(S)$ is a component of $\sigma(T)$ or $\sigma(S)$, respectively.

QED

A.6.4 Remark
Part (i)(a) was proved by Fialkow for $E = F = \mathcal{H}$, with $\mathcal{H}$ a Hilbert space [Fia, 2.4]. The other assertions first appeared in a paper of Grabiner under the assumption that E and F are Banach spaces [Gra, 2.1].

We conclude this section with an investigation of the case where $\lambda \in \mathbb{C}$ is an isolated point of both $\sigma(S)$ and $\sigma(T)$.

A.6.5 Proposition *Let E and F be non-zero sequentially complete and barrelled, and let $S \in \mathcal{L}(F)$, $Q \in \mathcal{L}(F, E)$, and $T \in \mathcal{L}(E)$ such that $TQ = QS$. Suppose that $\lambda \in \mathbb{C}$ is an isolated point in $\sigma(S)$ and in $\sigma(T)$. Then, provided that we identify all infinite cardinals, the following holds:*

(i) if Q is injective and λ is a pole of T, then λ is a pole of S, with $\nu_p(\lambda; S) \leq \nu_p(\lambda; T)$ and $m_{\mathrm{alg}}(\lambda; S) \leq m_{\mathrm{alg}}(\lambda; T)$;

(ii) if Q has dense range and λ is a pole of S, then λ is a pole of T, with $\nu_p(\lambda; T) \leq \nu_p(\lambda; S)$ and $m_{\mathrm{alg}}(\lambda; T) \leq m_{\mathrm{alg}}(\lambda; S)$.

Proof

(i) Let $A_n^{(S)}$ (respectively, $A_n^{(T)}$) denote the coefficients of the Laurent expansion of the resolvent of S (respectively, T) at λ. We know from lemma A.6.1 that Q intertwines $A_n^{(T)}$ and $A_n^{(S)}$:

$$A_n^{(T)} Q = Q A_n^{(S)} \text{ for } n \in \mathbb{Z}.$$

If Q is injective, then $A_n^{(T)} = 0$ implies that $A_n^{(S)} = 0$. Thus $\nu_p(\lambda; S) \leq \nu_p(\lambda; T)$. Furthermore, we have

$$Q(\mathcal{R}(A_{-1}^{(S)})) \subset \mathcal{R}(A_{-1}^{(T)}). \tag{A.24}$$

To see this, let $x \in \mathcal{R}(A_{-1}^{(S)})$. Then $x = A_{-1}^{(S)}x$ and $Qx = QA_{-1}^{(S)}x = A_{-1}^{(T)}Qx$. Thus, $Qx \in \mathcal{R}(A_{-1}^{(T)})$. Now, if Q is injective, then (A.24) implies $\dim \mathcal{R}(A_{-1}^{(S)}) \leq \dim \mathcal{R}(A_{-1}^{(T)})$. This finishes the proof of (i).

(ii) follows from (i) by taking adjoints, just as in the proof of the previous theorem. We only need to observe that λ is a pole of S of order m if and only if λ is a pole of S' of order m, which follows from theorem A.5.3 (iii), and use the fact that the range of a projection is finite-dimensional if and only if the range of its adjoint is finite-dimensional, in which case the two dimensions are equal (see [Sch, IV.1.1]).

QED

A.7 Inductive and projective limits

Inductive and projective limits play a significant part in modern functional analysis; notably, the theory of distributions developed by Laurent Schwartz relies heavily on these two constructions. One of the basic objects of his theory, namely the space of infinitely differentiable functions with compact support in $\mathbb{R}$, is an example of an inductive limit, while its topological dual, the space of distributions, is an example of a projective limit. Roughly speaking, these two 'limiting' processes generate and topologise a space E from a given sequence of locally convex topological vector spaces $\{E_n\}$. In this section we shall briefly review the basic properties of these constructions and then focus our attention on the relation between the spectra of operators on the generating spaces and the spectrum of their limit on the limit space. The section is divided into four parts, the first of which is devoted to a basic description of inductive limits. In the second part we shall consider maps of inductive systems and study their spectral properties. Parts three and four parallel the first two parts in the sense that they contain the analogous constructions for projective systems and limits. In what follows, we shall not assume that the topological vector spaces we consider are Hausdorff.

Inductive systems and limits

The definition of inductive systems requires the notion of a 'directed set'. Let us recall that a preordered set $(\mathcal{I}, \leq)$, that is a set $\mathcal{I}$ endowed with a reflexive and transitive relation '$\leq$', is called a *directed set* if for every pair $(n, m) \in \mathcal{I} \times \mathcal{I}$ there is an $l \in \mathcal{I}$ such that $n \leq l$ and $m \leq l$. A subset $\mathcal{J}$ of a directed set $\mathcal{I}$ is called *cofinal* if for every $n \in \mathcal{I}$ there is $m \in \mathcal{J}$ with $n \leq m$. The subset $\mathcal{J}$ is called *residual* if there is $n_0 \in \mathcal{I}$ such that $n_0 \leq n$ implies $n \in \mathcal{J}$.

A.7.1 Definition Let $\mathcal{I}$ be a countable directed set and let $\left\{ E_n \,\middle|\, n \in \mathcal{I} \right\}$ be a family of locally convex topological vector spaces indexed by $\mathcal{I}$. Assume that for each pair of indices (n, m) with $n \leq m$ there is a continuous linear transformation

$$\pi_{nm} : E_n \to E_m,$$

and that these transformations satisfy the conditions

$$\pi_{nn} = id_{E_n},$$

$$\pi_{nl} = \pi_{ml} \circ \pi_{nm} \text{ if } n \leq m \leq l.$$

Then the family $\{E_n; \pi_{nm}\}$ is called a *(countable) inductive system over* $\mathcal{I}$; the transformations π_{nm} are termed *connecting maps* .

Each inductive system $\{E_n; \pi_{nm}\}$ yields a limit space as follows. Let $\hat{E} := \oplus_{n \in \mathcal{I}} E_n$ be the locally convex direct sum of the spaces E_n (see [Sch, II.6.1]) and let $i_n : E_n \to \hat{E}$ denote the canonical embedding of E_n in $\hat{E}$. Let F denote the subspace generated by the ranges of the transformations $i_n - i_m \circ \pi_{nm}$ of E_n into $\hat{E}$, where (n, m) runs through all pairs such that $n \leq m$. We may now introduce the inductive limit as a quotient space (see [Sch, I.2.2]).

