On the Discrete Time Version of the Brussels Formalism

Oscar F Bandtlow

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Preface

The work described in this thesis was carried out in the TCM (Theory of Condensed Matter) group at the Cavendish Laboratory, Cambridge between October 1991 and September 1992. Except where explicitly stated, this thesis contains only results of my own work and includes nothing done in collaboration with others. The work has not been submitted for a degree or a diploma or any other degree at this or any other university.

I would particularly like to thank my supervisor Dr Peter Coveney for introducing me to the problems studied here, for his patience and guidance—and for proof-reading this thesis.

This work would not have been possible without the help of several other people at the Cavendish and other places. I am grateful to the *Freistaat Bayern* whose financial assistance made this year in Cambridge possible.

Finally, I thank my mother for her support in ways too numerous to mention.

Oscar F Bandtlow

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‘ Wie fang ich nach der Regel an?—
Ihr stellt sie selbst und folgt ihr dann.’
Richard Wagner, die Meistersinger

Introduction

One of the oldest and certainly most challenging problems in non-equilibrium statistical mechanics, not to say its *raison d'être*, is the reconciliation of the irreversible *macroscopic* laws governing the behaviour of matter in bulk with the basic *microscopic* laws of dynamics. A fertile branch in this endeavour was initiated by Ludwig Boltzmann, whose search for the origin of the second law of thermodynamics led him to a characterization of dynamical processes in many-body systems by a *kinetic equation*, which is now referred to as the Boltzmann equation. Since then numerous such equations have been derived; the Fokker-Planck equation for a Brownian particle, the Vlasov equation for a plasma or the diffusion equation, to name just a few. Their common feature is that they describe the dynamics of single particle distributions valid under certain *physical conditions*; the Boltzmann equation for example gives a correct description only for dilute gases. The above mentioned physical conditions are usually understood in terms of a limiting process for which an appropriately chosen scaling parameter of the system vanishes (see e.g. [Spo] and references therein)

It was in the late 1960's and early 1970's that a group working in Brussels, primarily I Prigogine, C George, R Balescu, F Henin and L Rosenfeld launched a formalism to dispose of this limiting procedure, i.e. to provide a means of derivation of kinetic equations valid over some finite range of values of the relevant parameter. *In nuce* their method is based on the construction of an idempotent operator Π, which commutes with the Liouville operator of the system. In other words: Π projects onto a subspace which is invariant under the Liouvillian. It is for this reason that the formalism is also referred to as *subdynamics*. Moreover, the elements of the subspace can be shown to obey an autonomous evolution equation, which is the desired kinetic equation.

Although this does not do justice to the sophistication of the approach it certainly represents the key aspects of the theory, as it stood in 1975. Details may be found in the book by Balescu [Bal] or in the original articles, e.g. [PGH], [GPR], [PGHR], [GGG]. The theory was later generalized to systems with a time-dependent Liouvillian, such as to include open systems (c.f. [BM], [Jow], [Cov] and references therein).

A new strand in this thinking emerged in the late 1980's, when the group meanwhile equipped with a further support leg in Austin (Texas) drew its attention to the study of so called *large Poincaré systems*, i.e. a special class of non-integrable systems, which are characterized by a continuous spectrum (c.f. [PH], [PPHT], [PPT]). While emphasis has shifted from the derivation of kinetic equations to obtaining a new spectral representation of the Liouvillian, such as to include the possibility of complex eigenvalues indicating decay, i.e. approach to equilibrium, the construction of the idempotent Π or rather a collection $\{\Pi^{(i)}\}_{i\in I}$ of them satisfying

$$\sum_{i\in I}^{p} \Pi^{(i)} = 1$$

$$\Pi^{(i)}\Pi^{(j)} = \delta_{ij}\Pi^{(i)}$$
$$L\Pi^{(i)} = \Pi^{(i)}L,$$

where L is the Liouvillian of the system, is still a central concern.

Unfortunately, the mathematical details of the formalism are so formidable that, despite of the overwhelming number of publications of the school, not many rigorous results have been obtained. The study by Lanz, Lugiato, and Ramella [LLR], who discussed the formalism in Liouville space, i.e. the space of Hilbert-Schmidt operators, should be mentioned here as well as Kummer's paper [Kum] for the more general Banach space case; Coveney and Penrose [CP] have only recently given conditions under which at least a part of the formalism rigorously holds in an arbitrary Hilbert space.

Another new development was put forward by Hasegawa and Saphir in a series of papers [HS1-5], who adopted the formalism to the investigation of chaotic mappings. In this thesis we will take up the idea of a *discrete time Brussels formalism.* The first chapter is devoted to a general description of disrete time dynamical systems and their evolution operators in Banach spaces. In a second chapter we will derive the discrete time analogue of the generalized master equation leading to the Brussels decomposition of the resolvent of the evolution operator used by Hasegawa and Saphir. In taking advantage of the fact that in the discrete time scenario a resolvent-like formalism with only bounded operators can be developed, and restricting the class of observables to compact operators, we will be able to solve the problems of the discrete time Brussels formalism: the main features of the continuous time formalism will be recovered (Theorem 2) and, in particular, a complete description of the class of evolution operators for which the formalism holds will be given (Theorem 1). Finally we hope to show how little is achieved when these problems are solved.

Chapter 1

Dynamical systems and their evolution operators

1.1 Discrete Time Dynamical Systems

The notion of a dynamical system naturally emerges from classical mechanics. There the evolution of a system represented by a point in phase space is governed by a one-parameter group of transformations which is volume preserving. A generalization should therefore include the following two aspects:

1. There is a space X and a family $\{T\}_{g \in G}$ of transformations of X with some structure on X, which is preserved by $\{T\}_{g \in G}$.
   The setting might for example be:

   (a) measure theoretic:
   X is a measure space and $T_g \colon X \to X, g \in G$ measurable transformations.

   (b) probabilistic:
   X is a probability space and $T_g \colon X \to X, g \in G$ measure preserving transformations[1].

   (c) topological:
   X is a topological space and $T_g \colon X \to X, g \in G$ continuous mappings.

[1] This is of course a special case of (a).

(d) or smooth:
X is a C^r-manifold and $T_g: X \to X, g \in G$ C^r-functions.

2. The family of transformations $\{T\}_{g\in G}$ constitutes a group action, i.e. G is a semigroup such that

$$T_{g_1} T_{g_2} = T_{g_1 g_2} \text{ for all } g_1, g_2 \in G. \tag{1.1}$$

We usually encounter two cases:

(a) Continuous time:
G is the additive semigroup of the positive or negative reals $(\mathbb{R}^{\pm}, +, 0)$, where $\{T_t\}_{n\in\mathbb{R}^{\pm}}$ is called a flow on X.

