2. Continuous-time stochastic processes

As before we have a collection of r.v's, $\{X(t) : t \in T\}$. but now we take $T = \mathbb{R}_{\geq 0} = [0, \infty)$. In our examples, $X(t)$ will always take on integer variables (i.e., the state space will be a subset of $\mathbb{Z}$).

2.1. The Poisson process.

Definition 2.1. A continuous-time stochastic process $X(t)$ us a *Poisson process of rate* λ (or *intensity* λ) if

- P1. $X(0) = 0$.
- P2. For all $s \geq 0$, $t > 0$, $X(s+t) - X(s) \sim \mathrm{Po}(\lambda t)$.
- P3. If $0 \leq t_1 < t_2 < \cdots < t_n$, then $X(t_2) - X(t_1)$, $X(t_3) - X(t_2)$, $\ldots$, $X(t_n) - X(t_{n-1})$ are mutually independent r.v's.

How might this process arise? suppose we want to count "events" occurring in $(0, \infty)$. Let $N(t)$ denote the number of events in $(0, t]$. (Note that $N(0) = 0$.) Suppose that

- I1. If $t > s$, the number $N(t) - N(s)$ of events in time interval $(s, t]$ is independent of the times of events during $(0, s]$.
- I2. Events are "rare" in the sense that

$$\mathbb{P}(N(t+h) = n + r \mid N(t) = n) = \begin{cases} 0, & \text{if } r < 0; \\ 1 - \lambda h + o(h), & \text{if } r = 0; \\ \lambda h + o(h), & \text{if } r = 1; \\ o(h), & \text{if } r > 1. \end{cases}$$

(The notation $o(h)$ stands for a function $f(x)$ such that $f(h)/h \to 0$ as $h \to 0$.)

Theorem 2.1. *The above conditions (1) and (2) imply that $N(t)$ is a Poisson process of rate λ.*

Proof. Property P1 is immediate and P3 is straightforward, so we concentrate on P2.

Let $p_k(t) = \mathbb{P}(N(t) = k)$. Our goal is to show that $p_k(t) = e^{-\lambda t}(\lambda t)^k/k!$ for all k, which will imply that $X(t) \sim \mathrm{Po}(\lambda t)$. As the process defined by (1) and (2) is time-homogeneous, it will follow that $X(s+t) - X(s) \sim X(t) - X(0) \sim \mathrm{Po}(\lambda t)$. So we'll be done if we can show that $p_k(t)$ is as given above.

Let's consider how p_k changes in a small interval $[t, t+h]$:

$$\begin{aligned} p_k(t+h) &= \mathbb{P}(N(t+h) = k) \\ &= \sum_{j=0}^{k} \mathbb{P}(N(t) = j)\, \mathbb{P}(N(t+h) = k \mid N(t) = j) \qquad \text{(Law of Total Probability)} \\ &= \sum_{j=0}^{k} p_j(t)\, \mathbb{P}(N(t+h) = k \mid N(t) = j) \\ &= \begin{cases} p_k(t)(1 - \lambda h + o(h)), & \text{if } k = 0; \\ p_{k-1}(t)(\lambda h + o(h)) + p_k(t)(1 - \lambda h + o(h)) + o(h), & \text{if } k \geq 1. \end{cases} \end{aligned}$$

So

$$p_0(t+h) = p_0(t) - \lambda h\, p_0(t) + o(h), \quad \text{and}$$
$$p_k(t+h) = \lambda h\, p_{k-1}(t) + p_k(t) - \lambda h\, p_k(t) + o(h), \quad \text{for } k \geq 1.$$

I.e.,

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda\, p_0(t) + \frac{o(h)}{h}, \quad \text{and}$$
$$\frac{p_k(t+h) - p_k(t)}{h} = \lambda\, p_{k-1}(t) - \lambda\, p_k(t) + \frac{o(h)}{h}, \quad \text{for } k \geq 1.$$

Letting $h \to 0$,

$$p_0'(t) = -\lambda\, p_0(t), \quad \text{and}$$
$$p_k'(t) = \lambda\, p_{k-1}(t) - \lambda\, p_k(t), \quad \text{for } k \geq 1.$$

We can solve these equations, one at a time, for $p_0, p_1, p_2, \ldots$. (Formally, we are using induction on k.) First, $p_0'(t) = -\lambda\, p_0(t)$, so $p_0(t) = ce^{-\lambda t}$ for some c. But $p_0(0) = 1$, so $c = 1$ and

$$p_0(t) = e^{-\lambda t}. \tag{2}$$

Now to $k = 1$. We have $p_1'(t) = \lambda\, p_0(t) - \lambda\, p_1(t)$, i.e., $p_1'(t) = \lambda e^{-\lambda t} - \lambda\, p_1(t)$ or, rearranging,

