Complementary Partial Orders and Rectangle Packing

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1 Introduction

Let A be a set, and P, Q be partial orders on A. The pair $\langle P, Q\rangle$ is said to be *complementary* if every pair of distinct elements of A is comparable in precisely one of P and Q. Complementary pairs of partial orders first appear in the work of Dushnik and Miller [4], where they are termed *conjugate* pairs.

Two results concerning complementary partial orders are presented in this paper. Although the proofs are straightforward, the results themselves are perhaps rather surprising. The first, amounting to not more than an observation, is that there is a natural bijection between complementary pairs of partial orders on A and pairs of linear orders on A. This observation, which is implicit in earlier literature, provides a very clean machine representation for complementary partial orders.

The second result appears to be new. Suppose that P, Q are partial orders on A such that every pair of elements of A is comparable in at least one of P and Q. The pair $\langle P, Q\rangle$ is not in general complementary, since there may be pairs of elements of A which are comparable in both P and Q. However, it can be proved that there exist partial orders P', Q', with $P' \subseteq P$ and $Q' \subseteq Q$, such that the pair $\langle P', Q'\rangle$ is indeed complementary. A simple counterexample shows that the result does not extend to the case of *three* partial orders sharing the same underlying set.

The final section of the paper draws a connection between complementary partial orders, and the problem of *rectangle packing*: given a large rectangle and a set of smaller rectangles, determine whether there is an orthogonal, non-overlapping packing of the small rectangles into the larger one. Depending on the exact specification of the problem, rectangles may be constrained to have a particular orientation, or may be free to rotate through right angles. There is also an optimisation version of the problem, in which a minimum height packing of rectangles into a semi-infinite 'bin' of fixed width is sought; this has been studied by several authors, beginning with Baker et al. [1] and Coffman et al. [3]. Aside from being of theoretical interest, the rectangle packing problem arises naturally in a number of practical applications: the layout of multi-project chips, screen management, and stock cutting, are examples. There are also some less obvious applications: Codd [2], for example, demonstrates that certain scheduling problems can be formulated in terms of rectangle packing.

The rectangle packing problem is a continuous one, in the sense that the smaller rectangles can potentially be placed at a continuous infinity of points within the larger rectangle. In the final section it is shown that the curious properties of complementary partial orders, described above, allow an elegant reduction from the continuous problem to an equivalent, discrete one. Once the problem has been reduced in this way, it is open to attack by the usual techniques in combinatorial optimisation. Application of the branch and bound strategy provides exact solutions to instances involving a small number (< 10) of rectangles; alternatively, hill-climbing or simulated annealing [6] heuristics can be used to obtain good approximate solutions to larger instances. Since the rectangle packing problem is NP-complete—as can be established by an easy reduction from BIN PACKING [5, p.226]—we cannot reasonably expect to find a solution which is simultaneously exact and efficient.

The reduction mentioned above can also be applied to more elaborate versions of the rectangle packing problem in which, for example, there are proximity constraints between certain pairs of rectangles. It thus provides an appealing approach to the phase in VLSI design known as floorplanning [7].

2 Complementary Partial Orders

In this section we explore some properties of complementary partial orders which are relevant to the rectangle packing problem. Some of the results are interesting in their own right. Firstly, let us fix our notation. Suppose A is a set, and P a partial order on A. We denote by P^{-1} the *inverse* of P, and by P^s ($= P \cup P^{-1}$) the *symmetric closure* of P. Let I_A stand for the identity relation on A. The *cover relation*, P^c, associated with P, is defined by $P^c = P - I_A - (P - I_A)^2$; thus, two distict elements $a, b \in A$ are related in P^c if they are related in P *and* there are no elements of A lying strictly between a and b in the partial order P. We call a pair $\langle P, Q \rangle$ of partial orders on A *complementary* iff $P^s \cup Q^s = A \times A$ and $P^s \cap Q^s = I_A$. Thus a pair of partial orders is complementary iff every pair of distinct elements of A is comparable in exactly one of the two orders. We denote by $\mathcal{C} = \mathcal{C}(A)$ the set of all complementary partial orders on A, and by $\mathcal{L} = \mathcal{L}(A)$ the set of all linear orders on A.