(b) Discrete time:
G is the additive semigroup of the positive or negative integers $(\mathbb{Z}^{\pm}, +, 0)$. Since for $n \in \mathbb{N}$

$$T_{n+1} = T_n T_1,$$

we have

$$T_n = T_1^n,$$

i.e. T_n is just the n-th 'iterate' of T_1, which generates the group.

If G is a *group*, then T_g is invertible for every $g \in G$; this describes a situation where there is no distinguished initial time, as the dynamics governing the evolution of the system are the same on $[0, t']_G$ and $[t, t+t']_G$: The system is *reversible*. If, however, G is a semigroup this invariance is broken: The system is *irreversible*. [2]

As we are interested in the possibility of a probabilistic interpretation, we will work with measure spaces and their morphisms, thereby keeping in touch with statistical mechanics and ergodic theory. Let us summarize and set the scenario with

[2] These two aspects are not independent, which is easily seen by rephrasing 1. in the language of the theory of categories: Let X be an object of a category; then the structure preserving transformations $T_g, g \in G$ belong to the set of endomorphisms of X, $T_g \in \mathrm{Hom}(X, X)$, which has a natural semigroup structure with unity Id_X.

Definition 1 A *discrete time dynamical system* is a quadruple $(X, \mathcal{A}, \mu, T)$, such that $(X, \mathcal{A}, \mu)$ is a normalized measure space, $\mathcal{A}$ denoting the σ-algebra of measurable subsets of X, μ being a measure on $\mathcal{A}$ with $\mu(X) = 1$, and $T: X \to X$ is a *measure preserving transformation*, i.e. $\mu(T^{-1}A) = \mu(A)$ for any $A \in \mathcal{A}$.

We will drop the epithet 'discrete time' in the sequel and sometimes refer to T as the dynamical system.

Remarks:

1. Given an arbitrary, not necessarily measure preserving transformation T on a measure space X, we may ask whether it is always possible to find an invariant measure; or in other words: can we turn any mapping into a dynamical system by adjusting the measure? This is generally a very difficult question (see [LM] for a survey). If, however, X is a compact metric space, then there always exists an invariant measure (see [CFS, Ch.1.8, Th.1] for details), which need not be unique. Existence and uniqueness proofs have, for example, been obtained for classes of transformations of the interval [0,1], amongst which are the piecewise monotonic Lasota-Yorke maps [LY], and a generalization studied by Hofbauer and Keller [HK].

2. Obviously we can always generate a discrete time system from a continuous time system in the following way:
   Fix $g_0 \in G$, then for $n \in \mathbb{N}$
   $$T_{ng_0} = T_{g_0} \circ \overset{n}{\cdots} \circ T_{g_0} = T_{g_0}^n.$$
   Another possibility is to introduce the notion of a mapping, which is (roughly speaking) effected by consecutive intersections of a trajectory with a 'surface of section', called a *Poincaré* or *first return map*. It is possible that an investigation of the discrete time system may yield some partial information about the continuous time system from which it was derived.
   The converse is not true: It is in general impossible to embed a discrete time system in a continuous time system without altering the phase space (see [Zdu]); a famous example is the quadratic map $T(x) = 4x(1-x), x \in [0, 1]$.

1.2 The Frobenius-Perron Operator

In statistical mechanics *densities* describe relevant aspects of dynamics. Rather than investigating the evolution of an individual object one considers a collection of them, the so called *ensemble* introduced by JW Gibbs. Given a dynamical system $(X, \mathcal{A}, \mu, T)$ we introduce a second measure ν on $\mathcal{A}$ absolutely continuous w.r.t. μ which specifies the weight to be ascribed to the presence of the object in any $A \in \mathcal{A}$. Thus, by the Radon-Nikodým theorem (see [Hal, §31, theorem B])

$$d\nu = f\, d\mu,$$

where $f \in L^1(X, \mathcal{A}, \mu)$ is a density, which we now treat as the fundamental object of our theory.

Before we go into details we review some properties of L^p spaces. Let $(X, \mathcal{A}, \mu)$ be a measure space and p a real number with $1 \leq p < \infty$. The family of all possible real-valued measurable functions $f\colon X \to R$ satisfying

$$\int_X |f(x)|^p\, d\mu < \infty$$

is the $L^p(X, \mathcal{A}, \mu)$ space. We shall suppress the lengthy notation $L^p(X, \mathcal{A}, \mu)$ and write L^p, when confusion is impossible. L^p is a *Banach* space together with the norm

$$\|f\|_{L^p} = \left[\int_X |f(x)|^p\, d\mu\right]^{1/p}.$$

It is well known that the *dual* space of L^p, i.e. the space of all continuous linear functionals on L^p, is L^q with $(1/p) + (1/q) = 1$, in the sense that every continuous linear functional l_g, $g \in L^q$ can be written as

$$l_g(f) = \int_X f(x) g(x)\, d\mu =: \langle f, g\rangle. \tag{1.2}$$

The dual of L^1 is by definition the family of all measurable functions f, such that $|f(x)| \leq M$ μ-a.e. for some $M < \infty$ and is denoted by L^∞ which becomes a Banach space if we take the norm to be the smallest such M.

We now try to reformulate the evolution of a dynamical system $(X, \mathcal{A}, \mu, T)$ in terms of transformations of densities. Consider to this end a random variable x with density f; we want to calculate the density f' of $T(x)$. Conservation of probability requires

$$\int_A f'(x)\, d\mu = \int_{T^{-1}A} f(x)\, d\mu \text{ for all events } A \in \mathcal{A}.$$

This indeed suffices to define an operator:

Definition 2 Let $(X, \mathcal{A}, \mu, T)$ be a discrete time dynamical system. The unique operator $U\colon L^1 \to L^1$ defined by

$$\int_A Uf(x)\, d\mu = \int_{T^{-1}A} f(x)\, d\mu \text{ for } A \in \mathcal{A}$$

is called the *Frobenius-Perron operator* associated with T and has the following properties:

(FP1) $U(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 U f_1 + \lambda_2 U f_2$ for all $f_1, f_2 \in L^1, \lambda_1, \lambda_2 \in \mathbb{R}$.