$$e^{\lambda t} p_1'(t) + \lambda e^{\lambda t} p_1(t) = \lambda$$

Noting that the l.h.s. of this equation is the derivative of a product, we may write $(p_1(t)e^{\lambda t})' = \lambda$ and, by integration, $p_1(t)e^{\lambda t} = \lambda t + c$. But $p_1(t) = 0$, so $c = 0$ and

$$p_1(t) = \lambda t e^{-\lambda t}.$$

Continuing to the general case, suppose we know that $p_{k-1}(t) = e^{-\lambda t}(\lambda t)^{k-1}/(k-1)!$. We saw earlier that $p_k'(t) = \lambda\, p_{k-1}(t) - \lambda\, p_k(t)$, so that

$$e^{\lambda t} p_k'(t) + \lambda e^{\lambda t} p_k(t) = \frac{\lambda^k t^{k-1}}{(k-1)!}.$$

Again, noticing that the l.h.s. is the derivative of a product, we arrive at

$$(p_k(t)e^{\lambda t})' = \frac{\lambda^k t^{k-1}}{(k-1)!}.$$

By integration, $p_k(t)e^{\lambda t} = \lambda^k t^k / k! + c$. But $p_k(t) = 0$, so $c = 0$ and

$$p_k(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}. \tag{3}$$

From (2) and (3) we see that $N(t)$ has Poisson distribution with parameter λt, as required. □

2.1.1. *Superposition and thinning.* Suppose $X(t)$ and $Y(t)$ are independent Poisson processes. The process $Z(t) = X(t)+Y(t)$ is the *superposition* of $X(t)$ and $Y(t)$, and counts the totality of X-events and Y-events.

Lemma 2.2. *Let $X(t)$ and $Y(t)$ be independent Poisson processes with rates λ and μ. The stochastic process $Z(t) = X(t) + Y(t)$ is a Poisson process with rate $\lambda + \mu$.*

Proof. In time interval $(t, t + h]$ there is an X-event with probability $\lambda h + o(h)$ and a Y-event with probability $\mu h + o(h)$. Thus

$$\mathbb{P}\big(Z(t+h) - Z(t) = r\big) = \begin{cases} 1 - (\lambda+\mu)h + o(h), & \text{if } r = 0; \\ (\lambda+\mu)h + o(h), & \text{if } r = 1; \\ o(h), & \text{otherwise.} \end{cases}$$

Comparing with the "infinitesimal description" of a Poisson process, we see that $Z(t)$ is a Poisson process of rate $\lambda + \mu$. □

Let $X(t)$ be a Poisson process of rate λ, and $p \in (0, 1]$. Consider the sequence of events associated with $X(t)$. Suppose that each event independently survives with probability p and is lost with probability $1 - p$. Denote by $\widehat{X}(t)$ the *thinned* process defined by the surviving events.

Lemma 2.3. *Let $X(t)$ be a Poisson processes with rate λ. The stochastic process $\widehat{X}(t)$ defined by the thinning procedure described above is a Poisson process with rate $p\lambda$.*

Proof. In time interval $(t, t + h]$ there is an event with probability $\lambda h + o(h)$ and this event survives with probability p. Thus

$$\mathbb{P}\big(\widehat{X}(t+h) - \widehat{X}(t) = r\big) = \begin{cases} 1 - p\lambda h + o(h), & \text{if } r = 0; \\ p\lambda h + o(h), & \text{if } r = 1; \\ o(h), & \text{otherwise.} \end{cases}$$

Comparing with the infinitesimal description of a Poisson process, we see that $\widehat{X}(t)$ is a Poisson process of rate $p\lambda$. □

2.1.2. *Random variables associated with the Poisson process.* Let $T_i = \inf\{t : X(t) = i\}$ be the time of occurrence of the ith event. The T_i are called *arrival times* (or waiting times). By convention, $T_0 = 0$. Let $S_i = T_i - T_{i-1}$, for $i = 1, 2, \ldots$, be the time between the $(i - 1$st and ith arrival. The S_i are called interarrival times.

Consider T_1 $(= S_1)$. We have

$$\begin{aligned} \mathbb{P}(T_1 \le t) &= \mathbb{P}(X(t) \ge 1) \\ &= 1 - \mathbb{P}(X(t) = 0) \\ &= 1 - e^{-\lambda t}, \end{aligned}$$

since the number of arrivals in $(0, t]$ is distributed as $\mathrm{Po}(\lambda t)$. So the cumulative distribution function (cdf) of T_1 is $F_{T_1}(t) = 1 - e^{-\lambda t}$. Differentiating, the probability density function (pdf) of T_1 is $f_{T_1} = \lambda e^{-\lambda t}$. Thus $T_1 \sim \mathrm{Exp}(\lambda)$.

Recall that the lack of memory property of the exponential distribution implies

$$\mathbb{P}(T_1 > t + s \mid T_1 > s) = \mathbb{P}(T_1 > t),$$

which agrees with the lack of memory of the Poisson process.