A.7.2 Definition Let $\{E_n; \pi_{nm}\}$ be a (countable) inductive system over $\mathcal{I}$. The quotient space $\hat{E}/F$ is called the *(countable) inductive limit* of the system, and is denoted by E^∞, or $\underrightarrow{\mathrm{Ind}}\, E_n$, or even $\mathcal{I}$-$\underrightarrow{\mathrm{Ind}}\, E_n$ if reference to $\mathcal{I}$ is desired.

A.7.3 Remark

(i) It is possible to define inductive systems and limits over uncountable directed sets. We restrict ourselves to countable systems, since some important characterisations of inductive limits, such as theorem A.7.6 below, no longer hold for uncountable systems.

(ii) The inductive limit is locally convex; however, even if each of the spaces E_n is Hausdorff, the inductive limit need not be Hausdorff. We shall see later that suitably well-behaved systems give rise to Hausdorff inductive limits.

(iii) To a certain extent, the inductive limit is independent of the index set $\mathcal{I}$. If $\mathcal{J}$ is a cofinal subset of $\mathcal{I}$, then one may easily verify that $\left\{ (E_n, \pi_{nm}) \,\middle|\, n, m \in \mathcal{J} \right\}$ is an inductive system as well. Furthermore, $\mathcal{J}\text{-}\underrightarrow{\mathrm{Ind}}\, E_n$ and $\mathcal{I}\text{-}\underrightarrow{\mathrm{Ind}}\, E_n$ are isomorphic as topological vector spaces (see [FW, §23; 3.5]).

Let $\pi_F : \hat{E} \to E^\infty$ denote the quotient map; its restriction $\pi_F|E_n$ to E_n is denoted by π_n and is called the *canonical map of E_n into E^∞*.

A.7.4 Lemma *Let $\{E_n; \pi_{nm}\}$ be an inductive system over $\mathcal{I}$. Then:*

(i) each canonical map $\pi_n : E_n \to E^\infty$ is continuous and satisfies

$$\pi_n = \pi_m \circ \pi_{nm} \text{ whenever } n \leq m;$$

(ii) $E^\infty = \bigcup \left\{ \pi_n(E_n) \,\middle|\, n \in \mathcal{I} \right\}$.

Proof

(i) We note that $\pi_n = \pi_F \circ i_n$. Since i_n and π_F are continuous by the very definition of the topologies on $\hat{E}$ and E^∞, the canonical map π_n is continuous for $n \in \mathcal{I}$. Furthermore, we have for $n \leq m$

$$\pi_F \circ (i_n - i_m \circ \pi_{nm}) = 0$$

by virtue of the construction of F, and the second assertion is proved.

(ii) Let $[x]_F \in E^\infty$ be the equivalence class arising from $x \in \hat{E}$. We only need to show that there is $k \in \mathcal{I}$ and $x_k \in E_k$ such that $x - i_k(x_k) \in F$, since then $\pi_k(x_k) = [x]_F$. Now, since x belongs to the locally convex direct sum $\hat{E}$, we may write $x = \sum_{j=0}^N i_{n_j}(x_{n_j})$, where N is an integer, $n_j \in \mathcal{I}$, and $x_{n_j} \in E_{n_j}$. Choose $k \in \mathcal{I}$ with $n_j \leq k$ for $0 \leq j \leq N$, and put

$$x_k := \sum_{j=0}^N \pi_{n_j k}(x_{n_j}).$$

Clearly, $x_k \in E_k$ and $x - i_k(x_k) = \sum_{j=0}^N (i_{n_j} - i_k \circ \pi_{n_j k})(x_{n_j}) \in F$, which is what we wanted to show.

QED

Let us call the image of some $x_n \in E_n$ under any connecting map a *successor* of x_n. A routine verification shows that for any n, m we have $\pi_n(x_n) = \pi_m(x_m)$ if and only if x_n and x_m have a common successor in the system. An element of the inductive limit may thus be visualised as an equivalence class of x_ns having a common successor. We also note that $E^\infty = \bigcup \left\{ \pi_n(E_n) \,\middle|\, n \in \mathcal{I} \right\}$ carries the inductive topology with respect to the family $\{E_n; \pi_n\}$ (see [Sch, p. 54]).

In practice, inductive systems often occur in the following special form. Suppose that $\left\{ E_n \,\middle|\, n \in \mathbb{N} \right\}$ is a family of subspaces of a vector space E such that $E_n \subset E_m$ for $n \leq m$ and $E = \bigcup \left\{ E_n \,\middle|\, n \in \mathbb{N} \right\}$. Assume, furthermore, that each E_n carries a locally convex topology τ_n such that for $n \leq m$ the topology induced by τ_m on E_n is weaker than τ_n. If we denote by π_{nm} the canonical embedding of E_n into E_m $(n \leq m)$, it is easily verified that $\{E_n; \pi_{nm}\}$ is an inductive system. In this case, π_n is the canonical embedding of E_n into E and $\underrightarrow{\operatorname{Ind}} E_n = E$ as a set. Inductive systems of this type will be called *injective*.

A.7.5 Definition An inductive system $\{(E_n, \tau_n); \pi_{nm}\}$ is called *Hausdorff* (respectively, *barrelled*; respectively, *complete*) if each (E_n, τ_n) is Hausdorff (respectively, barrelled; respectively, complete). The inductive system is *compact* if each connecting map π_{nm} is compact. Finally, the system is termed *strict* if it is injective and, for $n \leq m$, the topology τ_m induces the original topology τ_n on E_n.

The following theorem contains the fundamental characterisation of inductive limits, as it furnishes a relation between the topological properties of inductive systems and the topological properties of their limits.

A.7.6 Theorem *Let $\{E_n; \pi_{nm}\}$ be a Hausdorff inductive spectrum.*

(i) If $\{E_n; \pi_{nm}\}$ is either strict and complete, or injective and compact, then the associated inductive limit is Hausdorff and complete.

(ii) If (i) holds and $\{E_n; \pi_{nm}\}$ is barrelled, then the inductive limit is also barrelled.

Proof See [Sch, II.6.4 and II.6.6] and [FW, §25; 1.4 and §26; 2.3] for (i). Assertion (ii) is proved in [Sch, II.7.2 corollary 1].

QED

In view of this theorem, which guarantees the permanence of completeness and barrelledness under the operation of taking inductive limits, we may ask the question as to whether it is possible to establish a connection between the spectra of operators, which — in some sense — 'act' on inductive systems, and the spectrum of a certain 'limit' of these operators. This will be the content of the next section.