(FP2) $Uf \geq 0$ if $f \geq 0$

(FP3) $\int_X Uf(x)\, d\mu = \int_X f(x)\, d\mu$

(FP4) $\|Uf\|_{L^1} \leq \|f\|_{L^1}$

Proof Uniqueness is a consequence of the Radon-Nikodým theorem. Let $f \in L^1$, $f \geq 0$ and $A \in \mathcal{A}$. Since T is measure preserving $\int_{T^{-1}A} f(x)\, d\mu$ defines a finite measure absolutely continuous w.r.t. μ. Thus, by the Radon-Nikodým theorem there is a unique element in L^1, which we denote by Uf, such that

$$\int_A Uf(x)\, d\mu = \int_{T^{-1}A} f(x)\, d\mu \tag{1.3}$$

For $f \in L^1$ arbitrary set $f = f^+ - f^-$, where

$$f^+(x) := \begin{cases} f(x) & \text{for } x \text{ with } f(x) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define $Uf = Uf^+ - Uf^-$ and carry through the above procedure.
The linearity of U (FP1) is a consequence of the linearity of the Lebesgue integral, properties (FP2) and (FP3) obvious by construction and

$$\|Uf\|_{L^1} = \int_X |Uf(x)|\, d\mu \leq \int_X |Uf^+(x)|\, d\mu + \int_X |Uf^-|\, d\mu = \|f\|_{L^1},$$

which proves (FB4).

QED

Remarks

1. If U_n is the Frobenius-Perron operator associated with T^n, then $U_n = U^n$, where U is the Frobenius-Perron operator associated with T.

2. (FB2) and (FB3) guarantee that U takes densities into densities. Such operators are also called *Markov* operators and play an important rôle in ergodic theory. The set of all densities D, however, is not a linear space and we therefore investigate U on L^1.

3. If T is a piecewise invertible system on a compact manifold, then U can written in a more familiar form:

$$(Uf)(x) = \sum_{y:T(x)=y} \frac{1}{|\det DT(y)|} f(y),$$

where DT denotes the derivative of T.

We close this section by introducing another operator closely related to the Frobenius-Perron operator.

Definition 2 Let $(X, \mathcal{A}, \mu, T)$ be a dynamical system. The operator $\tilde{U}\colon L^\infty \to L^\infty$ defined by

$$\tilde{U}f(x) = f(T(x))$$

is called the *Koopman operator* [3] associated with T and has the following properties:

(K1) $\tilde{U}(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \tilde{U} f_1 + \lambda_2 \tilde{U} f_2$ for all $f_1, f_2 \in L^\infty, \lambda_1, \lambda_2 \in \mathbb{R}$.

(K2) $\left\|\tilde{U}f\right\|_{L^\infty} = \|f\|_{L^\infty}$.

(K3) For every $f \in L^1, g \in L^\infty$ $\langle Uf, g\rangle = \langle f, \tilde{U}g\rangle$, i.e. $\tilde{U}$ is the adjoint of U.

Proof The Koopman operator is well defined, since T is measure preserving and therefore $f_1(x) = f_2(x)$ μ-a.e. implies $f_1(T(x)) = f_2(T(x))$ μ-a.e.

[3] This operator was first used by Koopman [Kop] in his pioneering work initiating the Hilbert space approach to classical mechanics.

Property (K1) is trivial to check. For (K2) observe that

$$|f(x)| \leq \|f\|_{L^\infty} \ \mu\text{-a.e. implies } |f(T(x))| \leq \|f\|_{L^\infty} \ \mu\text{-a.e.}$$

and

$$|f(T(x))| \leq \left\|\tilde{U}f\right\|_{L^\infty} \ \mu\text{-a.e. implies } |f(x)| \leq \left\|\tilde{U}f\right\|_{L^\infty} \ \mu\text{-a.e. },$$

since T is measure preserving. Hence from the definition of $\|.\|_{L^\infty}$ we have $\left\|\tilde{U}f\right\|_{L^\infty} \leq \|f\|_{L^\infty}$ and $\|f\|_{L^\infty} \leq \left\|\tilde{U}f\right\|_{L^\infty}$.

Finally note that it suffices to prove (K3) for a fundamental set of L^∞. It is well known that the characteristic functions $\chi_A, A \in \mathcal{A}$ form a fundamental set of L^∞. But with $g = \chi_A, A \in \mathcal{A}$

$$\begin{aligned} \langle f, \tilde{U}g\rangle &= \int_X f(x)\chi_A(T(x))\,d\mu = \int_{T^{-1}X} f(x)\,d\mu \\ &= \int_X Uf(x)\chi_A(x)\,d\mu = \langle Uf, g\rangle. \end{aligned} \tag{1.4}$$

QED

Remarks

1. For $f \in L^1 \cap L^\infty$ the action of $\tilde{U}$ on f admits a nice interpretation: $\tilde{U}f$ is the density which will evolve into f. To be precise: the Koopman operator is the right inverse of the Frobenius-Perron operator

$$U\tilde{U} = \mathrm{Id}_{L^1},$$

since for $A \in \mathcal{A}$

$$\int_A U\tilde{U}\,d\mu = \int_A Uf(T(x))\,d\mu = \int_{TA} Uf(x)\,d\mu = \int_A f(x)\,d\mu.$$

It is, however, not the left inverse in general; this asymmetry reflects the asymmetry of the underlying transformation group for non-invertible T. As one might guess, if T is invertible then

$$\tilde{U}U = \mathrm{Id}_{L^\infty}$$

2. The Frobenius-Perron operator can be defined on an arbitrary L^p space, the properties (FP1)–(FP4) being unchanged.

We conclude this discussion of evolution operators in Banach spaces associated with dynamical systems with a brief analysis of their spectral properties which will be of major importance in the sequel.

Recall that, given a bounded operator A on a Banach space, a complex number z is said to belong to the *resolvent set* $\varrho(A)$, if $\lambda\mathrm{Id} - A$ is a bijection. $R_z(A) = (zI - A)^{-1}$ is called the *resolvent* of A at z. If $z \notin \varrho(A)$, then z is said to belong to the *spectrum* $\sigma(A)$ of A. Finally the *spectral radius* $r(A)$ of A is defined by

$$r(A) := \sup\{|z| : z \in \sigma(A)\}.$$

Let $(X, \mathcal{A}, \mu, T)$ be an invertible dynamical system and let $U: L^2 \to L^2$ be its associated Frobenius-Perron operator on the *Hilbert* space L^2. Then the Koopman operator is the Hilbert space adjoint of U with

$$\tilde{U} = U^{-1}.$$

Therefore U is unitary and its spectrum is a subset of the unit-circle. What happens if T is not invertible? There is a remarkable answer:

Proposition 1 *Let $(X, \mathcal{A}, \mu, T)$ be a non-invertible dynamical system and $1 \leq p < \infty$ a real number. Then the spectrum of the associated Frobenius-Perron operator $U: L^p \to L^p$ is the whole unit-disc.*