Theorem 2.4. *For $n \geq 1$, T_n has the Gamma distribution (see MTH 5121 Probability Models), which has pdf*

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}.$$

Proof. We'll treat the special case $n = 2$. (The case $n = 1$ was dealt with above.) The general case is left as an exercise. By analogy with the calculation above for T_1,

$$\begin{aligned} F_{T_2}(t) = \mathbb{P}(T_2 \leq t) &= \mathbb{P}(X(t) \geq 2) \\ &= 1 - \mathbb{P}(X(t) = 0) - \mathbb{P}(X(t) = 1) \\ &= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}. \end{aligned}$$

Differentiating,

$$\begin{aligned} f_{T_2}(t) &= \lambda e^{-\lambda t} - \lambda e^{-\lambda t} + \lambda^2 t e^{-\lambda t} \\ &= \lambda^2 t e^{-\lambda t}, \end{aligned}$$

as required. □

Theorem 2.5. *$S_1, S_2, \ldots$ are independent r.v's each with distribution* $\mathrm{Exp}(\lambda)$*.*

Sketch of Proof. For a fixed time s,

$$\begin{aligned} \mathbb{P}((\text{time to next arrival after } s) \leq t) &= 1 - \mathbb{P}(\text{no arrival in } (s, s+t]) \\ &= 1 - e^{-\lambda t}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{P}(S_n \leq t) &= 1 - \mathbb{P}(\text{no arrival in } (T_{n-1}, T_{n-1} + t]) \\ &= 1 - e^{-\lambda t}, \end{aligned}$$

independently of $S_1, S_2, \ldots, S_{n-1}$. (This is not entirely rigorous, as $(T_{n-1}, T_{n-1} + t]$ is not a fixed interval; its endpoints are r.v's. The fix is rather technical and beyond the scope of the module.) □

2.1.3. *Conditioning on $X(t) = n$.* If we know that $X(t) = n$ (i.e., there are n events in $(0, t]$), what can we say about how they occur in $(0, t]$?

Theorem 2.6. *If $0 \leq u \leq t$ and $0 \leq k \leq n$ then*

$$\mathbb{P}(X(u) = k \mid X(t) = n) = \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}.$$

In other words, the conditional distribution is $\mathrm{Bin}(n, \frac{u}{t})$ regardless of λ.

Proof.

$$\begin{aligned}
\mathbb{P}(X(u)=k \mid X(t)=n) &= \frac{\mathbb{P}(X(u)=k, X(t)=n)}{\mathbb{P}(X(t)=n)} \\
&= \frac{\mathbb{P}(X(u)-X(0)=k)\,\mathbb{P}(X(t)-X(u)=n-k)}{\mathbb{P}(X(t)-X(0)=n)} \\
&= \frac{\left[e^{-\lambda u}(\lambda u)^k/k!\right] \times \left[e^{-\lambda(t-u)}(\lambda(t-u))^{n-k}/(n-k)!\right]}{e^{-\lambda t}(\lambda t)^n/n!} \\
&= \frac{(\lambda u)^k(\lambda(t-u))^{n-k}\, n!}{(\lambda t)^n\, k!(n-k)!} \\
&= \binom{n}{k}\frac{u^k(t-u)^{n-k}}{t^n} \\
&= \binom{n}{k}\left(\frac{u}{t}\right)^k\left(1-\frac{u}{t}\right)^{n-k}.
\end{aligned}$$

□

One consequence is that

$$\mathbb{P}(T_1 \le u \mid X(t)=1) = \mathbb{P}(X(u)=1 \mid X(t)=1) = \frac{u}{t}.$$

In other words, conditioned on there being exactly one event in the interval $(0,t]$, that event is distributed uniformly in the interval. More generally

Theorem 2.7. *Let $T_1, T_2, \ldots$ be the arrival times of a Poisson process of rate λ, and f be a symmetric function on n variables. Then*

$$\mathbb{E}(f(T_1, T_2, \ldots, T_n) \mid X(t)=n) = \mathbb{E}(f(U_1, \ldots, U_n)),$$

where U_i are independent r.v's, uniform on $[0,t]$.

2.2. **Birth processes.** A *birth process* with parameters $\lambda_0, \lambda_1, \lambda_2, \ldots$ is a continuous-time process $X(t)$ satisfying

- $X(0) \ge 0$.
- $$\mathbb{P}(X(t+h)=n+r \mid X(t)=n) = \begin{cases} 0, & \text{if } r<0; \\ 1-\lambda_n h + o(h), & \text{if } r=0; \\ \lambda_n h + o(h), & \text{if } r=1; \\ o(h), & \text{if } r>1. \end{cases}$$
- If $s<t$ then $X(t)-X(s)$ conditioned on $X(s)$ is independent of the process prior to s.