Maps of inductive systems and their spectra

A.7.7 Definition Let $\{E_n; \pi_{nm}\}$ and $\{F_n; \rho_{nm}\}$ be two inductive systems over the same directed set $\mathcal{I}$. Assume that for each $n \in \mathcal{I}$ there is a continuous linear transformation $T_n : E_n \to F_n$ and that these transformations satisfy the condition

$$T_m \circ \pi_{nm} = \rho_{nm} \circ T_n \text{ whenever } n \leq m.$$

Then the family $\left\{ T_n \,\middle|\, n \in \mathcal{I} \right\}$ is called a *map of the inductive system* $\{E_n; \pi_{nm}\}$ *into the inductive system* $\{F_n; \rho_{nm}\}$; if this is the case, we write

$$\{T_n\} : \{E_n; \pi_{nm}\} \to \{F_n; \rho_{nm}\}.$$

It turns out that maps of inductive systems can be 'lifted' to the associated inductive limits in the following sense.

A.7.8 Proposition *Let* $\left\{ T_n \,\middle|\, n \in \mathcal{I} \right\}$ *be a map of the inductive system* $\{E_n; \pi_{nm}\}$ *into the inductive system* $\{F_n; \rho_{nm}\}$. *Then there is a unique* $T^\infty \in \mathcal{L}(E^\infty, F^\infty)$ *such that*

$$T^\infty \circ \pi_n = \rho_n \circ T_n \textit{ for } n \in \mathcal{I}. \tag{A.25}$$

Proof First we note that a T^∞ satisfying (A.25) is necessarily continuous; this follows immediately from the continuity criterion for inductive topologies (see [Sch, II.6.1]). Secondly, T^∞ — provided that it exists — must be unique. For suppose that there is $\tilde{T}^\infty$ such that (A.25) holds. Then $T^\infty \circ \pi_n = \tilde{T}^\infty \circ \pi_n$, whence it follows that $T^\infty = \tilde{T}^\infty$ by lemma A.7.4 (ii). Thus, we only need to show that T^∞ exists.

To this end, let $\hat{E} := \oplus_n E_n$ and $\hat{F} := \oplus_n F_n$. Let i_n (respectively, j_n) denote the canonical embedding of E_n in $\hat{E}$ (respectively, the embedding of F_n in $\hat{F}$), and let G (respectively, H) denote the subspace of $\hat{E}$ (respectively, $\hat{F}$) generated by the ranges of the transformations $i_n - i_m \circ \pi_{nm}$ (respectively, $j_n - j_m \circ \rho_{nm}$). Finally, we use π_G (respectively, π_H) for the corresponding quotient map. We define a linear transformation $\hat{T} : \hat{E} \to \hat{F}$ by $\hat{T}|E_n := T_n$. Obviously, $\hat{T}$ satisfies $\hat{T} \circ i_n = j_n \circ T_n$ for $n \in \mathcal{I}$. Moreover, we have

$$\hat{T}(G) \subset H, \tag{A.26}$$

since

$$\begin{aligned} \hat{T} \circ (i_n - i_m \circ \pi_{nm}) &= j_n \circ T_n - j_m \circ T_m \circ \pi_{nm} \\ &= j_n \circ T_n - j_m \circ \rho_{nm} \circ T_n \\ &= (j_n - j_m \circ \rho_{nm}) \circ T_n. \end{aligned}$$

Passing to the quotients we now define $T^\infty : E/G \to F/H$ by

$$T^\infty([x]_G) := [\hat{T}x]_H.$$

T^∞ is well-defined by virtue of (A.26), and trivially satisfies $T^\infty \circ \pi_G = \pi_H \circ \hat{T}$. Thus, $T^\infty \circ \pi_G \circ i_n = \pi_H \circ \hat{T} \circ i_n = \pi_H \circ j_n \circ T_n$, which is just (A.25).

QED

Given a map $\{T_n\} : \{E_n; \pi_{nm}\} \to \{F_n; \rho_{nm}\}$ we shall use $\underrightarrow{\mathrm{Ind}}\, T_n$ to denote the unique $T^\infty \in \mathcal{L}(E^\infty, F^\infty)$ constructed in the proposition above.

Suppose now, that $\{T_n\}$ is a map of a Hausdorff inductive system $\{E_m; \pi_{nm}\}$ into itself. For the rest of this section we shall be concerned with establishing a relation between the spectra of the operators T_n and the spectrum of $\underrightarrow{\mathrm{Ind}}\, T_n$. As a first step we shall give sufficient conditions for $\underrightarrow{\mathrm{Ind}}\, T_n$ to be invertible and regular.

A.7.9 Lemma *Let $\{E_n; \pi_{nm}\}$ be a Hausdorff inductive system over $\mathcal{I}$ such that $\underrightarrow{\mathrm{Ind}}\, E_n$ is Hausdorff. Suppose that $\{T_n\} : \{E_n; \pi_{nm}\} \to \{E_n; \pi_{nm}\}$ is a map of the inductive system into itself.*

(i) If $T_n^{-1} \in \mathcal{L}(E_n)$ for every $n \in \mathcal{I}$, then the inverse of $\underrightarrow{\mathrm{Ind}}\, T_n$ exists in $\mathcal{L}(\underrightarrow{\mathrm{Ind}}\, E_n)$ and is given by $(\underrightarrow{\mathrm{Ind}}\, T_n)^{-1} = \underrightarrow{\mathrm{Ind}}\, T_n^{-1}$.

(ii) If each T_n is regular as an element of $\mathcal{L}_s(E_n)$ and

$$\sup\left\{ \beta(T_n) \;\middle|\; n \in \mathcal{I} \right\} < \infty,$$

then $\underrightarrow{\mathrm{Ind}}\, T_n$ is regular as an element of $\mathcal{L}_s(\underrightarrow{\mathrm{Ind}}\, E_n)$.

Proof Put $T := \varinjlim\limits_{\text{Ind}} T_n$. In order to prove (i), suppose that $T_n^{-1} \in \mathcal{L}(E_n)$ for every $n \in \mathcal{I}$. Since $\{T_n\}$ is a map of the inductive system, so is $\{T_n^{-1}\}$. We may thus define $S := \underrightarrow{\text{Ind}}\, T_n^{-1} \in \mathcal{L}(E^\infty)$. Now $S \circ T \circ \pi_n = S \circ \pi_n \circ T_n = \pi_n \circ T_n^{-1} \circ T_n = \pi_n$, which together with lemma A.7.4 (ii) implies $ST = I$. A similar argument yields $TS = I$, so that S is indeed the inverse of T.