Proof Let λ be a complex number with $|\lambda| < 1$ and define $Q_\lambda: L^p \to L^p$ by

$$Q_\lambda = \sum_{k=0}^{\infty} \lambda^k \tilde{U}^k (I - \tilde{U}U), \tag{1.5}$$

where $\tilde{U}$ denotes the Koopman operator. Then

$$UQ_\lambda = (U - U) + \sum_{k=1}^{\infty} \lambda^k \tilde{U}^{k-1}(I - \tilde{U}U) = \lambda Q_\lambda,$$

and

$$Q_\lambda^2 \;=\; \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \lambda^k \tilde{U}^k (I - \tilde{U}U) \lambda^l \tilde{U}^l (I - \tilde{U}U)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \lambda^k \tilde{U}^k (I - \tilde{U}U)(I - \tilde{U}U) + \\
&\quad + \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \lambda^{k+l} (\tilde{U}^{k+l} - \tilde{U}^{k+l} - \tilde{U}^{k+l+1}U + \tilde{U}^{k+l+1}U) \\
&= Q_\lambda
\end{aligned}
$$

Furthermore

$$\sum_{k=0}^{\infty} \lambda^k \tilde{U}^k f - \sum_{k=0}^{\infty} \lambda^{k+1} \tilde{U}^{k+1} f = \lambda f$$

and therefore

$$Q_\lambda f = \lambda f \text{ if and only if}$$

$$\sum_{k=0}^{\infty} \lambda^k \tilde{U}^{k+1} (Uf - \lambda f) = 0 \text{ if and only if}$$

$$Uf = \lambda f.$$

Hence Q_λ is a projection onto $\{f \in L^p : Uf = \lambda f\}$, and λ is an eigenvalue of U if and only if $Q_\lambda \neq 0$. Since T is not bijective, there is f with $0 \neq f - \tilde{U}Uf$ and therefore $\tilde{U}^k(f - \tilde{U}Uf) \neq 0$ for all $k \in \mathbb{N}$ and there are arbitrary large n such that

$$Q_\lambda^{(n)} f = \sum_{k=0}^{n-1} \lambda^k \tilde{U}^k (I - \tilde{U}U) f \neq 0.$$

Now $Q_\lambda f \neq 0$ follows from

$$Q_\lambda f = Q_\lambda^{(n)} f + \sum_{k=1}^{\infty} \lambda^{kn} \tilde{U}^{kn} Q_\lambda^{(n)} f$$

and

$$\left\| \sum_{k=1}^{\infty} \lambda^{kn} \tilde{U}^{kn} Q_\lambda^{(n)} f \right\|_{L^p} \leq \frac{|\lambda|^n}{1 - |\lambda|^n} \left\| Q_\lambda^{(n)} f \right\|_{L^p}.$$

Hence λ is an eigenvalue of U, and the spectrum of U is the whole unit-disk. (This is a slightly modified version of an argument by Keller [Kel1, §IV]).

QED

Chapter 2

The discrete time Brussels formalism

2.1 The discrete time master equation

One of the earliest attempts to generalize Boltzmann's kinetic equation to arbitrary systems was made by Pauli [Pau] who derived a master equation, i.e. a linear Markovian rate equation for the probability distibution of the system. His methods were later improved by van Hove [Hov1], [Hov2], while Prigogine and his collaborators [PR] arrived at an exact master equation for an arbitrary system. Similar equations were derived by Nakajima [Nak], Zwanzig [Zwa1], and Montroll [Mon]. Their equivalence was shown by Zwanzig [Zwa2].

To keep the discussion general we take a Banach space $(B, \|.\|)$ with norm $\|.\|$ as a state space and a bounded operator U with $r(U) \leq 1$ as generator of the dynamical group $\{U^n\}_{n\in\mathbb{N}}$. It does no harm imagining U to be a Frobenius-Perron operator on a L^p space; it will, however, become clear in the sequel that this is not important and that the only essential features are those stated above. We shall furthermore not deal with a perturbative approach which although omnipresent in most of the work of the Brussels school (for example the famous Friedrichs model [Fri]) would enter the discrete time theory rather unmotivated and artificially.

Master equatons make use of *reduced descriptions*. The rationale for this is to bring an observational element into the theory. The act of observation may be mimicked with a bounded projector P

$$P \in \mathcal{B}(B) \tag{2.1}$$

$$P^2 = P \tag{2.2}$$

The idempotency of P (2.2) accounts for the fact that repeated applications will not have any further effect. As observations carried out by human beings have to yield results in low dimensional spaces we may ask for the range of P to be finite dimensional. This condition may be weakened in our approach to PU being compact

$$PU \in \mathcal{B}_0(B). \tag{2.3}$$

The derivation of the discrete time master equation starts by introducing P into the difference equation of the dynamical semigroup

$$U_{n+1} = UU_n,$$

which leads to the following pair of equations:

$$PU_{n+1} = PUPU_n + PUQU_n \tag{2.4}$$

$$QU_{n+1} = QUPU_n + QUQU_n. \tag{2.5}$$

where we made the definition

$$Q := I - P.$$

The Brussels school refers to the P and Q subspaces as to the 'vacuum' and the 'correlations', since in the original formalism P was meant to project on the diagonal part of the density matrix of the system.

Although this pair of operator equations can be solved by iteration, it is much easier to use z-transform techniques, which are the discrete time analogue of Laplace transforms (the appendix should be consulted for details). It is not difficult to see that all the terms occurring in (2.4) and (2.5) are of geometric order owing to the submultiplicativity of the operator norm in a Banach space. We have e.g.

$$\|PUPU^n\| \leq \|P\|^2 \|U\|^{n+1}.$$

We therefore may apply a z-transform and get:

$$z(P\mathcal{U}(z) - PU_0) = PUP\mathcal{U}(z) + PUQ\mathcal{U}(z) \tag{2.6}$$

$$z(Q\mathcal{U}(z) - PU_0) = QUP\mathcal{U}(z) + QUQ\mathcal{U}(z), \tag{2.7}$$

where we used the shifting theorem and the definition

$$\mathcal{U}(z) := \mathcal{Z}[U_n] = \frac{z}{z - U}.$$

For $\mathcal{U}(z)$ to represent a z-transform of the $\{U_n\}_{n \in N}$ its domain needs to be restricted to $A(U) := \{z : |z| > r(U)\}$, the annulus of convergence. We will, however, assume $\mathcal{U}(z)$ to be the analytic continuation of $\mathcal{Z}[U_n]$, such as to have the resolvent set $\varrho(U)$ as the new domain of $\mathcal{U}(z)$.

