MTH6141 MAY 2012 EXAMINATION: SPECIMEN SOLUTIONS

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Q1 (a) [Standard definitions.] (X_t) is a Markov chain if

$$\Pr(X_{t+1} = s \mid X_t = s_t, X_{t-1} = s_{t-1}, \dots, X_0 = s_0) = \Pr(X_{t+1} = s \mid X_t = s_t),$$

for all $t \in \mathbb{N}$ and all $s, s_t, s_{t-1}, \dots, s_0 \in S$ for which the conditional probabilities are defined.

$P = (p_{ij} : 1 \le i, j \le n)$ is the matrix with $p_{ij} = \Pr(X_{t+1} = j \mid X_t = i)$ for all i, j.

(b) [Special case of bookwork.] Using the Law of Total Probability and the Markov property,

$$\begin{aligned}
&\Pr(X_{t+2} = j \mid X_t = i)\\
&\quad= \sum_{k=1}^{n} \Pr(X_{t+1} = k \mid X_t = i) \Pr(X_{t+2} = j \mid X_{t+1} = k, X_t = i)\\
&\quad= \sum_{k=1}^{n} \Pr(X_{t+1} = k \mid X_t = i) \Pr(X_{t+2} = j \mid X_{t+1} = k)\\
&\quad= \sum_{k=1}^{n} p_{ik} p_{kj} = (P^2)_{ij}.
\end{aligned}$$

(c) [Easy calculations.]

(i) $\Pr(X_1 = 2 \mid X_0 = 1) = p_{1,2} = \frac{1}{6}$ [1 mark];

(ii) $\Pr(X_2 = 3 \mid X_1 = 2, X_0 = 1) = p_{2,3} = \frac{1}{3}$ (Markov property) [1 mark];

(iii) $\Pr(X_2 = 3, X_1 = 2 \mid X_0 = 1) = \Pr(X_1 = 2 \mid X_0 = 1) \Pr(X_2 = 3 \mid X_1 = 2, X_0 = 1) = p_{1,2} p_{2,3} = \frac{1}{6}\frac{1}{3} = \frac{1}{18}$ [2 marks]; and

(iv) $\Pr(X_2 = 3 \mid X_0 = 1) = (P^2)_{1,3} = \frac{1}{3}\frac{1}{2} + \frac{1}{6}\frac{1}{3} + \frac{1}{2}\frac{2}{3} = \frac{5}{9}$ [2 marks].

(d) [Similar things appeared in the coursework.]

$$\begin{aligned}
\Pr(X_{100} = 1 \mid X_0 = 1) &= \sum_{k=1}^{3} \Pr(X_{99} = k \mid X_0 = 1) \Pr(X_{100} = 1 \mid X_{99} = k)\\
&= \sum_{k=1}^{3} \Pr(X_{99} = k \mid X_0 = 1) \times \tfrac{1}{3}\\
&= \tfrac{1}{3}.
\end{aligned}$$

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Q2 (a) [Standard definition.] The t step transition probabilities are

$$p_{ij}^{(t)} = \Pr(X_t = j \mid X_0 = i),$$

for all $i, j \in S$.

(b) [A similar example appeared in lectures.] Let the states be labelled $\{0, 1, 2, 3\}$ according to the number of balls in the first urn. Then

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(c) [Standard definition.] A Markov chain is irreducible if, for all $i, j \in S$, there exists t such that $p_{ij}^{(t)} > 0$. There are transitions with non-zero probability $0 \to 1 \to 2 \to 3$ and $3 \to 2 \to 1 \to 0$, so it is possible to go from any state to any other in at most 3 transitions.

(d) [Easy calculation.]

$$\begin{aligned} w_0 &= \tfrac{1}{3} w_1, \\ w_1 &= w_0 + \tfrac{2}{3} w_2, \\ w_2 &= \tfrac{2}{3} w_1 + w_3, \text{ and} \\ w_3 &= \tfrac{1}{3} w_2. \end{aligned}$$

Thus $\boldsymbol{w} = (1, 3, 3, 1) w_0$. For $\boldsymbol{w}$ to be a probability distribution, $w_0 = \frac{1}{8}$. Thus $\boldsymbol{w} = (\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$ is an equilibrium distribution.

(e) [Standard fact from the course.] Every irreducible Markov chain has a unique equilibrium distribution, so the vector $\boldsymbol{w}$ from part (d) is unique.

Q3 (a) [Standard definition.] $f_{ii} = \sum_{t=1}^{\infty} f_{ii}^{(t)}$, where $f_{ii}^{(t)} = \Pr(X_t = i, X_{t-1} \neq i, \ldots, X_1 \neq i \mid X_0 = i)$. State i is recurrent iff $f_{ii} = 1$.

(b) [Standard result from the course.] State i is recurrent iff $\sum_{t=1}^{\infty} p_{ii}^{(t)} = \infty$.

(c) [Bookwork.] Since $\sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n}$, the average of the coefficients $\binom{2n}{k}$ taken over $0 \leq k \leq 2n$ is $2^{2n}/(2n+1)$. Now the largest of these binomial coefficients is $\binom{2n}{n}$ so it follows that $\binom{2n}{n} \geq 2^{2n}/(2n+1)$.

In order to be back at state 0 after $2n$ steps, the Markov chain must make n increasing transitions, and n decreasing. There are $\binom{2n}{n}$ possible ways to choose such a sequence of transitions. Each such sequence has a probability 2^{-2n} of occurring. Thus

$$p_{00}^{(2n)} = \binom{2n}{n} 2^{-2n} \geq \frac{2^{2n}}{2n+1} \times 2^{-2n} = \frac{1}{2n+1}.$$

Now

$$\sum_{t=1}^{\infty} p_{00}^{(t)} \geq \sum_{n=1}^{\infty} p_{00}^{(2n)} \geq \sum_{n=1}^{\infty} \frac{1}{2n+1} \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

By part (b), state 0 is recurrent.