(ii) Choose $k > \sup\left\{ \beta(T_n) \,\middle|\, n \in \mathcal{I} \right\}$. If we are able to show that, for every $x \in E^\infty$, the set

$$B_x := \bigcup \left\{ (T/k)^i x \,\middle|\, i \in \mathbb{N} \right\}$$

is bounded in E^∞, the assertion holds. Now let $x \in E^\infty$. Then, by lemma A.7.4 (ii), there is $n \in \mathcal{I}$ and $x_n \in E_n$ such that $x = \pi_n(x_n)$. Observe that, for $i \in \mathbb{N}$,

$$(T/k)^i x = (T/k)^i \circ \pi_n(x_n) = \pi_n \circ (T_n/k)^i (x_n).$$

By hypothesis and our choice of k, the set $B_n := \bigcup \left\{ (T_n/k)^i x_n \,\middle|\, i \in \mathbb{N} \right\}$ is bounded in E_n. However, since π_n is continuous, $\pi_n(B_n) = B_x$ must be bounded in E^∞.

QED

In order to state our main result we need to introduce the set-theoretic analogues of 'lim inf' and 'lim sup'. If $\mathcal{I}$ is a directed set and $\left\{ X_n \,\middle|\, n \in \mathcal{I} \right\}$ is a family of subsets of a set X, we put

$$\limsup_{n \in \mathcal{I}} X_n := \bigcap_{n \in \mathcal{I}} \bigcup_{n \leq k} X_k,$$

$$\liminf_{n \in \mathcal{I}} X_n := \bigcup_{n \in \mathcal{I}} \bigcap_{n \leq k} X_k.$$

It is not difficult to see that $x \in \limsup_{n \in \mathcal{I}} X_n$ (respectively, $x \in \liminf_{n \in \mathcal{I}} X_n$) if and only if $\left\{ n \in \mathcal{I} \,\middle|\, x_n \in X_n \right\}$ is a cofinal (respectively, residual) subset of $\mathcal{I}$. Furthermore, we have

$$\liminf_{n \in \mathcal{I}} X_n \subset \limsup_{n \in \mathcal{I}} X_n,$$

$$X \setminus \limsup_{n \in \mathcal{I}} X_n = \liminf_{n \in \mathcal{I}} (X \setminus X_n),$$

where $X \setminus \tilde{X}$ denotes the complement of $\tilde{X}$ in X.

A.7.10 Theorem *Let $\{E_n;\pi_{nm}\}$ be a complete and barrelled Hausdorff inductive system over $\mathcal{I}$ such that $\underrightarrow{\mathrm{Ind}}\, E_n$ is complete, barrelled and Hausdorff. Suppose that $\{T_n\} : \{E_n;\pi_{nm}\} \to \{E_n;\pi_{nm}\}$ is a map of the inductive system into itself. Then:*

(i) $\sigma'(\underrightarrow{\mathrm{Ind}}\, T_n) \subset \liminf_{n\in\mathcal{I}} \sigma'(T_n)$;

(ii) $\sigma(\underrightarrow{\mathrm{Ind}}\, T_n) \subset \mathrm{clo}\,(\liminf_{n\in\mathcal{I}} \sigma(T_n))$,

where 'clo' *denotes the closure of a set in $\overline{\mathbb{C}}$.*

Proof Throughout, we shall use the following abbreviations:

$$\begin{aligned} T &:= \underrightarrow{\mathrm{Ind}}\, T_n, \\ \varrho' &:= \limsup_{n\in\mathcal{I}} \varrho'(T_n), \\ \varrho &:= \limsup_{n\in\mathcal{I}} \varrho(T_n). \end{aligned}$$

We shall prove (i) by showing that $\varrho' \subset \varrho'(T)$. First, we note that ∞ always belongs to both ϱ' and $\varrho'(T)$. Now let $\lambda \in \varrho' \cap \mathbb{C}$. We may, with no loss of generality, assume that $\lambda \in \varrho'(T_n)$ for *every* $n \in \mathcal{I}$, since, by remark A.7.3 (iii), we can go over to a cofinal subset of $\mathcal{I}$ without changing E^∞. Hence, $(\lambda - T_n)^{-1} \in \mathcal{L}(E_n)$ for every $n \in \mathcal{I}$, which, by lemma A.7.9 (i), implies that $\lambda - \underrightarrow{\mathrm{Ind}}\, T_n = \underrightarrow{\mathrm{Ind}}(\lambda - T_n)$ is invertible in $\mathcal{L}(E^\infty)$.

(ii) will be proved by showing that $\mathrm{int}\,\varrho \subset \varrho(T)$, where $\mathrm{int}\,\varrho$ denotes the interior of ϱ. If $\infty \in \mathrm{int}\,\varrho$, then there is a cofinal subset $\mathcal{J}$ of $\mathcal{I}$ such that $\infty \in \varrho(T_n)$ for every $n \in \mathcal{J}$; moreover, since $\mathrm{int}\,\varrho$ is open, there is $K > 0$ such that $\left\{ z \in \mathbb{C} \,\middle|\, |z| > K \right\} \subset \varrho(T_n)$ for every $n \in \mathcal{J}$. Thus $\beta(T_n) = r(T_n) \le K$ for every $n \in \mathcal{J}$, and lemma A.7.9 (ii) together with remark A.7.3 (iii) imply that $\infty \in \varrho(T)$. Now let $\lambda \in \mathrm{int}\,\varrho \cap \mathbb{C}$. Then there is $\epsilon > 0$ such that $\mathrm{dist}(\lambda, \sigma(T_n)) \ge \epsilon$ for every $n \in \mathcal{I}$ — again, with no loss of generality. Theorem A.1.9 now implies that $r((\lambda - T_n)^{-1}) \le \epsilon^{-1}$ for every $n \in \mathcal{I}$. Yet another application of lemma A.7.9 shows that $\lambda - T$ has a regular inverse in $\mathcal{L}_s(E^\infty)$. Thus, $\lambda \in \varrho(T)$.

QED

The inclusions in (i) and (ii) may well be proper in general, although we do not yet know of any examples. In particular circumstances, however, the inclusion in (ii) can be replaced by equality. Suppose, for example, that the system is injective, and that $\sigma(T_n)$ is totally disconnected for every $n \in \mathcal{I}$. It is an immediate consequence of definition A.7.7 and proposition A.7.8 that

(a) π_{nm} intertwines T_m and T_n whenever $n \leq m$;

(b) π_n intertwines $\underrightarrow{\mathrm{Ind}}\, T_n$ and T_n for every $n \in \mathcal{I}$.