As with the Poisson process we can find differential equations defining the process. Let $p_n(t) = \mathbb{P}(X(t) = n)$ and suppose $X(0) = a$. Then for $n \geq a$,

$$\begin{aligned} p_n(t+h) &= \sum_{k=0}^{n} \mathbb{P}(X(t+h) = n \mid X(t) = k)\, \mathbb{P}(X(t) = k) \\ &= \mathbb{P}(X(t+h) = n \mid X(t) = n)\, \mathbb{P}(X(t) = n) \\ &\qquad + \mathbb{P}(X(t+h) = n \mid X(t) = n-1)\, \mathbb{P}(X(t) = n-1) + o(h) \\ &= (1 - \lambda_n h + o(h)) p_n(t) + (\lambda_{n-1} h + o(h)) p_{n-1}(t) + o(h). \end{aligned}$$

So

$$\frac{p_n(t+h) - p_n(t)}{h} = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$,

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t),$$

where for $n < a$ we let $p_n(t) = 0$. The initial conditions are $p_a(0) = 1$ and $p_n(0) = 0$ for $n > a$.

Theorem 2.8. *If $X(t)$ is the birth process with $X(0) = a$ and parameters $\lambda_a, \lambda_{a+1}, \ldots$, then the equations $p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t)$ for $n \geq a$ have a unique solution with the initial conditions $p_a(0) = 1$, and $p_n(0) = 0$, for $n > a$.*

The proof gives a method for finding the solution.

Proof. Solving $p_a'(t) = -\lambda_a p_a(t)$ we obtain $p_a(t) = Ce^{-\lambda_a t}$; but $p_a = 1$ so $C = 1$ and $p_a(t) = e^{-\lambda_a t}$. Suppose that we have solved for $p_{n-1}(t)$, where $n > a$. Rearranging $p_n'(t) = -\lambda_n p_n(t) = \lambda_{n-1} p_{n-1}(t)$ and multiplying through by $e^{\lambda_n t}$, we obtain

$$e^{\lambda_n t} p_n(t)' + \lambda_n e^{\lambda_n t} p_n(t) = \lambda_{n-1} e^{\lambda_n t} p_{n-1}(t),$$

i.e.,

$$(e^{\lambda_n t} p_n(t))' = \lambda_{n-1} e^{\lambda_n t} p_{n-1}(t).$$

Integrating and taking into account $p_n(0) = 0$,

$$e^{\lambda_n t} p_n(t) = \lambda_{n-1} \int_0^t e^{\lambda_n s} p_{n-1}(s)\, ds,$$

i.e.,

$$p_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n s} p_{n-1}(s)\, ds.$$

□

It can be shown (but not here) that, for any $t > 0$, $\sum_{n=a}^{\infty} p_n(t) = 1$ if and only if $\sum_{n=a}^{\infty} \lambda_i^{-1} = \infty$. (If the λ_n grow too fast then the process "explodes" at finite time.) To get an intuitive feel for what is going on, define arrival times $T_i = \min\{t : X(t) = i\}$ for $i = a, a+1, \ldots$, as for the Poisson process, and interarrival times $S_i = T_i - T_{i-1}$. Then

$$F_{T_{a+1}}(t) = \mathbb{P}(T_{a+1} \leq t) = \mathbb{P}(X(t) > a) = 1 - \mathbb{P}(X(t) = a) = 1 - p_a(t) = 1 - e^{-\lambda_a t},$$

and, differentiating,

$$f_{T_{a+1}} = \lambda_a e^{-\lambda_a t},$$

which is the pdf of the exponential function with parameter λ_a. Thus $S_{a+1} = T_{a+1} \sim \mathrm{Exp}(\lambda_a)$. Continuing as in Theorem 2.5, we see in general that $S_i \sim \mathrm{Exp}(\lambda_{i-1})$ for all $i > a$. Thus

$$\mathbb{E}\left(\sum_{i=a+1}^{\infty} S_i\right) = \sum_{i=a+1}^{\infty} \mathbb{E}(S_i) = \sum_{i=a}^{\infty} \frac{1}{\lambda_i}.$$

So if $\sum_{n=a}^{\infty} \lambda_i^{-1} < \infty$, with probability 1 the population will become infinite at finite time.

2.3. **Birth-death processes.** A *birth-death process* with *birth parameters* $\lambda_0, \lambda_1, \lambda_2, \dots$ and *death parameters* $\mu_1, \mu_2, \dots$ is a continuous-time process $X(t)$ on state space $\mathbb{N}$ satisfying the conditions