For $z \in A(QUQ)$ we can solve for $Q\mathcal{U}(z)$ in (2.7)

$$Q\mathcal{U}(z) = \frac{z}{z - QUQ}Q + \frac{1}{z - QUQ}QUP\mathcal{U}(z), \tag{2.8}$$

where we used $U_0 = 1$. Inserting this into (2.6) yields

$$z(P\mathcal{U}(z) - P) = PUP\mathcal{U}(z) + PUQ\frac{z}{z - QUQ} + PUQ\frac{1}{z - QUQ}QUP\mathcal{U}(z) \tag{2.9}$$

The Brussels school coined suggestive names for the continuous time analogues of the operators in (2.8) and (2.9), which shall be adopted:

The *collision operator*

$$\tilde{\psi}(z) := PUQ\frac{1}{z - QUQ}QUP, \tag{2.10}$$

the *destruction operator*

$$\mathcal{D}(z) := PUQ\frac{1}{z - QUQ}, \tag{2.11}$$

the *creation operator*

$$\mathcal{C}(z) := \frac{1}{z - QUQ}QUP, \tag{2.12}$$

and the *reduced resolvent*

$$\mathcal{S}(z) := Q\frac{1}{z - QUQ}Q, \tag{2.13}$$

all of them being $\mathcal{B}(B)$-valued functions holomorphic in $A(QUQ)$. For the sake of completeness we list their n-domain representations, i.e. their image under an inverse z-transform:

$$\psi_n := \mathcal{Z}_n^{-1}[\tilde{\psi}(z)] = \begin{cases} PUQ(QUQ)^{n-1}QUP & n \geq 1 \\ 0 & n = 0 \end{cases} \tag{2.14}$$

$$D_n := \mathcal{Z}_n^{-1}[\mathcal{D}(z)] = \begin{cases} PUQ(QUQ)^{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases} \tag{2.15}$$

$$C_n := \mathcal{Z}_n^{-1}[\mathcal{C}(z)] = \begin{cases} (QUQ)^{n-1}QUP & n \geq 1 \\ 0 & n = 0 \end{cases} \tag{2.16}$$

$$S_n := \mathcal{Z}_n^{-1}[\mathcal{S}(z)] = \begin{cases} Q(QUQ)^{n-1}Q & n \geq 1 \\ 0 & n = 0 \end{cases} \tag{2.17}$$

The desired master equation may now be obtained from (2.9) *quā* inverse z-transform:

$$PU_{n+1} = PUPU_n + D_{n+1}Q + \psi_n * PU_n, \tag{2.18}$$

where '$*$' denotes the convolution of two sequences:

$$\psi_n * PU_n := \sum_{i=0}^{n} \psi_i PU_{n-i}.$$

The above equation is an operator relation, which acting on an initial density f_0 reads

$$Pf_{n+1} = PUPf_n + D_{n+1}Qf_0 + \psi_n * Pf_n. \tag{2.19}$$

We have arrived at an *exact* equation for the evolution of the reduced densities Pf_n. The middle term in (2.19) desribes the influence of initial data about Qf_0 at time $n = 0$ on the subsequent time evolution of the system, hence the name 'memory term' for it. If Q is appropriately chosen it can be assumed that this effect should disappear, which explains why $\mathcal{D}(z)$ is called the destruction operator. This equation is, however, nonlocal in time, i.e. non-Markovian, due to the summation in the collision term. Thus, to pass from this equation to a Markovian equation, we need to restrict the influence of ψ_n for large n. We will deal with this problem in the next section.

2.2 The Brussels class

In order to understand the Brussels reasoning we need to derive an expression for $\mathcal{U}(z)$ in terms of the operators previously introduced. The z-transformed master equation reads

$$z(P\mathcal{U}(z) - P) = PUP\mathcal{U}(z) + z\mathcal{D}(z) + \tilde{\psi}(z)P\mathcal{U}(z).$$

We can formally solve for $P\mathcal{U}(z)$

$$P\mathcal{U}(z) = \frac{z}{z - PUP - \tilde{\psi}(z)}[P + \mathcal{D}(z)],$$

which added to the equation for $Q\mathcal{U}(z)$ (2.8)

$$Q\mathcal{U}(z) = z\mathcal{S}(z) + \mathcal{C}(z)P\mathcal{U}(z)$$

yields the *Brussels decomposition* of $\mathcal{U}(z)$:

$$\begin{aligned} \mathcal{U}(z) &= [P + \mathcal{C}(z)]P\mathcal{U}(z) + z\mathcal{S}(z) \\ &= [P + \mathcal{C}(z)]\frac{z}{z - PUP - \tilde{\psi}(z)}[P + \mathcal{D}(z)] + z\mathcal{S}(z). \end{aligned} \tag{2.20}$$

For a justification of the manipulations involved so far the existence of $[z - PUP - \tilde{\psi}(z)]^{-1}$ has to be ensured. This is done in the following

Lemma 1 *Let U, P, and Q be defined as above. If $r(QUQ) \neq 0$, then $z/[z - PUP - \tilde{\psi}(z)]$ is*

1. *meromorphic in the annulus $A(QUQ)$ with only a finite number of poles $z_i, i \in I$*
2. *holomorphic at infinity, i.e. $(1/z)/[1/z - PUP - \tilde{\psi}(1/z)]$ is holomorphic at 0.*

Proof For $z \in A(QUQ)$ the operator $\frac{1}{z}(PUP - \tilde{\psi}(z))$ is holomorphic. Since the product of a compact operator and a bounded operator is compact $\frac{1}{z}(PUP - \tilde{\psi}(z))$ is also compact due to PU being compact (2.3). Furthermore $1 - \frac{1}{z}(PUP - \tilde{\psi}(z))$ is invertible for z large enough. This is easily seen by taking into account that $\left\|\tilde{\psi}(z)\right\| \leq (\|P\| \, \|U\| \, \|Q\|)^2 \, (|z| - \|QUQ\|)^{-1}$

becomes arbitrary small for z large, and hence $\left\| \frac{1}{z}(PUP - \tilde{\psi}(z)) \right\| < 1$ for z large enough. The first assertion now easily follows from the analytic Fredholm theorem (see [RS, Theorem VI.14] and [DS, VII.11]). For the proof of the second part let $z \in \{z : |z| < 1/r(QUQ)\}$. Then

$$z\tilde{\psi}(1/z) = \sum_{n=0}^{\infty} z^{n+2} PUQ(QUQ)^n QUP,$$

hence $[1 - zPUP - z\tilde{\psi}(1/z)]^{-1}$ holomorphic at 0. Using the same expansion it is possible to show $\lim_{z\to 0} \left\| zPUP - z\tilde{\psi}(1/z) \right\| = 0$ and therefore $[1 - zPUP - z\tilde{\psi}(1/z)]^{-1}$ invertible at $z = 0$. This completes the proof.

QED

Remark The same arguments may be used to prove a slightly extended version of the lemma, in which the annulus $A(QUQ)$ is replaced by an arbitrary *connected* open subset of $\varrho(QUQ)$.