Since the system is injective, all connecting maps and all canonical maps are injective. We may now apply the intertwining theorem A.6.2 to (a) and (b) yielding $\sigma(T_n) \subset \sigma(T_m)$ for $n \leq m$, and $\sigma(T_n) \subset \sigma(\underrightarrow{\mathrm{Ind}}\, T_n)$ for every $n \in \mathcal{I}$. Thus,

$$\liminf_{n\in\mathcal{I}} \sigma(T_n) = \bigcup_{n\in\mathcal{I}} \sigma(T_n) \subset \sigma(\underrightarrow{\mathrm{Ind}}\, T_n).$$

Taking closures we obtain the following corollary.

A.7.11 Corollary *Let $\{E_n; \pi_{nm}\}$ and $\{T_n\} : \{E_n; \pi_{nm}\} \to \{E_n; \pi_{nm}\}$ be as in the theorem. If $\{E_n; \pi_{nm}\}$ is injective, and $\sigma(T_n)$ is totally disconnected for every $n \in \mathcal{I}$, then*

$$\mathrm{clo}\,(\liminf_{n\in\mathcal{I}} \sigma(T_n)) = \mathrm{clo}\,(\bigcup_{n\in\mathcal{I}} \sigma(T_n)) = \sigma(\underrightarrow{\mathrm{Ind}}\, T_n).$$

.

Projective systems and limits

Just like an inductive system, a projective system is a family of locally convex spaces together with a family of connecting maps. For projective systems, however, these maps go in the direction opposite to the order.

A.7.12 Definition Let $\mathcal{I}$ be a directed set and let $\left\{ (E_n, \tau_n) \,\middle|\, n \in \mathcal{I} \right\}$ be a family of locally convex topological vector spaces indexed by $\mathcal{I}$. Assume that for each pair of indices (n, m) with $n \leq m$ there is a continuous linear transformation

$$\mu_{mn} : E_m \to E_n,$$

and that these transformations satisfy the conditions

$$\mu_{nn} = id_{E_n},$$

$$\mu_{ln} = \mu_{mn} \circ \mu_{lm} \text{ if } n \leq m \leq l.$$

Then the family $\{(E_n, \tau_n); \mu_{mn}\}$ is called a *projective system over $\mathcal{I}$*; the transformations μ_{mn} are termed *connecting maps*.

With every projective system we may associate a limit space given by a certain subspace of the product $\prod_{n\in\mathcal{I}} E_n$ (see [Sch, I.2] for the definition).

A.7.13 Definition Let $\{E_n; \mu_{mn}\}$ be a projective system over $\mathcal{I}$. For each $m \in \mathcal{I}$, let $pr_m : \prod_n E_n \to E_m$ denote the projection onto the m-th factor. Then the subspace

$$\left\{ x \in \prod_n E_n \,\middle|\, pr_n(x) = \mu_{mn} \circ pr_m(x) \text{ whenever } n \le m \right\}$$

is called the *projective limit* of the system and is denoted by E_∞, or $\operatorname{Proj}_{\longleftarrow} E_n$, or even $\mathcal{I}\text{-}\operatorname{Proj}_{\longleftarrow} E_n$ if reference to $\mathcal{I}$ is desired.

A.7.14 Remark

(i) The projective limit of a family of complete Hausdorff spaces is Hausdorff and complete (see [Sch, II.5.3]). Moreover, if $\mathcal{I}$ is countable and each E_n is metrisable, then the projective limit is metrisable (see [FW, §6; 2.2]).

(ii) If $\mathcal{J}$ is a cofinal subset of $\mathcal{I}$ then $\mathcal{J}\text{-}\operatorname{Proj}_{\longleftarrow} E_n$ is isomorphic to $\mathcal{I}\text{-}\operatorname{Proj}_{\longleftarrow} E_n$ (see [FW, §6; 2.5]).

According to this definition, an element $x = \{x_n\} \in \prod_n E_n$ belongs to E_∞ if $x_n = \mu_{mn}(x_m)$ whenever $n \le m$. The restriction $pr_n|E_\infty \to E_n$ is denoted by μ_n and is called the *canonical map of E_∞ into E_n*.

A.7.15 Lemma *Let $\{E_n; \mu_{mn}\}$ be a projective system over $\mathcal{I}$. Then:*

(i) each canonical map μ_n is continuous and satisfies

$$\mu_n = \mu_{mn} \circ \mu_m \textit{ whenever } n \le m;$$

(ii) $\bigcap \left\{ \mathcal{N}(\mu_n) \,\middle|\, n \in \mathcal{I} \right\} = \{0\}$.

Proof (i) Since each pr_n is continuous by the very definition of the topology of the product, each connecting map μ_n must be continuous. The rest is obvious.

(ii) If $\mu_n(x) = 0$ for every $n \in \mathcal{I}$, then $pr_n(x) = 0$ for every $n \in \mathcal{I}$. Thus, $x = 0$.

QED

A particular instance of a projective system arises as follows. Let $\left\{ E_n \,\middle|\, n \in \mathbb{N} \right\}$ be a descending family of subspaces of a vector space, that is $E_m \subset E_n$ whenever $n \leq m$. Suppose that each E_n carries a locally convex topology τ_n and that, for $n \leq m$, the topology induced by τ_n on E_m is weaker than τ_m. Denoting by μ_{mn} the canonical embedding of E_m into E_n $(n \leq m)$, it is not difficult to see that $\{(E_n, \tau_n); \mu_{mn}\}$ is a projective system. In this case, $\underset{\longleftarrow}{\operatorname{Proj}} E_n$ can be identified with $\bigcap \left\{ E_n \,\middle|\, n \in \mathbb{N} \right\}$ and the canonical maps μ_n are just the canonical embeddings of $\bigcap \left\{ E_n \,\middle|\, n \in \mathbb{N} \right\}$ into E_n. Projective systems of this type will be called *injective*.

Maps of projective systems and their spectra

A.7.16 Definition Let $\{E_n; \mu_{mn}\}$ and $\{F_n; \nu_{mn}\}$ be two projective systems over the same directed set $\mathcal{I}$. Assume that for each $n \in \mathcal{I}$ there is a continuous linear transformation $T_n : E_n \to F_n$ and that these transformations satisfy the condition

$$T_n \circ \mu_{mn} = \nu_{mn} \circ T_m \text{ whenever } n \leq m.$$

Then the family $\left\{ T_n \,\middle|\, n \in \mathcal{I} \right\}$ is called a *map of the projective system* $\{E_n; \mu_{mn}\}$ *into the projective system* $\{F_n; \nu_{mn}\}$; if this is the case, we write

$$\{T_n\} : \{E_n; \mu_{mn}\} \to \{F_n; \nu_{mn}\}.$$

A map of a projective system yields a continuous linear transformation on the projective limits as follows.