- The probabilities
$$p_{ij}(t) = \mathbb{P}(X(s+t) = j \mid X(s) = i)$$
are independent of s, and of the process up to time s.
- For $h > 0$,
$$p_{ij}(h) = \begin{cases} \lambda_i h + o(h), & \text{if } i \geq 0 \text{ and } j = i+1; \\ \mu_i h + o(h), & \text{if } i \geq 1 \text{ and } j = i-1; \\ 1 - (\lambda_i + \mu_i)h + o(h), & \text{if } i \geq 1 \text{ and } j = i; \\ 1 - \lambda_0 h + o(h), & \text{if } i = j = 0; \\ o(h), & \text{otherwise.} \end{cases}$$
Also,
$$p_{ij}(0) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.9 (Chapman-Kolmogorov relations). *For all $s, t \geq 0$ and $i, j \in \mathbb{N}$,*

$$p_{ij}(s+t) = \sum_{k=0}^{\infty} p_{ik}(s) p_{kj}(t).$$

Proof. Condition on $X(s)$, as for the discrete case (Theorem 1.2). □

We can use Theorem 2.9 to derive differential equations for $p_{ij}(t)$. For $j \geq 1$,

$$\begin{aligned} p_{ij}(t+h) &= \sum_{k=0}^{\infty} p_{ik}(t) p_{kj}(h) \\ &= p_{i,j-1}(t)\,(\lambda_{j-1} h) + p_{ij}(t)\,(1 - (\lambda_j + \mu_j)h) + p_{i,j+1}(t)\,(\mu_{j+1} h) + o(h). \end{aligned}$$

Rearranging,

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \lambda_{j-1}\, p_{i,j-1}(t) - (\lambda_j + \mu_j)\, p_{ij}(t) + \mu_{j+1}\, p_{i,j+1}(t) + \frac{o(h)}{h}.$$

So

$$p'_{ij}(t) = \lambda_{j-1}\, p_{i,j-1}(t) - (\lambda_j + \mu_j)\, p_{ij}(t) + \mu_{j+1}\, p_{i,j+1}(t) \qquad (\text{for } j \geq 1),$$

and

$$p'_{i0}(t) = -\lambda_0\, p_{i0}(t) + \mu_1\, p_{i1}(t).$$

The special case $j = 0$ arises because there is no possibility of a death. These are the *forward equations.*

Similarly, for $i \geq 1$

$$\begin{aligned} p_{ij}(t+h) &= \sum_{k=0}^{\infty} p_{ik}(h) p_{kj}(t) \\ &= \mu_i h\, p_{i-1,j}(t) + (1 - (\lambda_i + \mu_i)h)\, p_{ij}(t) + \lambda_i h\, p_{i+1,j}(t) + o(h). \end{aligned}$$

Rearranging,

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \mu_i\, p_{i-1,j}(t) - (\lambda_i + \mu_i)\, p_{ij}(t) + \lambda_i\, p_{i+1,j}(t) + \frac{o(h)}{h}.$$

So

$$p'_{ij}(t) = \mu_i\, p_{i-1,j}(t) - (\lambda_i + \mu_i)\, p_{ij}(t) + \lambda_i\, p_{i+1,j}(t) \qquad \text{(for } i \geq 1\text{)},$$

and

$$p'_{0j}(t) = -\lambda_0\, p_{0j}(t) + \lambda_0\, p_{1j}(t).$$

These are the *backwards equations.*

Theorem 2.10. *Suppose $\lambda_0, \lambda_1, \ldots > 0$ and $\mu_1, \mu_2, \ldots > 0$. There exists a probability vector $\boldsymbol{w} = (w_0, w_1, w_2, \ldots)$ such that*

1. *$p_{ij}(t) \to w_j$ as $t \to \infty$, for every $i, j \in \mathbb{N}$.*
2. *Either (a) $w_j = 0$ for all j, or (b) $\sum_{j=0}^{\infty} w_j = 1$ and $\boldsymbol{w}$ is the limiting distribution of $X(t)$.*
3. *If we are in case 2(b) then $\boldsymbol{w}$ is the unique equilibrium distribution, i.e., the solution to $w_j = \sum_i w_i p_{ij}(t)$.*

Proof. Omitted. □

Letting $t \to \infty$ in the backwards equations,

$$\begin{aligned} &\lim_{t\to\infty} p'_{ij}(t) = \mu_i w_j - (\lambda_i + \mu_i) w_j + \lambda_i w_j = 0, \quad \text{for } i \geq 1, \text{ and} \\ &\lim_{t\to\infty} p'_{0j}(t) = -\lambda_0 w_j + \lambda_0 w_j = 0. \end{aligned}$$

Now consider the forward equations, and let $t \to \infty$:

$$\begin{aligned} 0 &= \lambda_{j-1} w_{j-1} - (\lambda_j + \mu_j) w_j + \mu_{j+1} w_{j+1}, \quad \text{and} \\ 0 &= -\lambda_0 w_0 + \mu_1 w_1. \end{aligned}$$

Lemma 2.11. *These equations have the unique solution (given w_0)*

$$w_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j}\, w_0.$$

Proof. To see this use induction on j. The base case $j = 1$ follows from $0 = \lambda_0 w_0 + \mu_1 w_1$. Now suppose $j \geq 1$, and we know the result for w_{j-1}, w_j. Then

$$\begin{aligned}\mu_{j+1} w_{j+1} &= (\lambda_j + \mu_j) w_j - \lambda_{j-1} w_{j-1} \\ &= \frac{\lambda_0 \lambda_1 \cdots \lambda_j}{\mu_1 \mu_2 \cdots \mu_j} w_0 + \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_{j-1}} w_0 - \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_{j-1}} w_0 \quad = \frac{\lambda_0 \lambda_1 \cdots \lambda_j}{\mu_1 \mu_2 \cdots \mu_j} w_0.\end{aligned}$$

