Queen Mary
University of London

B.Sc. EXAMINATION BY COURSE UNITS

MTH6126 Metric Spaces

Specimen paper

Disclaimer. This paper has not been subjected to the rigorous scrutiny and checking regime that is being applied to the actual paper. Please notify the course organiser of any errors. I do not know at this stage exactly what the rubric of the actual paper will say. However, Section A will contain four questions carrying 10 marks each, all of which will count; and Section B will contain three questions carrying 30 marks each, of which the two best will count.

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

SECTION A

This section carries 40 marks and each question carries 10 marks. You should attempt ALL FOUR questions.

A1. Consider the following three possible definitions of a function $\varrho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. (In each case, the independent variables $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are points in $\mathbb{R}^2$.) None of the functions defined in (a)–(c) are metrics on $\mathbb{R}^2$. In each case identify one of the axioms for a metric that is violated, and present an example to show that it is violated.

(a) $\varrho(a, b) = \min\{|a_1 - b_1|, |a_2 - b_2|\}$.

(b) $\varrho(a, b) = (a_1 - b_1)^2 + (a_2 - b_2)^2$.

(c) $\varrho(a, b) = \max\{b_1 - a_1, 2(a_1 - b_1), b_2 - a_2, 2(a_2 - b_2)\}$.

A2. Explain what it means for a sequence x_n in a metric space (X, ϱ) to *converge to a limit* $\alpha \in X$.

Let $B[0, 1]$ denote the space of bounded real functions on $[0, 1]$ with the sup (or uniform) metric. Which of the following sequences of functions converges to a limit in $B[0, 1]$? For those that converge, state what that limit is.

(a) $f_n(x) = x \exp(-nx)$.

(b) $f_n(x) = x \exp(-x/n)$.

(c) $f_n(x) = nx \exp(-nx)$.

A3. Suppose $f : X \to Y$ is a map between metric spaces (X, ϱ) and (Y, σ).

(a) Define the inverse image $f^{-1}(B)$ of a set $B \subseteq Y$.

(b) Write down a definition of *continuity* of f in terms of inverse images of sets.

(c) Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/x$ if $x \neq 0$, and $f(0) = 0$. (Assume $\mathbb{R}$ has the usual metric.) Give an open set $B \subseteq \mathbb{R}$ such that $A = f^{-1}(B)$ is *not* open. Briefly justify your claim that B is open and A is not.

A4. Explain what it means for a function f to be a *contraction* on a metric space.

State the contraction mapping theorem.

Suppose that $f : [a, b] \to \mathbb{R}$ is a contraction on the closed interval $[a, b]$ (with the usual metric). Prove that the function $g(x) = \frac{1}{2}x + \frac{1}{2}f(x)$ is also a contraction on $[a, b]$.

[Next section overleaf]

SECTION B

This section carries 60 marks and each question carries 30 marks. You may attempt all three questions but only marks for the BEST TWO questions will be counted.

B1. (a) [7 marks] Define $\delta : \mathbb{R}^2 \to \mathbb{R}$ by

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise.} \end{cases}$$

Prove that δ is a metric on $\mathbb{R}$.

(b) [5 marks] Define $\varrho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $\varrho(a, b) = |a_1 - b_1| + \delta(a_2, b_2)$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Prove that ϱ satisfies the triangle inequality on $\mathbb{R}^2$.

(c) [8 marks] In fact, ϱ is a metric on $\mathbb{R}^2$. Describe the open balls $B_1(\mathbf{0})$ and $B_2(\mathbf{0})$ of radius 1 and 2 centred at the origin $\mathbf{0} = (0, 0)$.

(d) [10 marks] Which of the following sets are open in the metric space $(\mathbb{R}^2, \varrho)$?

- $(0, 1) \times (0, 1)$,
- $(0, 1) \times [0, 1]$, and
- $[0, 1] \times [0, 1]$.

Briefly explain each of your answers.

B2. For any set S, let $B(S)$ denote the space of bounded real functions on S with the sup (or uniform) metric.

(a) [6 marks] Explain what it means for a sequence of functions f_n in $B(S)$ to converge *pointwise* to a function $f \in B(S)$.

(b) [10 marks] Let f_n be any Cauchy sequence of functions in $B(S)$. Prove that f_n converges pointwise to some function $f : S \to \mathbb{R}$. [Hint. Start by showing that $f_n(\alpha)$ is a Cauchy sequence in $\mathbb{R}$, for any $\alpha \in S$.]

(c) [10 marks] Let f_n be as in part (b). Let $\varepsilon > 0$, and suppose N_ε is chosen so that

$$\sup_{x \in S} |f_n(x) - f_m(x)| < \varepsilon/2, \quad \text{for all } n, m \geq N_\varepsilon.$$

Prove that

$$\sup_{x \in S} |f_n(x) - f(x)| < \varepsilon, \quad \text{for all } n \geq N_\varepsilon.$$

(d) [4 marks] Deduce that $B(S)$ is a complete metric space.

[Next question overleaf]

B3. (a) [4 marks] Explain what it means for a subset K of a metric space (X, ϱ) to be (sequentially) *compact*.

(b) [9 marks] From first principles — i.e., directly from the definition you gave in part (a) — prove that the following subsets of $\mathbb{R}$ are not compact (with the usual metric).

- $[0, \infty)$.
- $[0, 1)$,
- $\mathbb{Q} \cap [0, 1]$.

(c) [10 marks] Prove that any closed subset of a compact set is compact.