A.7.17 Proposition *Let* $\left\{ T_n \,\middle|\, n \in \mathcal{I} \right\}$ *be a map of the projective system* $\{E_n; \mu_{mn}\}$ *into the projective system* $\{F_n; \nu_{mn}\}$. *Then there is a unique* $T_\infty \in \mathcal{L}(E_\infty, F_\infty)$ *such that*

$$T_n \circ \mu_n = \nu_n \circ T_\infty \textit{ for } n \in \mathcal{I}. \tag{A.27}$$

Proof Define $\hat{T} : \prod_n E_n \to \prod_n F_n$ by $\hat{T} : \{x_n\} \mapsto \{T_n(x_n)\}$. It turns out that $\hat{T}$ maps E_∞ into F_∞. To see this let $x = \{x_n\}$ belong to E_∞. Then, since $x_n = \mu_{mn}(x_m)$ for $n \leq m$, we must have $\nu_{mn} \circ T_m(x_m) = T_n \circ \mu_{mn}(x_m) = T_n(x_n)$ for $n \leq m$. Thus $\{T_n(x_n)\} \in F_\infty$. Letting $T_\infty := \hat{T}|E_\infty$, we have $T_\infty : E_\infty \to F_\infty$, and T_∞ trivially satisfies (A.27). This relation also implies that T_∞ is continuous, by the continuity criterion for projective topologies (see [Sch, II.5.2]), while uniqueness follows from lemma A.7.15 (ii).

QED

We write $\underleftarrow{\mathrm{Proj}}\, T_n$ to denote the unique operator on the limit spaces. Our final aim will be to derive the analogue of theorem A.7.10 for projective limits. This requires the following.

A.7.18 Lemma *Let $\{E_n; \mu_{mn}\}$ be a Hausdorff projective system over $\mathcal{I}$, that is, we assume that each E_n is Hausdorff. Suppose that*

$$\{T_n\} : \{E_n; \mu_{mn}\} \to \{E_n; \mu_{mn}\}$$

is a map of the projective system.

(i) If $T_n^{-1} \in \mathcal{L}(E_n)$ for every $n \in \mathcal{I}$, then the inverse of $\underleftarrow{\mathrm{Proj}}\, T_n$ exists in $\mathcal{L}(\underleftarrow{\mathrm{Proj}}\, E_n)$ and is given by $(\underleftarrow{\mathrm{Proj}}\, T_n)^{-1} = \underleftarrow{\mathrm{Proj}}\, T_n^{-1}$.

(ii) If each T_n is regular as an element of $\mathcal{L}_s(E_n)$ and

$$\sup\Big\{\, \beta(T_n) \,\Big|\, n \in \mathcal{I} \,\Big\} < \infty,$$

then $\underleftarrow{\mathrm{Proj}}\, T_n$ is regular as an element of $\mathcal{L}_s(\underleftarrow{\mathrm{Proj}}\, E_n)$.

Proof

(i) Put $T := \underleftarrow{\mathrm{Proj}}\, T_n$ and $S := \underleftarrow{\mathrm{Proj}}\, T_n^{-1}$, and note that $\mu_n \circ S \circ T = \mu_n \circ T \circ S = \mu_n$ for every $n \in \mathcal{I}$. By lemma A.7.15 (ii), we have $ST = TS = I$.

(ii) Choose $k > \sup\Big\{\, \beta(T_n) \,\Big|\, n \in \mathcal{I} \,\Big\}$. We shall show that

$$B_x := \bigcup \Big\{\, (T/k)^i x \,\Big|\, i \in \mathbb{N} \,\Big\}$$

is bounded in E_∞, for every $x \in E_\infty$. Now let $x \in E_\infty$ and note that $\mu_n \circ (T/k)^i = (T_n/k)^i \circ \mu_n$ for $n \in \mathcal{I}$ and $i \in \mathbb{N}$. By hypothesis and our choice of k, the set $B_n := \bigcup \Big\{\, (T_n/k)^i \circ \mu_n(x) \,\Big|\, i \in \mathbb{N} \,\Big\}$ is bounded in E_n for every $n \in \mathbb{N}$. Thus, $\mu_n(B_x) = B_n$ is bounded for every $n \in \mathcal{I}$, which by [Sch, I.5.5] implies that B_x is bounded in E_∞.

QED

The envisaged theorem will be stated for a *countable projective system of Fréchet spaces*, that is, a projective system $\{E_n; \mu_{mn}\}$ over $\mathcal{I}$ such that $\mathcal{I}$ is countable and each E_n is a Fréchet space, since then the associated projective limit is a Fréchet space as well (see remark A.7.14).

A.7.19 Theorem *Let $\{E_n; \mu_{mn}\}$ be a countable projective system of Fréchet spaces. Suppose that $\{T_n\} : \{E_n; \mu_{mn}\} \to \{E_n; \mu_{mn}\}$ is a map of the projective system into itself. Then:*

(i) $\sigma'(\underset{\longleftarrow}{\mathrm{Proj}}\, T_n) \subset \liminf_{n\in\mathcal{I}} \sigma'(T_n)$;

(ii) $\sigma(\underset{\longleftarrow}{\mathrm{Proj}}\, T_n) \subset \mathrm{clo}\,(\liminf_{n\in\mathcal{I}} \sigma(T_n))$.

The *proof* is — *mutatis mutandis* — the same as the proof of theorem A.7.10.

Finally, we shall give sufficient conditions for equality in (ii) above. These conditions rely on the concept of the 'essential spectrum' of a continuous operator on a Banach sapce. Various definitions of the essential spectrum of an operator on a Banach space have occurred in the literature (see [Gus2]), and, in general, they are not equivalent (see [Gus1]). Fortunately, the radius of the essential spectrum is the same for most of the standard definitions (see [EE, I.4]). We shall use the *Browder-essential spectrum* for a continuous T acting on a Banach space, which is given by

$$\sigma_e(T) := \left\{ \lambda \in \sigma(T) \,\middle|\, \lambda \notin \{\text{poles of } T \text{ with finite rank residue}\} \right\}.$$

Accordingly, the radius of the essential spectrum $r_e(T)$ is defined as

$$r_e(T) := \sup \left\{ |\lambda| \,\middle|\, \lambda \in \sigma_e(T) \right\}.$$

Obviously, every spectral point of T not contained in the closed disk of radius $r_e(T)$ is an isolated eigenvalue of finite algebraic multiplicity.