So

$$w_{j+1} = \frac{\lambda_0 \lambda_1 \cdots \lambda_j}{\mu_1 \mu_2 \cdots \mu_{j+1}} w_0.$$

□

2.4. **Queueing systems.** Customers wait in a queue to be served by a certain number of servers. Denote by $Q(t)$ the number of customers at time t. We assume $Q(0) = 0$. In this module we assume:

- If the nth customer arrives at time T_n, then the interarrival times $S_n = T_n - T_{n-1}$ are independent and identically distributed.
- Service is first-come, first-served, with a single queue.
- Service times are independent, identically distributed r.v's.

A queue is thus described by a triple $A/B/s$, where A describes the arrivals distribution, B the service time distribution, and s is the number of servers. Typically, A, B are:

- $M(\lambda)$ (memoryless or Markovian), i.e., following an $\text{Exp}(\lambda)$ distribution. (If $A = M(\lambda)$ the arrivals form a Poisson process of rate λ.)
- $D(d)$ (deterministic), i.e., taking value d with probability 1.
- G (general), i.e., some fixed but unspecified distribution.

2.4.1. $M(\lambda)/M(\mu)/1$ *queue.* In this case, interarrival times are $\text{Exp}(\lambda)$, service times are $\text{Exp}(\mu)$ and there is one server.

We claim that, for a $M(\lambda)/M(\mu)/1$ queue, $Q(t)$ is a birth-death process with $\lambda_n = \lambda$ for all $n \geq 0$, and $\mu_n = \mu$ for all $n \geq 1$. As usual, we consider what happens in a (short) time interval $(t, t+h]$.

- $\mathbb{P}\big(\text{one arrival in } (t, t+h]\big) = \lambda h + o(h)$. (Arrivals form a Poisson process of rate λ.)
- $\mathbb{P}\big(\text{service is completed in } (t, t+h]\big) = 1 - e^{-\mu h} = \mu h + o(h)$, assuming $Q(t) > 1$. (Service time is $\text{Exp}(\mu)$.)

The probability of more than one arrival/service-end in $(t, t+h]$ is $o(h)$. So, for $n \geq 1$,

$$\begin{aligned}\mathbb{P}(Q(t+h) = n+1 \mid Q(t) = n) &= \lambda h(1 - \mu h) + o(h) \\ &= \lambda h + o(h), \\ \mathbb{P}(Q(t+h) = n \mid Q(t) = n) &= (1 - \lambda h)(1 - \mu h) + o(h) \\ &= 1 - (\lambda + \mu) h + o(h), \\ \mathbb{P}(Q(t+h) = n-1 \mid Q(t) = n) &= (1 - \lambda h)\mu h + o(h) \\ &= \mu h + o(h).\end{aligned}$$

Also,

$$\mathbb{P}(Q(t+h) = 1 \mid Q(t) = 0) = \lambda h + o(h),$$
$$\mathbb{P}(Q(t+h) = 0 \mid Q(t) = 0) = 1 - \lambda h + o(h).$$

So we do have a birth-death process with the specified parameters.

We saw earlier that $p_{0j}(t) = \mathbb{P}(Q(t) = j) \to w_j$ as $t \to \infty$, where

$$\frac{\lambda_0\lambda_1 \dots \lambda_{j-1}}{\mu_1\mu_2 \dots \mu_j} w_0, \quad \text{as } t \to \infty.$$

In this case, $w_j = (\lambda/\mu)^j w_0$, for $j \geq 0$. For a limiting distribution we need $\sum_{j=0}^{\infty} w_j = 1$, i.e.,

$$w_0 \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = 1.$$

If $\lambda \geq \mu$ then the sum does not converge and we do not have a limiting distribution. (The expected length of the queue will tend to infinity with time.) If $\lambda < \mu$ then the geometric series converges to $\mu/(\mu - \lambda)$, and hence $w_0 = 1 - \lambda/\mu$. In this case there is a limiting distribution given by

$$\mathbb{P}(Q(t) = j) \to w_j = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j.$$

The limiting distribution of $Q(t)$ is essentially geometric; specifically $Q(t)+1 \sim \mathrm{Geom}(1-\lambda/\mu)$. At equilibrium, letting $\varrho = \lambda/\mu$,

$$\begin{aligned}
\mathbb{E}(Q(t)) &= \sum_{j=1}^{\infty} w_j j \\
&= \sum_{j=1}^{\infty} \varrho^j (1-\varrho) j \\
&= \varrho \sum_{j=1}^{\infty} \varrho^{j-1} (1-\varrho) j \\
&= \frac{\varrho}{1-\varrho}.
\end{aligned}$$