With the help of this lemma we are now able to derive a new expression for U_n:

$$U_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} z^n \left\{ (P + \mathcal{C}(z)) \frac{1}{z - PUP - \tilde{\psi}(z)} (P + \mathcal{D}(z)) + \mathcal{S}(z) \right\} dz.$$

The contour $\mathcal{C}$ has to enclose the poles $z_i, i \in I$ as well as $\sigma(QUQ)$. As the integrand is meromorphic in $A(QUQ)$ we can deform $\mathcal{C}$ such as to separate the contributions from the poles resulting in a splitting of the integral :

$$U_n = \Sigma_n^{(1)} + \cdots + \Sigma_n^{(p)} + \hat{\Sigma}_n$$

with

$$\Sigma_n^{(i)} := \frac{1}{2\pi i} \oint_{\mathcal{C}_i} z^n \left\{ [P + \mathcal{C}(z)] \frac{1}{z - PUP - \tilde{\psi}(z)} [P + \mathcal{D}(z)] \right\} dz \text{ for } 1 \le i \le p$$

$$\hat{\Sigma}_n := \frac{1}{2\pi i} \oint_{\mathcal{C}'} z^n \left\{ (P + \mathcal{C}(z)) \frac{1}{z - PUP - \tilde{\psi}(z)} (P + \mathcal{D}(z)) + \mathcal{S}(z) \right\} dz$$

and $p = \text{card}I$ being the number of poles in $A(QUQ)$; the new contours $\mathcal{C}_i$ enclose the poles z_i only and $\mathcal{C}'$ is a circle around the origin with radius $r(QUQ) + \epsilon$ with $\epsilon > 0$ small enough.

The operators $\Sigma_n^{(i)}$ are the so called *asymptotic evolution operators*, which play an important rôle in the Brussels approach and will be studied in some detail later on. For the moment we only remark that they are supposed to describe the dominant long time behaviour of U_n, which is, as should have become clear by now, associated with the poles z_i.

It may happen that there are no poles at all. If there is at least one, and this is obviously important for the formalism to make sense at all, then as a consequence of the lemma its modulus needs to be strictly greater than the spectral radius of QUQ and hence

$$r(QUQ) < 1, \tag{2.21}$$

since a pole with modulus greater than one would result in an expansion of U_n which is impossible to occur as $r(U) = 1$. This is the famous *hypothesis of rapid decay of correlations*. It is as a *necessary* condition for the existence of a pole in $[z - PUP - \tilde{\psi}(z)]^{-1}$ and therefore for the existence of the theory. If we deal with contractive operators ($r(U) < 1$) condition (2.21) has to be replaced by

$$r(QUQ) < r(U).$$

We cast these observations into the

Definition Let $(B, \|.\|)$ be a Banach space. An operator $U \in \mathcal{B}(B)$ is said to belong to the *Brussels class* $Br(B)$ of B if there is a projector $Q \in \mathcal{B}(B)$ such that $(I - Q)U$ is compact and $r(QUQ) < r(U)$.

An explicit characterization of the Brussels class will be given in the next section.

2.3 The Brussels class as the space of quasi-compact operators

To give a characterization of the Brussels class put forward in the previous section it will be inevitable to provide some tools from functional analysis.

Let $(B, \|.\|)$ be a Banach space. Let $\mathcal{B}(B)$ denote the algebra of bounded operators on B and $\mathcal{B}_0(B)$ the ideal of compact operators on B. For $A \in \mathcal{B}(B)$ an element $z \in \sigma(A)$ is said to belong to the *essential spectrum* $\sigma_{ess}(A)$ if one or more of the following is true

(i) $\text{Ran}(z - A)$ is not closed in B

(ii) z is a limit point of $\sigma(A)$

(iii) $\bigcup_{r=1}^{\infty} \text{Ker}(z - A)^r$ is finite dimensional,

where $\text{Ker}A = \{f \in B : Af = 0\}$ denotes the kernel of A and $\text{Ran}A = \{Af : f \in B\}$ the range of A.

In particular $\sigma_{ess}(A)$ does not contain isolated eigenvalues of finite multiplicity. By analogy with the spectral radius of A the *essential spectral radius* $r_{ess}(A)$ is defined to be

$$r_{ess}(A) = \sup\{|z| : z \in \sigma_{ess}(A)\}.$$

Nussbaum [Nus] showed that there is an *essential spectral radius formula* given by

$$r_{ess}(A) := \lim_{n\to\infty} \left(\inf\{\|A^n - K\| : K \in \mathcal{B}_0(B)\}\right)^{1/n}$$

This motivates a definition for a new class of operators given by Keller [Kel2]:

Definition A bounded operator U is said to be *quasicompact* if there is a $k \in \mathbb{N}$ and a compact operator $K \in \mathcal{B}_0(B)$ such that

$$\left\|U^k - K\right\| < r(U)^k.$$

The space of all quasicompact operators on B will be denoted by $\mathcal{B}_1(B)$.

Roughly speaking quasicompact operators are characterized by their approximability by compact operators. The main purpose for their appearance in this context is that their spectra are of a particular structure outlined in the

Proposition 2 *Let U be a quasicompact operator. Then for every $0 < \epsilon \leq (r(U) - r_{ess}(U))$ the set $\sigma_\epsilon(U) := \sigma(U) \cap \{z : |z| \geq r_{ess}(U) + \epsilon\}$ is not empty and consists of a finite set of eigenvalues with finite multiplicity.*

Proof Note that it is sufficient to prove $r_{ess}(U) < r(U)$; because then $\sigma_\epsilon(U)$ has got the desired properties by the definition of $r_{ess}(U)$, the finiteness of the number of eigenvalues being a consequence of the compactness of $\sigma_\epsilon(U)$.

Since $r_{ess}(U) \leq r(U)$ in general we only need to show $r_{ess} \neq r(U)$, or equivalently: There is $\epsilon > 0$ and $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ there are compact operators K_n with

$$\|U^n - K_n\|^{1/n} \leq r(U) - \epsilon.$$

Since U is quasicompact there is $k \in \mathbb{N}$ and $K \in \mathcal{B}_0(B)$ with $\left\|U^k - K\right\| < r(U)^k$. Now let ϵ be such that $\left\|U^k - K\right\| = (r(U) - \epsilon)^k$ and define $K_n := U^{nk} - (U^k - K)^n$ for $n \in \mathbb{N}$. It is not difficult to see that K_n is compact and

$$\left\|U^{nk} - K_n\right\|^{1/nk} \leq \left\|U^k - K\right\|^{1/k} = r(U) - \epsilon$$

for $n \in \mathbb{N}$.

QED

We are now able to prove our main result which constitutes a complete description of the Brussels class.