A.7.20 Proposition *Let $\{E_n; \mu_{mn}\}$ be a countable injective projective system of Banach spaces such that $E_\infty = \bigcap \left\{ E_n \,\middle|\, n \in \mathbb{N} \right\}$ is dense in each E_n, and let $\{T_n\}$ be a map of the projective system. If the radius of the essential spectrum $r_e(T_n)$ shrinks to zero as $n \to \infty$ then, with $T_\infty := \underset{\longleftarrow}{\mathrm{Proj}}\, T_n$*

$$\liminf_{n\in\mathbb{N}} \sigma(T_n) = \bigcap_{n\in\mathbb{N}} \sigma(T_n) = \sigma(T_\infty). \tag{A.28}$$

Moreover, $z \mapsto (I - zT_\infty)^{-1}$ is meromorphic in $\mathbb{C}$ with each residue being a finite rank operator.

Proof As a first step we shall show that

$$\bigcap_{n\leq k} \sigma(T_k) = \bigcap_{k\in\mathbb{N}} \sigma(T_k) \text{ for any } n \in \mathbb{N}. \tag{A.29}$$

Let $n \in \mathbb{N}$ and let $\lambda \in \bigcap_{n\leq k} \sigma(T_k)$. Since $r_e(T_k) \to 0$, there is m with $n \leq m$ such that $\{\lambda\}$ is a spectral set of $\sigma(T_m)$. Since, by hypothesis, the connecting maps are injective an application of the intertwining theorem A.6.2 shows that $\lambda \in \sigma(T_k)$ for $k \leq m$. Thus $\bigcap_{n\leq k} \sigma(T_k) \subset \bigcap_{k\in\mathbb{N}} \sigma(T_k)$. This proves (A.29), as the opposite inclusion is trivially satisfied.

Next we show that

$$\bigcap_{k\in\mathbb{N}} \sigma(T_k) \subset \sigma(T_\infty). \tag{A.30}$$

If $\lambda \in \bigcap_{k\in\mathbb{N}} \sigma(T_k)$, then $\{\lambda\}$ is a spectral set of $\sigma(T_m)$ for m large enough. Since the canonical map μ_m has dense range, we have $\lambda \in \sigma(T_\infty)$, again by theorem A.6.2. Now, (A.28) follows from (A.29) and (A.30) by virtue of the previous theorem.

In order to prove the last assertion let $\lambda \in \bigcap_{n\in\mathbb{N}} \sigma(T_n)$. Then there is m such that $\lambda \in \sigma(T_m)$ is a pole of T_m with finite rank residue. Thus, λ is isolated in $\sigma(T_\infty)$; moreover, since the canonical map μ_m is injective, λ is a pole of T_∞ with finite rank residue by proposition A.6.5

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Appendix B

The $\mathcal{Z}$-transform

The $\mathcal{Z}$-transform is a mapping which sends a function $T : \mathbb{N}_0 \to F$ (that is, a function defined on the non-negative integers taking values in some space F), or equivalently a sequence $\{T_n\}$ of objects belonging to F, to a function $\tilde{T} : D \to F$ defined on a subset D of the complex plane. The main reason to introduce this transform is that certain operations which are difficult to handle in the domain of sequences are transformed into much simpler operations on the image of the sequence. In this respect, the $\mathcal{Z}$-transform is the discrete time analogue of the Laplace-transform. In this appendix we shall show how the classical $\mathcal{Z}$-transform of sequences of complex numbers (see, for example, [Jur, Mut]) can be extended to deal with sequences of operators on locally convex spaces.

We shall assume that E is a sequentially complete and barrelled space. We also assume that $\mathfrak{S}$ is admissible for E so that $\mathcal{L}_{\mathfrak{S}}(E)$ is a locally convex algebra. The symbol P will denote a directed family of seminorms generating the topology of $\mathcal{L}_{\mathfrak{S}}(E)$.

Suppose now that $\left\{ T_n \in \mathcal{L}(E) \,\middle|\, n \in \mathbb{N}_0 \right\}$ is a sequence of operators on E, and define the following quantities:

$$\begin{aligned}
\beta[T_n] &:= \inf\left\{ r > 0 \,\middle|\, \{r^{-n}T_n\}_{n\in\mathbb{N}_0} \text{ is bounded in } \mathcal{L}_{\mathfrak{S}}(E) \right\};\\
\beta'[T_n] &:= \sup\left\{ \limsup_{n\to\infty} |\langle T_n x, x'\rangle|^{1/n} \,\middle|\, x \in E, x' \in E' \right\};\\
\beta''[T_n] &:= \sup\left\{ \limsup_{n\to\infty} p(T_n)^{1/n} \,\middle|\, p \in P \right\}.
\end{aligned}$$

These quantities, which were introduced by Allan [All] for the case where

$T_n := T^n$, $n \in \mathbb{N}_0$, provide a generalisation of the Cauchy-Hadamard formula for the radius of convergence of power series.

B.0.1 Lemma *Let E and $\{T_n\}$ be as above. Then*

$$\beta[T_n] = \beta'[T_n] = \beta''[T_n].$$

Proof The proof is essentially the same as the proof of proposition 2.18 in [All].

QED

The sequence $\{T_n\}$ is called *regular* if $\beta[T_n]$ is finite. With a given regular sequence of operators we may associate a holomorphic operator-valued function by virtue of the following theorem.

B.0.2 Theorem *Let $\left\{ T_n \in \mathcal{L}(E) \,\middle|\, n \in \mathbb{N}_0 \right\}$ be a regular sequence. Then the function*

$$\tilde{T} : A \to \mathcal{L}_{\mathfrak{S}}(E)$$
$$\tilde{T}(z) := \sum_{n=0}^{\infty} z^{-n} T_n$$

is holomorphic on the annulus $A := \left\{ z \in \mathbb{C} \,\middle|\, |z| > \beta[T_n] \right\}$.

Proof We shall prove the theorem in two steps. First we show that $\tilde{T}(z)$ defined above belongs to $\mathcal{L}(E)$ for every $z \in A$. Next we prove that $\tilde{T}$ is holomorphic on A.

Let $z \in A$. Since $\mathcal{L}_{\mathfrak{S}}(E)$ is sequentially complete (see [Sch, III.4.4]) and

$$\sum_{n=0}^{\infty} |z|^{-n} \, p(T_n) < \infty \text{ for } p \in P,$$

$\tilde{T}(z) = \sum_{n=0}^{\infty} z^{-n} T_n$ defines a continuous linear transformation. This finishes the first part of the proof.