(d) [7 marks] Prove that $K = \{1/n : n = 1, 2, 3, \ldots\} \cup \{0\}$ is a compact subset of $\mathbb{R}$ with the usual metric. You may assume that any closed interval $[a, b]$ in $\mathbb{R}$ is compact.

[Next section overleaf]

Specimen solutions

SECTION A

A1. (a) $\varrho((0,1),(0,0)) = \min\{|0-0|,|1-0|\} = 0$, violating the axiom $\varrho(x,y) = 0 \implies x = y$. (The triangle inequality is also violated.)

(b) $\varrho((-1,0),(0,0)) + \varrho((0,0),(1,0)) = 1+1 = 2 < 4 = \varrho((-1,0),(1,0))$, violating the triangle inequality.

(c) $\varrho((0,0),(1,0)) = \max\{1,-2,0,0\} = 1 \neq 2 = \max\{-1,2,0,0\} = \varrho((1,0),(0,0))$, violating symmetry.

A2. (a) Converges to the function $f(x) = 0$.

(b) Converges to the function $f(x) = x$.

(c) Does not converge.

A3. (a) [Standard definition.]

(b) [Bookwork.]

(c) Let $B = (-1,1)$. Then $A = (-\infty,1) \cup \{0\} \cup (1,\infty)$. B is an open ball in $\mathbb{R}$ and hence an open set. A contains no open ball centred at 0 and so is not open.

A4. [Standard definition.]

[Bookwork.]

$$a = \tfrac{1}{2}a + \tfrac{1}{2}a \leq \tfrac{1}{2}x + \tfrac{1}{2}f(x) = g(x) = \tfrac{1}{2}x + \tfrac{1}{2}f(x) \leq \tfrac{1}{2}b + \tfrac{1}{2}b = b$$

and

$$|g(x) - g(y)| = \left|\tfrac{1}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}f(x) - \tfrac{1}{2}f(y)\right| \leq \tfrac{1}{2}|x-y| + \tfrac{1}{2}c|x-y| = \tfrac{1}{2}(1+c)|x-y|,$$

where c is the contraction constant for f. But $c < 1$ and hence $(1+c) < 1$.

SECTION B

B1. (a) By inspection $\delta(x,y) \geq 0$ and $\delta(x,y) = 0 \implies x = y$; also $\delta(x,y) = \delta(y,x)$. (This part just tests whether you recall the definition of metric.) So we just need to check the triangle inequality

$$\delta(x,y) + \delta(y,z) \geq \delta(x,z).$$

If $x = z$ then the r.h.s. is 0 and the inequality is automatically satisfied. So assume $x \neq z$, i.e., $\delta(x,z) = 1$. In this case, either $x \neq y$ or $y \neq z$ (or both) and hence the l.h.s. is at least 1.

[This question continues overleaf . . .]

(b)

$$\begin{aligned}\varrho(a,b)+\varrho(b,c) &= |a_1-b_1|+\delta(a_2,b_2)+|b_1-c_1|+\delta(b_2,c_2)\\ &= |a_1-b_1|+|b_1-c_1|+\delta(a_2,b_2)+\delta(b_2,c_2)\\ &\le |a_1-c_1|+\delta(a_2,c_2)\\ &= \varrho(a,c),\end{aligned}$$

where we have used the triangle inequalities for $|\cdot|$ and δ

(c)
- $B_1(\mathbf{0}) = (-1,1)\times\{0\}$.
- $B_2(\mathbf{0}) = (-2,2)\times\{0\}\ \cup\ (-1,1)\times\mathbb{R}$.

(d) Note first that the set $B_\varepsilon((x,y)) = (x-\varepsilon, x+\varepsilon)\times\{y\}$ is an open ball in the metric ϱ, for any $0<\varepsilon<1$.

- Yes. Each point (x,y) in the set contains the open ball $B_\varepsilon((x,y))$ where $\varepsilon = \min\{x, 1-x\} \ge 0$.
- Yes. Same reason.
- No. The point $(0,0)$ in not the centre of any open ball (which would necessarily contain points of the form $(x,0)$ for $x<0$).

B2. I try to avoid setting questions which are largely or wholly bookwork, but sometimes it happens anyway.

(a) [Standard definition.]

(b) [This part and the following two lead you through a proof from the course (Proof of Theorem 6.7 in the printed notes). One strategy is to memorise the proofs, but it's much better to memorise the key ideas. The steps in the question are there to help you reconstruct the proof.]

B3. (a) [Standard definition.]

(b)
- Consider the sequence $x_n = n$. Since $|x_n - x_m| \ge 1$ for all $n \ne m$, this sequence is not Cauchy, nor is any subsequence. Since x_n has no Cauchy subsequence, it has no convergent subsequence.
- Consider the sequence $x_n = 1 - 1/n$. This sequence converges to 1 in $\mathbb{R}$, and hence so does any subsequence. But $1 \notin (0,1)$.
- Consider any sequence x_n of rationals in the interval $[0,1]$ converging to $1/\sqrt{2}$ in $\mathbb{R}$. E.g., for each n chose some rational number $x_n \in [0,1]$ with $|x_n - 1/\sqrt{2}| \le 1/n$. Any subsequence of x_n converges also to $1/\sqrt{2} \notin mathbbQ \cap [0,1]$

[This question continues overleaf . . .]

(c) [Bookwork.]

(d) K is a subset of $[0,1]$ so by (c) we just need to show that K is closed. The complement of K in $[0,1]$ is $(1/2,1) \cup (1/3,1/2) \cup (1/4,1/3) \cdots$, which is a union of open sets and hence is open. Hence K is closed in $[0,1]$. (Also OK to argue that K is closed in $\mathbb{R}$.)

[End of examination paper]