Theorem 1 *A bounded operator on a Banach space belongs to the Brussels class if and only if it is quasicompact:*

$$Br(B) = \mathcal{B}_1(B).$$

Proof '$\Rightarrow$' Let U belong to the Brussels class. Then there is a projector $Q \in \mathcal{B}(B)$ such that $r(QUQ) < r(U)$ and $P := I - Q$ with $PU \in \mathcal{B}_0(B)$. We show by induction on n:

$$U^n - (QUQ)^{n-1}U \text{ is compact for all } n \in \mathbb{N}$$

For $n = 1$ this is trivial. Assume (2.3) holds for n, then

$$\begin{aligned} & U^{n+1} - (QUQ)^n U = \\ = \; & PUU^n + QUU^n - QU(QUQ)^{n-1}U + QUP(QUQ)^{n-1}U \\ = \; & PUU^n + QUP(QUQ)^{n-1}U + QU(U^n - (QUQ)^{n-1}U). \end{aligned}$$

The first term in this sum is compact since PU is compact. The second term is only different from 0 for $n = 1$, in which case its compacity follows from that of PU, whereas the last term is compact by the induction assumption. That U is quasicompact now follows from

$$\lim_{n\to\infty} \|(QUQ)^{n-1}U)\|)^{1/n} = r(QUQ) < r(U).$$

'$\Leftarrow$' Let U be quasicompact and $\epsilon > 0$. Choose P to be the projector onto the eigenspaces of the eigenvalues of U in $\sigma_\epsilon(U)$. Then by the lemma P is a finite rank operator , which implies that PU is compact. The inequality $r(QUQ) < r(U)$ follows from the fact that $QUQ = QU = UQ$ has no eigenvalues in $\sigma_\epsilon(U)$.

QED

2.4 Subdynamics

Let us return to equation (2.2). We show that this splitting of the evolution operator gives rise to independent 'subdynamics' in the following sense.

Theorem 2 *Let U belong to the Brussels class and let $z_i, i \in \{1,\ldots,p\}$ denote the poles of $[z - PUP - \tilde{\psi}(z)]^{-1}$ in $A(QUQ)$. Then $p \geq 1$ and there are $p+1$ bounded projectors $\Pi^{(i)}, i \in \{0,\ldots,p\}$ with*

$$\sum_{i=0}^{p} \Pi^{(i)} = 1 \tag{2.22}$$

$$\Pi^{(i)}\Pi^{(j)} = \delta_{ij}\Pi^{(i)} \text{ for } i,j \in \{0,\ldots,p\} \tag{2.23}$$

$$U\Pi^{(i)} = \Pi^{(i)}U \text{ for } i \in \{0,\ldots,p\}. \tag{2.24}$$

Moreover the asymptotic evolution operators $\Sigma_n^{(i)}$ can be written

$$\Sigma_0^{(i)} = \Pi^{(i)} \tag{2.25}$$

$$\Sigma_n^{(i)} = z_i^n \Pi^{(i)} + \sum_{l=1}^{n} \binom{n}{l} z_i^{n-l} \Delta_i^l, \tag{2.26}$$

where

$$\Delta_i := \frac{1}{2\pi i}\oint_{\mathcal{C}_i} z^n[P+\mathcal{C}(z)]\frac{1}{z-PUP-\tilde{\psi}(z)}[P+\mathcal{D}(z)]\,dz$$

and $\mathcal{C}_i$ *is a contour enclosing the pole* z_i *only. The remainder operator* $\hat{\Sigma}_n$ *obeys*

$$\left\|\hat{\Sigma}_n\right\| \leq Ka^n \tag{2.27}$$

for some constant K *and* $r(QUQ) < a \leq \min_{i\in\{1,\ldots,p\}}\{|z_i|\}$.

The *Proof* is almost trivial; let $0 < \epsilon < \min_{i\in\{1,\ldots,p\}}\{|z_i|\} - r(QUQ)$ and observe that the Brussels decomposition of the resolvent (2.20) is valid for a connected subset of the complex plain, i.e. for $\{z : |z| \geq r(QUQ)+\epsilon\}$ except for a finite number of points due to Lemma 1 and U being quasicompact. We may therefore replace the integrand in (2.2) by $z^n/(z-U)$ and use the following Laurent series expansion:

$$\frac{1}{z-U} = \sum_{i=1}^{p}\left[\frac{\Pi^{(i)}}{z-z_i} + \sum_{l=1}^{\infty}\frac{\Delta_i^n}{(z-z_i)^{n+1}}\right] + \frac{1}{z-U}\Pi^{(0)},$$

where $\Pi^{(i)}$ and Δ_i are the eigenprojection and the eigennilpotent associated with z_i and $\frac{1}{z-U}\Pi^{(0)}$ is holomorphic in $z_i, i \in \{1,\ldots,p\}$ with

$$\Pi^{(0)} := 1 - \left(\Pi^{(1)} + \cdots + \Pi^{(p)}\right).$$

This is a standard result and may be found in [Kat, III.6.5]. Now trivially (2.22), (2.23) and (2.24) hold. Finally (2.25) and (2.26) follow from a simple integration

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_i}\frac{z^n}{(z-z_0)^{l+1}}\,dz \begin{cases} \binom{n}{l} z_i^{n-l} & \text{for } l \leq n \\ 0 & \text{otherwise} \end{cases}$$

and (2.27) is a consequence of

$$\hat{\Sigma}_n = \frac{1}{2\pi i}\oint_{\mathcal{C}'}\frac{z^n}{z-U}\Pi^{(0)}\,dz$$

and the fact that $\frac{1}{z-U}\Pi^{(0)}$ is analytic in $\{z : |z| \geq r(QUQ)+\epsilon\}$.

QED

We have recovered the basic features of subdynamics. The temporal evolution of the system may be separated into independently evolving parts by virtue of the projectors $\Pi^{(i)}$. The long time behaviour in the subspace $\Pi^{(i)}B$ is governed by (2.26). Note that since

$$\Delta_i^n = 0 \text{ for } n \geq \nu_i,$$

where ν_i is the algebraic multiplicity of z_i , i.e. the dimension of $\Pi^{(i)}B$, the evolution of a density f_0 entirely lying in $\Pi^{(i)}B$ for n large (i.e. for $n \geq \nu_i$) is given by

$$f_n = z_i^n f_0 + n z_i^{n-1} + \cdots + \frac{n!}{(n-\nu+1)!(\nu-1)!} z_i^{n-\nu+1} \Delta_i^{\nu-1} f_0.$$

Thus we encounter the discrete time analogue of the *long-time tails*, i.e. the occurrence of non-exponentially decaying states. In other words only modes associated with *semisimple eigenvalues*, i.e. eigenvalues with vanishing eigennilpotent, will give rise to exponentially [1] decaying states, and hence to Markovian master equations.