In order to show that $\tilde{T}$ is holomorphic on A we observe that by the previous lemma the scalar-valued function

$$z \mapsto \left\langle \tilde{T}(z)x, x' \right\rangle = \sum_{n=0}^{\infty} z^{-n} \left\langle T_n x, x' \right\rangle$$

is holomorphic on A for every $x \in E$ and $x' \in E'$.

Thus, $\tilde{T}$ is holomorphic on A with respect to the weak topology of $\mathcal{L}_s(E)$. This, however, implies that $\tilde{T}$ is holomorphic on A with respect to the b-topology of $\mathcal{L}(E)$ (see, for example, [Uss]).

QED

The holomorphic function $\tilde{T}$ given by the theorem is called the *$\mathcal{Z}$-transform* of the (regular) sequence $\{T_n\}$. We write

$$\mathcal{Z}[T_n] := \tilde{T}$$

to denote the $\mathcal{Z}$-transform of the sequence. The $\mathcal{Z}$-transform is an operator-valued Laurent series at the origin. Consequently, the $\mathcal{Z}$-transform $\tilde{T}$ is uniquely determined by the series $\{T_n\}$ and *vice versa* (see the inversion formula below).

We note that, if the sequence $\{T_n\}$ is defined by

$$T_n = T^n \text{ for } n \in \mathbb{N}_0,$$

for some $T \in \mathcal{L}(E)$, then $\{T_n\}$ is regular if and only if T is a regular element of the algebra $\mathcal{L}_{\mathfrak{S}}(E)$ and $\mathcal{Z}[T_n]$ is up to a factor z identical with the Neumann series of the resolvent of T:

$$\mathcal{Z}[T^n](z) = \sum_{n=0}^{\infty} z^{-n} T^n = z(z-T)^{-1}.$$

In this case $\beta[T_n]$ is equal to the spectral radius of T.

B.0.3 Theorem (Inversion formula) *If $\tilde{T} : A \to \mathcal{L}_{\mathfrak{S}}(E)$ is holomorphic on the annulus $A := \left\{ z \,\middle|\, |z| > r \right\}$, then there is a regular sequence $\left\{ T_n \in \mathcal{L}_{\mathfrak{S}}(E) \,\middle|\, n \in \mathbb{N}_0 \right\}$ such that*

$$\mathcal{Z}[T_n] = \tilde{T}.$$

This sequence is given by

$$T_n := \frac{1}{2\pi i} \oint_{\mathcal{C}} \zeta^{n-1} \tilde{T}(\zeta)\, d\zeta \text{ for } n \in \mathbb{N}_0,$$

where the contour $\mathcal{C}$ can be chosen to be any circle $|\zeta| = r'$ with $r' > r$.

Proof This is simply the expression for the coefficients of a Laurent series at the origin.

QED

In the following theorems, which are particularly useful for handling difference equations, $\{T_n\}$ and $\{S_n\}$ denote regular sequences of operators in $\mathcal{L}_{\mathfrak{S}}(E)$, and $\tilde{T}$ and $\tilde{S}$ the corresponding $\mathcal{Z}$-transforms. Furthermore, R denotes an operator on E.

B.0.4 Theorem (Linearity) *The sequences* $\{T_n + S_n\}$, $\{RT_n\}$ *and* $\{T_nR\}$ *are regular and the following relations hold:*

$$\mathcal{Z}[T_n + S_n] = \mathcal{Z}[T_n] + [S_n];$$

$$\mathcal{Z}[RT_n] = R\mathcal{Z}[T_n];$$

$$\mathcal{Z}[T_nR] = \mathcal{Z}[T_n]R.$$

Proof The assertions concerning the sum of the sequences are obvious. The other assertions follow from the fact that multiplication is separately continuous in the algebra $\mathcal{L}_{\mathfrak{S}}(E)$.

QED

B.0.5 Theorem (Shifting Theorem) *If* $k \in \mathbb{N}_0$, *then*

$$\mathcal{Z}[T_{n+k}](z) = z^k \left(\tilde{T}(z) - \sum_{n=0}^{k-1} z^{-n} T_n \right).$$

Proof This follows from

$$\begin{aligned} \sum_{n=0}^{\infty} z^{-n} T_{n+k} &= z^k \sum_{n=0}^{\infty} z^{-(n+k)} T_{n+k} \\ &= z^k \left(\sum_{n=0}^{\infty} z^{-n} T_n - \sum_{n=0}^{k-1} z^{-n} T_n \right). \end{aligned}$$

QED

B.0.6 Theorem (Convolution Theorem) *The convolution* $\{(T * S)_n\}$ *of the regular sequences* $\{T_n\}$ *and* $\{S_n\}$, *which is the sequence defined by*

$$(T * S)_n := \sum_{k=0}^{n} T_k S_{n-k} \text{ for } n \in \mathbb{N}_0,$$

is regular and

$$\mathcal{Z}[(T * S)_n] = \tilde{T}\tilde{S}.$$

Proof The only difficult part of the proof is to show that $\{(T * S)_n\}$ is regular. Once this is established, the $\mathcal{Z}$-transform of the convolution exists and satisfies

$$\begin{aligned}\tilde{T}(z)\tilde{S}(z) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} z^{-(n+m)}T_n S_m \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} z^{-m}T_n S_{m-n} = \mathcal{Z}[(T * S)_n](z).\end{aligned}$$

We shall now prove that $\{(T * S)_n\}$ is regular in $\mathcal{L}_s(E)$. Choose positive real numbers r and r' with $r > r' > \max\{\beta[T_n], \beta[S_n]\}$. Suppose now that $x \in E$ and that q is a continuous seminorm on E. Since $\{S_n\}$ is regular in $\mathcal{L}_s(E)$ the set $B := \{r^{-n}S_n x\}$ is bounded in E. Since $\{T_n\}$ is regular in $\mathcal{L}_b(E)$ there is $M > 0$ such that

$$\sup_{y\in B} q(T_k y) \leq M(r')^k \text{ for every } k \in \mathbb{N}_0.$$

Thus, we have for every $n \in \mathbb{N}_0$

$$\begin{aligned} q(r^{-n}\sum_{k=0}^{n} T_k S_{n-k}x) &\leq \sum_{k=0}^{n} r^{-k} q(T_k(r^{-(n-k)}S_{n-k}x)) \\ &\leq M\sum_{k=0}^{n}(r'/r)^k \leq M\frac{r}{r-r'}.\end{aligned}$$

This implies that the set $\{r^{-n}(T * S)_n\}$ is bounded in $\mathcal{L}_s(E)$ and hence that $\{(T * S)_n\}$ is regular.

QED

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