Lemma 1 If P, Q are complementary partial orders on a set A, then $L = P \cup Q$ is a linear order on A.

Proof: Reflexivity, antisymmetry and totality of L are straightforward, so we may proceed immediately to transitivity. It is enough to show that $Q \circ P \subseteq P \cup Q$ (and hence by symmetry $P \circ Q \subseteq P \cup Q$); transitivity of L then follows from the chain of inclusions

$$\begin{aligned} L^2 &= (P \cup Q)^2 \\ &= P^2 \cup Q^2 \cup Q \circ P \cup P \circ Q \\ &\subseteq P \cup Q \cup Q \circ P \cup P \circ Q \qquad \text{(by transitivity of } P \text{ and } Q\text{)} \\ &= L. \end{aligned}$$

So let $\langle a, b\rangle \in Q \circ P$, $a \neq b$, and choose $c \in A$ such that aPc and cQb. If c is equal either to a or b, then it is immediate that $\langle a, b\rangle \in P \cup Q$; so suppose that c is distinct from a and b. It cannot be the case that $aP^{-1}b$, since that would imply bPc and hence b and c comparable in both partial orders; similarly, the possibility $aQ^{-1}b$ may be eliminated. Since a and b are comparable in one or other partial order, we are forced to conclude $\langle a, b\rangle \in P \cup Q$. Hence $Q \circ P \subseteq P \cup Q$ as claimed. □

Theorem 2 Let $f \in \mathcal{L}^2 \to \mathcal{C}$ be defined by $f(\langle L, M\rangle) = \langle L \cap M, L \cap M^{-1}\rangle$ for all $L, M \in \mathcal{L}$. Then the function f is well-defined, i.e. the pair $\langle L \cap M, L \cap M^{-1}\rangle$ is indeed complementary for all choices of L, M. Furthermore, f is a bijection.

Proof: Since the intersection of any two partial orders is a partial order, $P = L \cap M$ and $Q = L \cap M^{-1}$ are certainly partial orders. Now let a, b be arbitrary distinct elements of A. Precisely one of the following conditions must hold: (i) aLb, aMb (ii) aLb, bMa (iii) bLa, aMb (iv) bLa, bMa. In cases (i) and (iv), a and b are comparable in P but not Q, and in cases (ii) and (iii), a and b are comparable in Q but not P. Thus P and Q are indeed complementary, as claimed.

We show that f is a bijection by giving its inverse explicitly. Define $g \in C \to \mathcal{L}^2$ by $g(\langle P, Q\rangle) = \langle P \cup Q, P \cup Q^{-1}\rangle$ for all complementary pairs $\langle P, Q\rangle$. By the previous lemma, g is well-defined: $P \cup Q$ and $P \cup Q^{-1}$ are indeed linear orders. It is a straightforward matter to check that $g \circ f$ and $f \circ g$ are the identity functions on $\mathcal{L}^2$ and C respectively. □

Corollary 3 The number of complementary pairs of partial orders on a set of cardinality n is precisely $(n!)^2$.

Remark 1 The correspondence described in the previous theorem is implicit in the work of Dushnik and Miller [4].

The following theorem gives an alternative characterisation of complementary partial orders in terms of a minimality condition.

Theorem 4 Let A be a finite set, and P, Q be partial orders on A satisfying $P^s \cup Q^s = A \times A$. Suppose that P and Q are minimal in the following sense: for all partial orders $P_1 \subseteq P$ and $Q_1 \subseteq Q$, the condition $P_1^s \cup Q_1^s = A \times A$ implies $P_1 = P$ and $Q_1 = Q$. Then the pair $\langle P, Q\rangle$ is complementary.