(d) [Unseen.] The communicating classes are $\{0\}$, $\{1, 2, 3, \ldots\}$ and $\{-1, -2, -3, \ldots\}$. (State 0 is absorbing, so the Markov chain cannot escape or pass through it.) $\{0\}$ is clearly recurrent. State 1 is transient since $f_{11} \leq \frac{1}{2} < 1$, so the communicating class $\{1, 2, 3, \ldots\}$ is transient, as transience is a class property. The communicating class $\{-1, -2, -3, \ldots\}$ is transient, by symmetry.

Q4 (a) [Standard fact.] $X(t) \sim \mathrm{Po}(\lambda t)$.

(b) [Special case of a result from the course.]

$$\begin{aligned}\Pr(X(t)=k \mid X(2t)=n) &= \frac{\Pr(X(t)=k, X(2t)=n)}{\Pr(X(2t)=n)} \\ &= \frac{\Pr(X(t)-X(0)=k)\Pr(X(2t)-X(t)=n-k)}{\Pr(X(2t)-X(0)=n)} \\ &= \frac{\left[e^{-\lambda t}(\lambda t)^k/k!\right] \times \left[e^{-\lambda t}(\lambda t)^{n-k}/(n-k)!\right]}{e^{-2\lambda t}(2\lambda t)^n/n!} \\ &= \binom{n}{k}2^{-n}.\end{aligned}$$

So conditioned on $X(2t)=n$, $X(t) \sim \mathrm{Bin}(n, \frac{1}{2})$.

(c) [Routine calculations, such as the students have sen before.] Let $A(t)$ and $B(t)$ be the Poisson processes representing the goals scored by the two teams.

(i) $B(\frac{3}{2}) - B(0) \sim \mathrm{Po}(\frac{3}{2})$, so the expected number is $\frac{3}{2}$.

(ii)

$$\begin{aligned}\Pr(A(\tfrac{3}{4})=1, A(\tfrac{3}{2})-A(\tfrac{3}{4})=1) &= \Pr(A(\tfrac{3}{4})=1)\Pr(A(\tfrac{3}{2})-A(\tfrac{3}{4})=1) \\ &= \tfrac{3}{2}e^{-\frac{3}{2}} \times \tfrac{3}{2}e^{-\frac{3}{2}} \\ &= \tfrac{9}{4}e^{-3}.\end{aligned}$$

(iii) $A(\frac{3}{2}) + B(\frac{3}{2}) \sim \mathrm{Po}(3 \times \frac{3}{2}) = \mathrm{Po}(\frac{9}{4})$ (superposition of two Poisson processes).

(d) [Unseen.] The half-time score must have been 1–0 to B. So the required probability is

$$\Pr(A(\tfrac{3}{4})=0 \mid A(\tfrac{3}{2})=3)\Pr(B(\tfrac{3}{4})=1 \mid B(\tfrac{3}{2})=1) = (\tfrac{1}{2})^3 \times \tfrac{1}{2} = \tfrac{1}{16},$$

using part (b).

Q5 (a) [Standard definition.] An $M(\lambda)/M(\mu)/1$ queueing system has a single queue, arrivals forming a $\mathrm{Po}(\lambda)$ process, and service times distributed $\mathrm{Exp}(\mu)$.

(b) [Bookwork.] The appropriate parameters are $\lambda_0, \lambda_1, \lambda_2, \cdots = \lambda$ (in time interval $(t, t+h]$ the probability on an arrival is $\lambda h + o(h)$), and $\mu_0 = 0$ and $\mu_1 = \mu_2 = \cdots = \mu$ (in time interval $(t, t+h]$ the probability on a departure is $\mu h + o(h)$, provided the queue is non-empty).

(c) [Bookwork.] According to the fact given in the question, $w_j = (\lambda/\mu)^j w_0$, for $j \geq 0$. For a limiting distribution we need $\sum_{j=0}^{\infty} w_j = 1$, i.e.,

$$w_0 \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = 1.$$

If $\lambda < \mu$ then the geometric series converges to $\mu/(\mu - \lambda)$, and hence $w_0 = 1 - \lambda/\mu$. In this case there is a limiting distribution given by

$$\Pr(Q(t) = j) \to w_j = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j.$$

The limiting distribution of $Q(t)$ is essentially geometric; specifically $Q(t)+1 \sim \mathrm{Geom}(1-\lambda/\mu)$.

(d) [Unseen, though equivalent to an $M(\lambda)/M(\mu)/\infty$ queueing system.] The appropriate parameters are $\lambda_0, \lambda_1, \lambda_2, \cdots = \lambda$ (in time interval $(t, t+h]$ the probability on a arrival is $\lambda h + o(h)$), and $\mu_0 = 0$ and $\mu_1 = \mu$, $\mu_2 = 2\mu$, $\mu_3 = 3\mu$, etc. (in time interval $(t, t+h]$ the probability of a death is $k\mu h + o(h)$, where k is the current population). For a limiting distribution we need $\sum_{j=0}^{\infty} w_j = 1$, i.e.,

$$w_0 \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j = 1.$$

Thus $w_0 = \exp(-\lambda/\mu)$ and there is always a limiting distribution, namely $\mathrm{Po}(\lambda/\mu)$.