

B.Sc. EXAMINATION BY COURSE UNITS

MTH6126 Metric Spaces

Specimen paper

Disclaimer. This paper has not been subjected to the rigorous scrutiny and checking regime that is being applied to the actual paper. Please notify the course organiser of any errors. I do not know at this stage exactly what the rubric of the actual paper will say. However, Section A will contain four questions carrying 10 marks each, all of which will count; and Section B will contain three questions carrying 30 marks each, of which the two best will count.

The duration of this examination is 2 hours.

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

SECTION A

This section carries 40 marks and each question carries 10 marks. You should attempt ALL FOUR questions.

- A1.** Consider the following three possible definitions of a function $\varrho : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$. (In each case, the independent variables $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are points in \mathbb{R}^2 .) None of the functions defined in (a)–(c) are metrics on \mathbb{R}^2 . In each case identify one of the axioms for a metric that is violated, and present an example to show that it is violated.

(a) $\varrho(a, b) = \min\{|a_1 - b_1|, |a_2 - b_2|\}$.

(b) $\varrho(a, b) = (a_1 - b_1)^2 + (a_2 - b_2)^2$.

(c) $\varrho(a, b) = \max\{b_1 - a_1, 2(a_1 - b_1), b_2 - a_2, 2(a_2 - b_2)\}$.

- A2.** Explain what it means for a sequence x_n in a metric space (X, ϱ) to *converge to a limit* $\alpha \in X$.

Let $B[0, 1]$ denote the space of bounded real functions on $[0, 1]$ with the sup (or uniform) metric. Which of the following sequences of functions converges to a limit in $B[0, 1]$? For those that converge, state what that limit is.

(a) $f_n(x) = x \exp(-nx)$.

(b) $f_n(x) = x \exp(-x/n)$.

(c) $f_n(x) = nx \exp(-nx)$.

- A3.** Suppose $f : X \rightarrow Y$ is a map between metric spaces (X, ϱ) and (Y, σ) .

(a) Define the inverse image $f^{-1}(B)$ of a set $B \subseteq Y$.

(b) Write down a definition of *continuity* of f in terms of inverse images of sets.

(c) Consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 1/x$ if $x \neq 0$, and $f(0) = 0$. (Assume \mathbb{R} has the usual metric.) Give an open set $B \subseteq \mathbb{R}$ such that $A = f^{-1}(B)$ is *not* open. Briefly justify your claim that B is open and A is not.

- A4.** Explain what it means for a function f to be a *contraction* on a metric space.

State the contraction mapping theorem.

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a contraction on the closed interval $[a, b]$ (with the usual metric). Prove that the function $g(x) = \frac{1}{2}x + \frac{1}{2}f(x)$ is also a contraction on $[a, b]$.

SECTION B

This section carries 60 marks and each question carries 30 marks. You may attempt all three questions but only marks for the BEST TWO questions will be counted.

B1. (a) [7 marks] Define $\delta : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise.} \end{cases}$$

Prove that δ is a metric on \mathbb{R}^2 .

- (b) [5 marks] Define $\varrho : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by $\varrho(a, b) = |a_1 - b_1| + \delta(a_2, b_2)$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Prove that ϱ satisfies the triangle inequality on \mathbb{R}^2 .
- (c) [8 marks] In fact, ϱ is a metric on \mathbb{R}^2 . Describe the open balls $B_1(\mathbf{0})$ and $B_2(\mathbf{0})$ of radius 1 and 2 centred at the origin $\mathbf{0} = (0, 0)$.
- (d) [10 marks] Which of the following sets are open in the metric space (\mathbb{R}^2, ϱ) ?
- $(0, 1) \times (0, 1)$,
 - $(0, 1) \times [0, 1]$, and
 - $[0, 1] \times [0, 1]$.

Briefly explain each of your answers.

B2. For any set S , let $B(S)$ denote the space of bounded real functions on S with the sup (or uniform) metric.

- (a) [6 marks] Explain what it means for a sequence of functions f_n in $B(S)$ to converge *pointwise* to a function $f \in B(S)$.
- (b) [10 marks] Let f_n be any Cauchy sequence of functions in $B(S)$. Prove that f_n converges pointwise to some function $f : S \rightarrow \mathbb{R}$. [Hint. Start by showing that $f_n(\alpha)$ is a Cauchy sequence in \mathbb{R} , for any $\alpha \in S$.]
- (c) [10 marks] Let f_n be as in part (b). Let $\varepsilon > 0$, and suppose N_ε is chosen so that

$$\sup_{x \in S} |f_n(x) - f_m(x)| < \varepsilon/2, \quad \text{for all } n, m \geq N_\varepsilon.$$

Prove that

$$\sup_{x \in S} |f_n(x) - f(x)| < \varepsilon, \quad \text{for all } n \geq N_\varepsilon.$$

- (d) [4 marks] Deduce that $B(S)$ is a complete metric space.

[Next question overleaf]

- B3.** (a) [4 marks] Explain what it means for a subset K of a metric space (X, ρ) to be (sequentially) *compact*.
- (b) [9 marks] From first principles — i.e., directly from the definition you gave in part (a) — prove that the following subsets of \mathbb{R} are not compact (with the usual metric).
- $[0, \infty)$.
 - $[0, 1)$,
 - $\mathbb{Q} \cap [0, 1]$.
- (c) [10 marks] Prove that any closed subset of a compact set is compact.
- (d) [7 marks] Prove that $K = \{1/n : n = 1, 2, 3, \dots\} \cup \{0\}$ is a compact subset of \mathbb{R} with the usual metric. You may assume that any closed interval $[a, b]$ in \mathbb{R} is compact.