Q1. (a) Probability of being absorbed at state k given the MC starts in state i. (3)

(b) $a_k = 1$ and $a_i = 0$ for $i \in A \setminus \{k\}$

For ~~$i \notin A$~~ $i \in S \setminus A$, condition on the first step, say $X_1 = j$ and apply the Law of Total Probability:

$$a_i = Pr(X_T = k \mid X_0 = i) = \sum_{j \in S} Pr(X_T = k \mid X_1 = j) Pr(X_1 = j \mid X_0 = i)$$

$$= \sum_{j \in S} p_{ij} a_j = \sum_{j \in S \setminus A} p_{ij} a_j + p_{ik}.$$ ~~(4)~~ (5)

(c) From every state it is possible to reach an absorbing state (all states transient). (2)

(d)
$$a_1 = \tfrac{3}{5} a_1 + \tfrac{1}{5} a_2 \quad \Rightarrow \quad 2a_1 = a_2 \quad (A)$$
$$a_2 = \tfrac{1}{4} a_1 + \tfrac{1}{2} a_2 + \tfrac{1}{4} \quad \Rightarrow \quad 2a_2 = a_1 + 1 \quad (B)$$

(A) + (B) $\Rightarrow$ $4a_1 = a_1 + 1$ $\Rightarrow$ $a_1 = 1/3$ (5)

Prob^{y} absorption in state 4 given $X_0 = 1$ is $1/3$

(e) Make i, j into absorbing states by setting $p_{ii} = 1$, $p_{jj} = 1$ & $p_{ii'} = 0 \ \forall i' \neq i$, $p_{jj'} = 0 \ \forall j' \neq j$.

Required prob^{y} is prob^{y} of absorption in state i (4)

[(a)–(c) bookwork, (d) routine calculation, (e) unseen.]

Q2(a) (i)

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

(3 marks)

(ii) $\underline{w} P = \underline{w} \Rightarrow \left.\begin{array}{l} w_1 = \frac{1}{2}(w_2 + w_3) = w_4 \\ w_2 = \frac{1}{2}(w_1 + w_4) = w_3 \end{array}\right\} \Rightarrow w_1 = w_2 = w_3 = w_4$

(3 marks)

For $\underline{w}$ to be a prob^y dist^n, $w_1 + w_2 + w_3 + w_4 = 1 \Rightarrow$ $w_1 = w_2 = w_3 = w_4 = 1/4$. So $(1/4, 1/4, 1/4, 1/4)$ is the (unique) equilibrium dist^n.

(iii) $\underline{\mu}^{(1)} = (0, 1/2, 1/2, 0)$, $\underline{\mu}^{(2)} = (1/2, 0, 0, 1/2)$. Since $\underline{\mu}^{(t+2)} = \underline{\mu}^{(t)}$ $\forall t$ and $\underline{\mu}^{(1)} \neq \underline{\mu}^{(0)}$, $\mu^{(t)}$ does not converge. So P does not have a limiting dist^n. (4 marks)

(b) (i)

$$P = \begin{pmatrix} 2/3 & 1/6 & 1/6 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/6 & 1/6 & 2/3 \end{pmatrix}$$

(4 marks)

(ii) $(w, w', w', w) P = (w, w', w', w) \Rightarrow w = \frac{2}{3} w + \frac{1}{3} w' + \frac{1}{3} w'$
$\Rightarrow w = 2w'$. Also $2w + 2w' = 1$. So
$\underline{w} = (\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3})$ is an equilibrium dist^n. ~~[It is unique as P is irreducible (see (iii))]~~. (3 marks)

(iii) P is regular (P^2 has all entries > 0). So the (unique) equilibrium dist^n from (ii) is also the limiting dist^n. (3 marks)

[Fairly routine application of definitions, concepts.]

Q3 (a) $i, j \in S$ intercommunicate if $\exists\, r, s \in \mathbb{N}$ st $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$

(b) $f_{kk}^{(t)} = \mathbb{P}(X_1 \neq k, X_2 \neq k, \ldots, X_{t-1} \neq k, X_t = k \mid X_0 = k)$

$f_{kk} = \sum_{t=1}^{\infty} f_{kk}^{(t)}$. k is recurrent if $f_{kk} = 1$.

(d) Communicating classes are $\{1,2\}$ & $\{3,4,5\}$.
$(1 \leftrightarrow 2;\ 3 \to 4 \to 5 \to 3,\ 3 \not\to 1)$ (2 marks)

$f_{11}^{(1)} = \frac{1}{2}$, $f_{11}^{(2)} = p_{12} p_{21} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$, $f_{11}^{(t)} = 0 \ (t \geq 3)$

$f_{11} = f_{11}^{(1)} + f_{11}^{(2)} = \frac{1}{2} + \frac{1}{3} = 5/6 < 1$. (3 marks)

State 1 is transient since $f_{11} < 1$. (1 mark)

$$p_{33}^{(t)} = \sum_{k=1}^{5} p_{3k}^{(t-1)} \cdot p_{k3}$$
$$= p_{33}^{(t-1)} \cdot \frac{1}{3} + p_{34}^{(t-1)} \cdot \frac{1}{3} + p_{35}^{(t-1)} \frac{1}{3}.$$

Since $p_{31}^{(t-1)} = p_{32}^{(t-1)} = 0$,

$$p_{33}^{(t)} = \frac{1}{3}\left(p_{33}^{(t-1)} + p_{34}^{(t-1)} + p_{35}^{(t-1)}\right) = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

for all $t \geq 1$. (4 marks)

Thus $\sum_{t=1}^{\infty} p_{33}^{(t)}$ diverges, and 3 is recurrent by (c). (1 mark)

(c) k is recurrent iff $\sum_{t=1}^{\infty} p_{kk}^{(t)}$ diverges.