Proof: Let P, Q be as in the statement of the theorem. We make the preliminary observation that $P^c \cap Q = P \cap Q^c = \emptyset$. For suppose $\langle a, b\rangle \in P^c \cap Q$, and let $P_1 = P - \{\langle a, b\rangle\}$; then P_1 is a partial order satisfying $P_1^s \cup Q^s = A \times A$, contradicting minimality of P.

We shall now assume that $\langle P, Q\rangle$ is not a complementary pair and obtain a contradiction. Because P and Q are not complementary, there must exist pairs of elements of A which are comparable in both P and Q. Let $a_0, \ldots, a_l$ be a minimum length sequence of distinct elements of A with $a_i P^c a_{i+1}$ for $0 \leq i \leq l-1$, and such that a_0 and a_l are comparable in Q. Suppose, without loss of generality, that $a_0 Q a_l$. (If the converse relation holds, then the proof goes through with Q replaced by Q^{-1} thoughout.) Let $b_0, \ldots, b_m$ be a sequence of distinct elements of A with $b_0 = a_0$, $b_m = a_l$, and $b_i Q^c b_{i+1}$ for $0 \leq i \leq m-1$. Since $P^c \cap Q = P \cap Q^c = \emptyset$, both l and m must be greater than 1.

Consider the relationships holding between a_1 and b_i for $1 \leq i \leq m-1$. Since the sequence $a_1, \ldots, a_l$ is of minimal length, none of these pairs is comparable in Q. Thus for each i, $1 \leq i \leq m-1$, either $a_1 P b_i$ or $b_i P a_1$ must hold; moreover, we know that $b_0 P a_1$ and $a_1 P b_m$. Hence there exists j, with $0 \leq j \leq m-1$, for which $b_j P a_1$ and $a_1 P b_{j+1}$. By transitivity of P, $b_j P b_{j+1}$, and hence $\langle b_j, b_{j+1} \rangle \in P \cap Q^c$, a contradiction. □

The following corollary plays a significant role in the application of complementary partial orders to the rectangle packing problem.

Corollary 5 Let A be a finite set, and P, Q be partial orders on A satisfying $P^s \cup Q^s = A \times A$. Then there exist partial orders $P_0 \subseteq P$ and $Q_0 \subseteq Q$ such that the pair $\langle P_0, Q_0 \rangle$ is complementary.

Proof: Consider the set $\mathcal{P}$ of all pairs $\langle R, S \rangle$ of partial orders which satisfy $R \subseteq P$, $S \subseteq Q$ and $R^s \cup S^s = A \times A$. Choose a pair $\langle P_0, Q_0 \rangle \in \mathcal{P}$ which is minimal with respect to set inclusion. (Since A is finite, minimal elements certainly exist.) Then by theorem 4, the pair $\langle P_0, Q_0 \rangle$ is complementary. □

Remark 2 It has been observed by Gordon Plotkin that the corollary extends to arbitrary sets A (not just finite ones) by application of the compactness theorem of propositional logic.

Remark 3 That the corollary does not extend to the case of *three* partial orders on the same underlying set A can be seen from the following counterexample. Let $A = \{a, b, c, d\}$, and let

$$P = I_A \cup \{\langle a, b \rangle, \langle b, d \rangle, \langle a, d \rangle\}$$

$$Q = I_A \cup \{\langle a, c \rangle, \langle c, d \rangle, \langle a, d \rangle\}$$

and

$$R = I_A \cup \{\langle b, c\rangle\}$$

be partial orders on A. It is easy to check that $P^s \cup Q^s \cup R^s = A \times A$, and that of all triples of partial orders satisfying this property, P, Q, R are minimal with respect to set inclusion. However, a and d are comparable in both P and Q.