2.4.2. *$M(\lambda)/M(\mu)/s$ queue, $s > 1$.* If k servers are operating at time t then the probability that one becomes available in time interval $(t, t+h]$ is

$$k(\mu h)(1-\mu h)^{k-1} + o(h) = k\mu h + o(h),$$

and the probability that more than one becomes available is $o(h)$. The situation with arrivals is as with the $M(\lambda)/M(\mu)/1$ queue. Arguing as before,

$$\begin{aligned}
\mathbb{P}(Q(t+h) = n+1 \mid Q(t) = n) &= \lambda h + o(h), \\
\mathbb{P}(Q(t+h) = n-1 \mid Q(t) = n) &= \min\{n, s\}\, \mu h + o(h), \\
\mathbb{P}(Q(t+h) = n \mid Q(t) = n) &= 1 - (\lambda + \min\{n, s\}\mu)h + o(h).
\end{aligned}$$

So we see that $Q(t)$ is a birth-death process with $\lambda_k = \lambda$ for all $k \geq 0$, and

$$\mu_k = \begin{cases} s\mu, & \text{for } k \geq s; \\ k\mu, & \text{for } 0 \leq k < s. \end{cases}$$

So $\mathbb{P}(Q(t) = j) \to w_j$, where

$$w_j = \frac{\lambda_0 \lambda_1 \ldots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j} w_0 = \begin{cases} \dfrac{\lambda^j}{\mu(2\mu)\cdots(j\mu)} w_0 = \Big(\dfrac{\lambda}{\mu}\Big)^j \dfrac{1}{j!} w_0, & \text{if } 0 \leq j < s; \\ \dfrac{\lambda^j}{\mu(2\mu)\cdots(s\mu) \times (s\mu)^{j-s}} w_0 = \Big(\dfrac{\lambda}{\mu s}\Big)^j \dfrac{s^s}{s!} w_0, & \text{if } j \geq s. \end{cases}$$

For $(w_0, w_1, w_2, \ldots)$ to be a probability distribution we need

$$\sum_{j=0}^{\infty} w_j = \left[\sum_{j=0}^{s-1} \Big(\frac{\lambda}{\mu}\Big)^j \frac{1}{j!} + \frac{s^s}{s!} \sum_{j=s}^{\infty} \Big(\frac{\lambda}{\mu s}\Big)^j \right] w_0 = 1. \tag{4}$$

So a limiting distribution exists exactly when the geometric series above converges, i.e., when $\varrho = \lambda/s\mu < 1$. The parameter ϱ is the *traffic intensity.*

In principle, for any s, we can solve (4) for w_0, and hence determine $w_1, w_2, \ldots$. In practice, the working would get a little complicated for large s. Here we just look at $s = 2$ (i.e., the case of two servers); see Exercise Sheet 9 for the case $s = 3$.

2.4.3. *$M(\lambda)/M(\mu)/2$ queue.* In this case, $\lambda_k = \lambda$ for all k, $\mu_1 = \mu$, and $\mu_k = 2\mu$, for all $k \geq 2$. Then, for all $j \geq 1$,

$$w_j = \frac{\lambda_0 \lambda_1 \ldots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j} w_0 = 2\Big(\frac{\lambda}{2\mu}\Big)^j w_0 = 2\varrho^j w_0,$$

where $\varrho = \lambda/2\mu$. A limiting distribution exists when $\varrho < 1$. We require $\sum_{j=0}^{\infty} w_j = 1$, i.e.,

$$\left[1 + 2\sum_{j=1}^{\infty} \varrho^j\right] w_0 = \left[1 + \frac{2\varrho}{1-\varrho}\right] w_0 = \frac{1+\varrho}{1-\varrho} w_0 = 1,$$

and hence

$$w_0 = \frac{1-\varrho}{1+\varrho}.$$

So

$$\mathbb{P}(Q(t) = j) \to w_j = \begin{cases} \dfrac{1-\varrho}{1+\varrho}, & \text{if } j = 0; \\ 2\Big(\dfrac{1-\varrho}{1+\varrho}\Big)\varrho^j, & \text{if } j \geq 1. \end{cases}$$

At equilibrium,

$$\begin{aligned}\mathbb{E}(Q(t)) &= \sum_{j=1}^{\infty} 2\Big(\frac{1-\varrho}{1+\varrho}\Big)\varrho^j j \\ &= \frac{2\varrho}{1+\varrho}\sum_{j=1}^{\infty}(1-\varrho)\varrho^{j-1}j \\ &= \frac{2\varrho}{1-\varrho^2}.\end{aligned}$$

2.4.4. $M(\lambda)/M(\mu)/\infty$ *queue.* (Not physically reasonable, but might be a good approximation for large s.) We have

$$\begin{aligned}\lambda_k &= \lambda, \quad \text{for } k \geq 0; \\ \mu_k &= k\mu, \quad \text{for } k \geq 1.\end{aligned}$$