[(a)–(c) are bookwork, (d) is routine calculation, though students need to perceive the structure in P!]

Q4 (a) $p_0(t+h) = \mathbb{P}(X(t+h)=0)$
$= \mathbb{P}(X(t)=0)\,\mathbb{P}(X(t+h)=0 \mid X(t)=0)$
$= p_0(t)(1-\lambda h + o(h))$ (3 marks)

Thus, $\dfrac{p_0(t+h)-p_0(t)}{h} = -\lambda p_0(t) + \dfrac{o(h)}{h}$

and, letting $h \to 0$, $p_0'(t) = -\lambda p_0(t)$. (2 marks)

The sol$\underline{n}$ is $p_0(t) = ce^{-\lambda t}$ for some c
But $X(0) = 1$, so $p_0(0) = 1$ and $c = 1$. (2 marks)

(b) In general, $p_k'(t) = \lambda p_{k-1}(t) - \lambda p_k(t)$, for $k \geq 1$.

(c) $X(t) \sim Po(\lambda t)$.

(d) (i) $\lambda = 12$ and $t = 1/12$, so $\mathbb{P}(X(\frac{1}{12})=0) =$
$e^{-\lambda t} = e^{-1}$. So $\mathbb{P}(X(\frac{1}{12})>0) = 1-e^{-1}$.
(Prob$\underline{y}$ of at least one #25 is $1-e^{-1}$) (3 marks)

(ii) $X(\frac{1}{6}) - X(\frac{1}{12})$ and $X(\frac{1}{12}) - X(0)$ are independent r.v's with same dist$\underline{n}$ so prob$\underline{y}$ of at least one #25 is $1-e^{-1}$ as in (d)(i) (2 marks)

(iii) Arrivals of all buses is the superposition of indepentent Poisson processes with rates $\frac{\cancel{\ }}{12}$ & $\frac{\cancel{\ }}{8}$, and hence is a P.P. of rate $12+8 = 20$. By same independence argument as (d)(ii), prob$\underline{y}$ of at least one bus in the five-minute interval is $1-e^{-20/12} = 1-e^{-5/3}$. (4 marks)

[(a)-(c) are bookwork, (d) is similar to coursework.]

Q5 (a) In this case $w_k = \frac{\lambda^k}{\mu^k} w_0 = \left(\frac{\lambda}{\mu}\right)^k w_0$. For $(w_0, w_1, w_2, \dots)$ to be a prob^{y} dist^{n} we need $\sum_{k=0}^{\infty} w_k = 1$ i.e., $\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k w_0 = 1$. If $\left(\frac{\lambda}{\mu}\right) < 1$ then $w_0^{-1} = \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k < \infty$ and we have a limiting dist^{n}. Otherwise, $\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k$ diverges and we do not. So condition is $\lambda < \mu$.

(b) A new customer arrives in time interval $(t, t+h]$ with prob^{y} $\lambda h + o(h)$; thus $\lambda_k = \lambda \ \forall k \geq 0$. If one customer is being served at time t, the prob^{y} he exits during $(t, t+h]$ is $\mu h + o(h)$. If two or more are being served then the prob^{y} ~~that~~ one exits is $2\mu h + o(h)$. So $\mu_1 = \mu$ and $\mu_k = 2\mu$, $k \geq 2$. 4 marks

From the formula, $w_1 = \frac{\lambda_0}{\mu_1} w_0 = \frac{\lambda}{\mu} w_0$, $w_2 = \frac{\lambda_1}{\mu_2} w_1 = \frac{\lambda}{2\mu} w_1$ and, more generally, $w_k = \frac{\lambda_{k-1}}{\mu_k} w_{k-1} = \frac{\lambda}{2\mu} w_{k-1} \ \forall k \geq 2$. For $(w_0, w_1, \dots)$ to be a prob^{y} dist^{n}, $1 = w_0 + w_1 + w_2 + \dots = w_0\left(1 + \frac{\lambda}{\mu} + \frac{\lambda}{\mu}\left(\frac{\lambda}{2\mu}\right) + \frac{\lambda}{\mu}\left(\frac{\lambda}{2\mu}\right)^2 + \ \right) = w_0\left[1 + \frac{\lambda}{\mu}\left(1 + \frac{\lambda}{2\mu} + \left(\frac{\lambda}{2\mu}\right)^2 + \dots\right)\right]$

If $\left(\frac{\lambda}{2\mu}\right) < 1$ then the geometric series converges and we get an equilibrium dist^{n}. Otherwise it diverges, and we don't. So the condition is $\lambda < 2\mu$. 6 marks

(c) When $\lambda = \mu = 1$, the limiting dist^{n} is $\underline{w} = w_0\left(1, \frac{\lambda}{\mu}, \frac{\lambda}{\mu}\left(\frac{\lambda}{2\mu}\right), \dots\right)$ ie $w_0(1, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$. Hence $w_0 = 1/3$ and $\underline{w} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\cdot\frac{1}{2}, \frac{1}{3}\cdot\frac{1}{4}, \frac{1}{3}\cdot\frac{1}{8}, \frac{1}{3}\cdot\frac{1}{16}, \dots\right)$ ~~The expected # customers waiting to be served is~~ $\cancel{\frac{1}{3}\left(\frac{1}{2}\cdot 1 + \frac{1}{4}2 + \frac{1}{8}3 + \right) = 2/3}$.

[(a) + (b) bookwork, (c) routine calculation.]