3 Rectangle Packing

We shall consider the version of the rectangle packing problem in which orientations of rectangles are fixed—the extension to the case where rotations are allowed will be clear. It is convenient to describe the positions of rectangles in a Cartesian coordinate system in which the large rectangle, with sides of length X and Y say, has vertices at points $\langle 0,0\rangle$, $\langle X,0\rangle$, $\langle 0,Y\rangle$ and $\langle X,Y\rangle$. The positions of the small rectangles can be specified by giving the coordinates of the vertex nearest the origin. Let the set of small rectangles be A. For each $a \in A$, $d_x(a)$ and $d_y(a)$ give, respectively, the x- and y-dimensions of the rectangle a.

Now suppose that, for each pair of rectangles, an x- or y-constraint (possibly both) is specified. The existence of an x-constraint between rectangles a and b will be denoted $a <_x b$, and the existence of a y-constraint by $a <_y b$. A particular layout of rectangles satisfies the constraint $a <_x b$ iff there exists a constant α such that the rectangle a is contained entirely within the region $x \le \alpha$ and b entirely within the region $x \ge \alpha$. If a layout satisfies *all* the constraints, then, clearly, none of the rectangles can overlap. Since the satisfaction of the constraints $a <_x b$ and $b <_x c$ automatically ensures satisfaction of the constraint $a <_x c$, we lose nothing by assuming that $<_x$ and $<_y$ are both strict partial orders (i.e. obtained from a partial order by subtraction of the identity relation I_A).

The rectangle packing problem with the additional constraints specified by $<_x$ and $<_y$ is easily solved. (Note that non-overlapping of rectangles is an automatic consequence of satisfaction of the sets of constraints $<_x$ and $<_y$.) For all $a \in A$, denote by $\mu_x(a, <_x)$ the minimum, over all packings satisfying the constraints $<_x$, of the x-coordinate of rectangle a; the analogue in the y-dimension may be denoted $\mu_y(a, <_y)$. For a given strict partial order $<_x$, the values $\mu_x(a, <_x)$, for $a \in A$, can be computed by dynamic programming [8, p.483] from the inductive rule

$$\mu_x(a,<_x) = \begin{cases} 0, & \text{if } a \text{ is minimal in } <_x; \\ \max\{\mu_x(b,<_x) + d_x(b) : b <_x a\}, & \text{otherwise.} \end{cases}$$

A similar procedure can be applied in the y-dimension. Notice that the minimum x- and y-coordinates specified by μ_x and μ_y, are achievable by all rectangles *simultaneously* in a single packing. Thus the small rectangles may be packed into the larger one, subject to the additional constraints $<_x$ and $<_y$, iff

$$\begin{aligned} \max\{\mu_x(a,<_x) + d_x(a) : a \in A\} &\leq X \\ \max\{\mu_y(a,<_y) + d_y(a) : a \in A\} &\leq Y \end{aligned}$$

This deals with the (artificially) constrained problem; for the (original) unconstrained problem it is sufficient to solve the constrained version for all possible pairs $\langle <_x, <_y \rangle$ of strict partial orders which are jointly total. (We shall say that a pair of strict partial orders on A is *jointly total* if every pair of distinct elements of A is comparable in at least one order.) Clearly, if the unconstrained problem has no solution, then nor do any of the constrained problems. Conversely, suppose the unconstrained problem does have a solution. Consider any packing of the small rectangles into the larger, and let $<_x$ be the maximal set of x-constraints consistent with the packing, $<_y$ be the maximal set of y-constraints. (The relations $<_x$ and $<_y$ will necessarily be strict partial orders; moreover they are jointly total.) For this choice of $<_x$ and $<_y$ it is easy to see that the constrained problem has a feasible solution.