2.5 Conclusion

A critical assessment of this work requires a second look at Theorem 2. The crucial ingredient of its proof was the insight that, for suitably chosen analytic continuations of the Brussels operators comprising the reduced resolvent $Q[z-QUQ]^{-1}Q$ with domain $A(QUQ)$ the singularities of $[z-U]^{-1}$ and $[z-PUP-\tilde{\psi}(z)]^{-1}$ coincide. Being intuitively clear in the finite dimensional case this is well known for situations in an infinite dimensional space with the evolution operator having a discrete spectrum (c.f. [GGG, p. 429]). In other words, no *new* representations for the long time behaviour of the systems may be derived. In particular, given a unitary evolution, no decaying states may be obtained. The symmetry of the underlying transformations prohibit the derivation of time-asymmetric phenomenological laws. This is another version of *Loschmidt's Umkehreinwand* (see [Los]), which constitutes a severe obstacle for the derivation of irreversible kinetic equations. In the Brussels

[1] We suppress the expression 'geometrically' decaying state, which would be consistent with our terminology, as it may appear to be an oxymoron.

approach it is, however, assumed that the situation drastically changes if the spectra of the evolution operators become continuous. For quantum systems this may be achieved by taking the famous *thermodynamic limit*; in classical mechanics a necessary condition is, that the system be *mixing*.

In this work the possibility of continuous spectra is not excluded *a priori*, since infinite dimensional spaces are permitted. However, Theorem 1 shows that the choice of P and Q together with the analyticity requirements of the theory restricts the class of allowed evolution operators.

So far we did not touch the question which systems allow of a description in terms of quasicompact evolution operators. Proposition 1 already excludes non-invertible transformations on L^p spaces. The well known Baker's transformation is invertible, but its Frobenius-Perron operator on $L^2(\Omega, dx)$, where Ω is the 2-torus and dx the Lebesgue measure, is known to have purely absolutely continuous spectrum on $\{1\}^\perp$ (c.f. [RS, VII.4 Example 2]). So far we do not know of any nontrivial system with a quasicompact evolution operator [2]. The work of Ruelle *et al.* on power spectra of correlation functions, however, suggests that the Frobenius-Perron operator of the Baker's transformation as an operator on the Banach space of *Lipschitz continuous functions on subshifts of finite type* is quasicompact (see for example [Rue], [PP]). Since the spectral properties of linear operators crucially depend on the underlying space, this should not be surprising. Different state spaces will lead to different phenomena. In the work of Hasegawa and Saphir, this is epitomized by their recourse to rigged Hilbert spaces to obtain decaying eigenstates for systems with a unitary evolution operator. The description of the systems is shifted to a new state space, thereby changing the spectral properties of the relevant operator.

It is the author's conviction that this is the only accessible way of a reconciliation of reversible microscopic laws with irreversible macroscopic phenomena, i.e. irreversibility has to be included in the laws of physics on a fundamental level.

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[2] This problem is currently under investigation

Appendix A

Z-Transforms

The material covered here is an easy generalization of the standard methods (see e.g. [Jur], [Mut]).

Let $(B, \|.\|)$ be a Banach space with norm $\|.\|$. A sequence of $\{A_n\}_{n\in\mathbb{N}}$ of bounded operators $A_n \in \mathcal{B}(B)$ is said to be of *geometric order*, if there exist positive reals A and a and an integer n_0, such that for all $n \geq n_0$

$$\|A_n\| \leq Aa^n. \tag{A.1}$$

Then the *z-transform* of $\{A_n\}_{n\in\mathbb{N}}$ is defined by

$$\mathcal{Z}[A_n] := \mathcal{A}(z) := \sum_{n=0}^{\infty} A_n z^{-n}.$$

Theorem 3 (Properties) *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of geometric order with constant a as in (A.1). The z-transform $\mathcal{Z}[A_n]$ is unique and holomorphic in the extended annulus $\{z : |z| > a\} \cup \{\infty\}$.*

Proof The assertion follows from the fact that $\mathcal{Z}[A_n]$ is a Laurent series with no positive powers and radius of convergence r not exceeding a by the Cauchy-Hadamard formula:

$$r = \lim_{n\to\infty} \|A_n\|^{1/n} \leq a.$$

QED

Note that if

$$A_n = A^n,$$

then $\mathcal{Z}[A_n]$ is up to a factor z identical with the *von Neumann series* of the resolvent of A:

$$\mathcal{Z}[A^n] = \sum_{n=0}^{\infty} A^n z^{-n} = \frac{z}{z-A}.$$

Theorem 4 (Inversion formula) *The inverse z-transform $\mathcal{Z}^{-1}$ is given by*

$$\mathcal{Z}_n^{-1}[\mathcal{A}(z)] = \frac{1}{2\pi i} \oint_{\mathcal{C}} z^{n-1} \mathcal{A}(z)\, dz,$$

where $\mathcal{C}$ may be any contour enclosing all singularities of $\mathcal{A}(z)$.

Proof This is just the expression for the coefficients of a Laurent series.

QED

The following theorems are particularly useful for handling difference equations:

Theorem 5 (Shifting Theorem) *If $\mathcal{Z}[A_n] = \mathcal{A}(z)$, then for $k \geq 0$*

$$\mathcal{Z}[A_{n+k}] = z^k \left(\mathcal{A}(z) - \sum_{n=0}^{k-1} A_n z^{-n} \right).$$

Proof This follows from

$$\begin{aligned} \sum_{n=0}^{\infty} A_{n+k} z^{-n} &= z^k \sum_{n=0}^{\infty} A_{n+k} z^{-(n+k)} \\ &= z^k \left(\sum_{n=0}^{\infty} A_n z^{-n} - \sum_{n=0}^{k-1} A_n z^{-n} \right). \end{aligned}$$

QED

Theorem 6 (Convolution Theorem) *Given $\{A_n\}_{n\in\mathbb{N}}$ and $\{A'_n\}_{n\in\mathbb{N}}$ with z-transforms $\mathcal{A}(z)$ and $\mathcal{A}'(z)$ respectively we can define the convolution of the two series by*

$$\{A_n * A'_n\}_{n\in\mathbb{N}} = \left\{ \textstyle\sum_{i=0}^{n} A_i A'_{n-i} \right\}_{n\in\mathbb{N}},$$

its z-transform being given by

$$\mathcal{Z}[A_n * A'_n] = \mathcal{A}(z)\mathcal{A}'(z).$$

Proof We only need to take into account that $A_n = 0$ for $n < 0$ by definition. Then

$$\begin{aligned} \mathcal{A}(z)\mathcal{A}'(z) &= \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} A_n A_{n'} z^{-(n+n')} = \sum_{n=0}^{\infty}\sum_{n'=n}^{\infty} A_n A_{n'-n} z^{-n'} \\ &= \sum_{n'=0}^{\infty}\sum_{n=0}^{n'} A_n A_{n'-n} z^{-n'} = \mathcal{Z}[A_n * A'_n]. \end{aligned}$$

QED

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