Thus,

$$w_j = \frac{\lambda^j}{j!\mu^j}\, w_0.$$

For a limiting distribution, we require $\sum_{j=0}^{\infty} w_j = 1$, i.e.,

$$w_0 \sum_{j=0}^{\infty} \frac{1}{j!}\Big(\frac{\lambda}{\mu}\Big)^j = 1.$$

So $w_0 = e^{-\lambda/\mu}$ and there is always a limiting distribution:

$$\mathbb{P}(Q(t)=j) \to w_j = e^{-\lambda/\mu}\frac{1}{j!}\Big(\frac{\lambda}{\mu}\Big)^j.$$

In other words, $Q(t) \sim \mathrm{Po}(\lambda/\mu)$ at equilibrium. The expected number of customers in the system is thus $\mathbb{E}(Q(t)) = \lambda/\mu$.

The queues we studied above ($M(\lambda)/M(\mu)/s$ and $M(\lambda)/M(\mu)/\infty$) are the only ones that can be modelled as birth-death processes. But some further examples can be treated using other ideas.

2.4.5. $M(\lambda)/D(d)/\infty$ *queue.* Suppose that $t \geq d$. The customers who are being processed at time t are the ones who arrived in the time interval $(t-d, t]$. (Compare this with the shop with Poisson arrivals in the coursework.) Thus

$$\mathbb{P}(Q(t)=j) = \mathbb{P}\big(j \text{ arrivals in } (t-d,t]\big) = e^{-\lambda d}\frac{(\lambda d)^j}{j!};$$

in other words, $Q(t) \sim \mathrm{Po}(\lambda d)$. (This is an exact result, not just a description of what happens in the limit.)

2.4.6. *$M(\lambda)/G/\infty$ queue.* The distribution of service times is $\mathcal{Y}$, where $\mathcal{Y}$ is an arbitrary distribution with finite expectation. Let $A(t)$ be the number of arrivals in $(0,t]$. By the Law of Total Probability

$$\mathbb{P}(Q(t)=m)=\sum_{n=m}^{\infty}\mathbb{P}(Q(t)=m\mid A(t)=n)\,\mathbb{P}(A(t)=n). \tag{5}$$

From earlier work on the Poisson process, we know that, conditioned on $A(t)=n$, the n arrivals in the interval $(0,t]$ are distributed as n independent, Uniform$(0,t]$, random variables. Consider one of these arrivals. The probability that it is still present at time t is $p=\mathbb{P}(U+Y>t)$, where $U\sim$ Uniform$(0,t]$ and $Y\sim\mathcal{Y}$, and U and Y are independent. The probability that m of the n arrivals are still being processed at time t is distributed binomially, with success probability p, thus

$$\mathbb{P}(Q(t)=m\mid A(t)=n)=\binom{n}{m}p^m(1-p)^{n-m}.$$

Also, since the arrivals form a Poisson process, $A(t)\sim\mathrm{Po}(\lambda t)$, so that

$$\mathbb{P}(A(t)=n)=e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Substituting the above two expressions in (5),

$$\begin{aligned}\mathbb{P}(Q(t)=m)&=\sum_{n=m}^{\infty}\binom{n}{m}p^m(1-p)^{n-m}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\\&=\frac{p^m(\lambda t)^m}{m!}\,e^{-\lambda t}\sum_{n=m}^{\infty}\frac{(1-p)^{n-m}(\lambda t)^{n-m}}{(n-m)!}\\&=\frac{p^m(\lambda t)^m}{m!}\,e^{-\lambda t}e^{(1-p)\lambda t}\\&=e^{-p\lambda t}\,\frac{(p\lambda t)^m}{m!}.\end{aligned}$$

Equivalently, $Q(t)\sim\mathrm{Po}(p\lambda t)$.

We still need to determine $p=\mathbb{P}(U+Y>t)$.

$$\begin{aligned}p&=\mathbb{P}(Y>t-U)\\&=\int_0^t\frac{1}{t}\,\mathbb{P}(Y>t-u)\,du\\&=\frac{1}{t}\int_0^t\mathbb{P}(Y>s)\,ds\end{aligned}$$

Now, $\mathbb{E}(Y)=\int_0^\infty(1-F_Y(s))\,ds=\int_0^\infty\mathbb{P}(Y>s)\,ds$, so $pt\to\mathbb{E}(Y)$ as $t\to\infty$. Thus, the limiting distribution for $Q(t)$ is $\mathrm{Po}(\lambda\,\mathbb{E}(Y))$. Note that this agrees with our earlier results for $M(\lambda)/M(\mu)/\infty$ (where $\mathbb{E}(Y)=\mu^{-1}$) and $M(\lambda)/D(d)/\infty$ (where $\mathbb{E}(Y)=d$).