Now corollary 5 tells us that we do not need to consider *all* possible pairs of strict partial orders $\langle <_x, <_y \rangle$ which are jointly total. Those pairs which allow distinct elements of A to be compared in both $<_x$ and $<_y$ orders correspond to over-constrained problems—corollary 5 assures that weaker strict partial orders exist which are still jointly total. Thus, an exact solution to the rectangle packing problem can be obtained by solving the constrained problem for all possible pairs of complementary strict partial orders. The correspondence, made explicit in theorem 2, between complementary pairs of strict partial orders, and pairs of linear orders on the same set, makes cycling through the possibilities very easy. Of course, the very large number of possibilities to be tested prohibits the application of this naïve approach to any but the smallest problem instances. (By corollary [3], the number of constrained problems to be solved is $(n!)^2$, where n is the number of rectangles.) Building the constraints $<_x$ and $<_y$ incrementally, and applying branch-and-bound [8, p.519], allows the range of applicability to be extended, but only up to $n \approx 15$ (or $n \approx 10$ if rotations are allowed).

Fortunately, the method presented here lends itself well to approximation algorithms using heuristic search. For each non-overlapping arrangement of rectangles in the positive quadrant of the coordinate system, associate a *cost* which expresses how 'far away' the arrangement is from being a feasible solution. Starting with a randomly selected pair of complementary partial orders $<_x$ and $<_y$, solve the constrained problem determined by $<_x$ and $<_y$, and compute the associated cost. Then attempt to reduce the cost by applying a succession of pertubations to $<_x$ and $<_y$. A simple heuristic is to accept only perturbations which reduce the cost function, but there are more complex techniques [6] which allow the cost function sometimes to increase. Because the current constraints $<_x, <_y$ can be represented by a very simple data structure, namely a pair of linear orders, the perturbations are very easy to effect: a simple example is to transpose a pair of elements in one of the linear orders.

An appropriate cost function for layouts can be computed as follows: Find the smallest rectangle which bounds the layout, letting its dimensions be X' and Y'. Then assign cost

$$\left(\max\{X' - X, 0\}\right)^2 + \left(\max\{Y' - Y, 0\}\right)^2$$

to the layout. (Recall that X, Y are the dimensions of the large rectangle into which the smaller ones are to be packed.) The squaring of terms in the cost function tends to penalise layouts whose aspect ratio varies substantially from the ideal. In practice, it is advantagous to introduce a secondary cost function which can distinguish between layouts which are equally good according to the primary criterion. The following secondary cost function appears to give good results:

$$\iint_R \max\{x, y\}\, dx\, dy$$

Here R is the region formed by the union of the small rectangles. Under the influence of the secondary cost fuction, rectangles tend to migrate towards the origin, until, in due course, an improvement in the primary cost function can be made.

Some small scale experiments were performed, using the cost functions described above. Starting from a randomly chosen pair of linear orders, simple hill-climbing was employed to find a packing optimal with respect to the following perturbations:

- transposing a pair of adjacent rectangles in one of the linear orders,
- interchanging two rectangles, and
- rotating a rectangle.

Ten runs of the algorithm were made, using as input the 55 rectangles with integer side lengths less than or equal to 10, which are distinct up to rotation. On grounds of area alone, no packing of these rectangles into a square of side 41 is possible. On two of the runs, the packing obtained was of size 42×44, on five runs of size 43×43, on two runs of size 42×43, and on a single run an optimal square packing of size 42×42 was achieved. For this last packing, the utilisation of area within the large square is better than 96.6%. The time taken to reach a local optimum in a problem of this size, using an unsophisticated Pascal implementation running on a Vax 11/780, is about 5–10 minutes. This could be much improved by careful programming.

It is tempting to think that the reduction described here could be applied to more complex problems, such as VLSI floorplanning, by making suitable modifications to the cost function. In the case of floorplanning, it would be necessary to incorporate an additional component into the cost function, to account for wiring costs between modules. It is by no means clear how this might best be done.

Acknowledgement

A rectangle packing algorithm, based on the reduction to complementary partial orders, was implemented by Jerry Scott, then a final year undergraduate. Without the benefit of the correspondence presented in theorem 2, the task was much more difficult than it ought to have been. The experience gained while implementing the algorithm prompted the rediscovery of theorem 2.

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