

7 Compactness

Intervals which are bounded and closed figure prominently in analysis on the real line. The appropriate generalization of their essential properties that are relevant to analysis in more general spaces is compactness. There are two definitions of compactness which can be shown to be equivalent.

Definition 7.1. Suppose (x_n) is a sequence in X . For any sequence of increasing indices $0 < n_1 < n_2 < n_3 < \dots$ we say that (x_{n_k}) , i.e., $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$, is a *subsequence* of x_n .

Note that if a sequence (x_n) converges to a limit α in (X, ϱ) then any subsequence also converges to α .

Definition 7.2. A subset K of a metric space (X, ϱ) is said to be (*sequentially*) *compact* if any sequence of elements of K has a subsequence that converges to a limit in K .

It is clear from the definition that K is compact in (X, ϱ) if and only if it is compact in (X, σ) for any metric σ equivalent to ϱ .

Definition 7.3. A subset K of a metric space (X, ϱ) is *bounded* if, for some $\alpha \in X$ and $r > 0$, we have $K \subseteq B_r(\alpha)$.

Theorem 7.4. *A compact set K is bounded and closed.*

Proof. Let K be any compact set in an arbitrary metric space (X, ϱ) .

If K is not bounded then, for every $n \in \mathbb{N}$, we may choose a point $x_n \in K \setminus B_n(\alpha)$. Since K is compact, the sequence (x_n) has a subsequence (x_{n_k}) converging to β (say) in K . Since the subsequence is convergent, there is a number N such that $\varrho(\beta, x_{n_k}) < 1$ for all $k \geq N$. Now note that

$$\varrho(\alpha, x_{n_k}) \leq \varrho(\alpha, \beta) + \varrho(\beta, x_{n_k}) < \varrho(\alpha, \beta) + 1$$

for all $k \geq N$. In other words $x_{n_k} \in B_r(\alpha)$ for all $k \geq N$, where $r = \varrho(\alpha, \beta) + 1$. But this contradicts the construction of the sequence x_n .

If K is not closed, it has a limit point, say α , that is not contained in K . Consider a sequence of elements of K which converges to α . Since K is compact, this sequence has a subsequence converging to a point $\beta \in K$. But this is a contradiction, since the sequence and its subsequence must converge to the same point, i.e., β must equal α . \square

Lemma 7.5. *A closed subset of a compact set is compact.*

Proof. Let K be compact, K_0 be a closed subset of K and (x_n) be a sequence of elements of K_0 . Since K is compact and (x_n) is a sequence in K , it follows that (x_n) has a convergent subsequence. Since K_0 is closed, the limit of this subsequence lies in K_0 . Therefore any sequence of elements of K_0 has a subsequence which converges to a limit in K_0 , which implies that K_0 is compact. \square

Lemma 7.6. *If K and L are compact subsets of metric spaces (X, ρ) and (Y, σ) respectively then $K \times L$ is a compact subset of $X \times Y$ with the metric*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2}.$$

is compact.

Proof. Let (x_n, y_n) be an arbitrary sequence in $K \times L$. Since K is compact, there is a subsequence (x_{n_k}) which converges to a limit $x \in K$ as $k \rightarrow \infty$. Write this sequence (\tilde{x}_k) , where $\tilde{x}_k = x_{n_k}$, and let (\tilde{y}_k) be the corresponding subsequence of y_n ; that is $\tilde{y}_k = y_{n_k}$. Since L is compact, the sequence (\tilde{y}_k) has a subsequence (\tilde{y}_{k_i}) that converges to a limit $y \in L$ as $i \rightarrow \infty$. Since $\tilde{x}_k \rightarrow x$ as $k \rightarrow \infty$, we also have $\tilde{x}_{k_i} \rightarrow x$ as $i \rightarrow \infty$. By definition of convergence, $\rho(\tilde{x}_{k_i}, x) \rightarrow 0$ and $\sigma(\tilde{y}_{k_i}, y) \rightarrow 0$ as $i \rightarrow \infty$. This implies that $d((\tilde{x}_{k_i}, \tilde{y}_{k_i}), (x, y)) \rightarrow 0$ as $i \rightarrow \infty$, that is, $(\tilde{x}_{k_i}, \tilde{y}_{k_i}) \rightarrow (x, y) \in K \times L$. Therefore any sequence (x_n, y_n) of elements of $K \times L$ has a subsequence which converges to a limit in $K \times L$. \square

Lemma 7.7. *A closed interval $[a, b]$ is a compact subset of \mathbb{R} (with the usual metric).*

Proof. Let (x_n) be an arbitrary sequence of numbers lying in a closed interval $[a, b]$. Let us split $[a, b]$ into the union of two intervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$ of length $L/2$, where $L = b - a$. At least one of these intervals contains infinitely many elements x_n of our sequence. Let us choose one of these elements, say x_{n_1} , and denote it by y_1 . Now we split the interval of length $L/2$ which contains infinitely many elements x_n into the union of two intervals of length $L/4$. Again, at least one of these intervals contains infinitely many elements of x_n . We choose one of these elements, say x_{n_2} , taking care that $n_2 > n_1$, and denote it by y_2 . Repeating this procedure, we obtain a subsequence (y_k) of the sequence x_n such that y_k lies in an interval of length $2^{-k}L$ for all $k \geq 1$. Clearly, (y_k) is a Cauchy sequence. Since \mathbb{R} is a complete metric space, (y_k) converges to a limit. Since a closed interval is a closed set, this limit belongs to $[a, b]$. Thus, any sequence of elements of $[a, b]$ has a subsequence that converges to a limit in $[a, b]$, which means that the closed interval is compact. \square

Theorem 7.8. *A bounded and closed subset of \mathbb{R}^n is compact.*

Proof. Suppose K is a bounded and closed subset of \mathbb{R}^n . Let Q be the closed interval $[-L, L]$ in \mathbb{R} . Since K is bounded, it is contained in the cube $Q^n \subset \mathbb{R}^n$ for some sufficiently large L . (Specifically, we may take $L = d_2(0, \alpha) + R$, where α and R are as in Definition 7.3 of boundedness, and $d_2(0, \alpha)$ is the distance of α from the origin in the Euclidean metric.) By Lemma 7.7, Q is a compact subset of \mathbb{R} . So by Lemma 7.6, $Q^2 = Q \times Q$ is a compact subset of \mathbb{R}^2 with the Euclidean metric, $Q^3 = Q^2 \times Q$ is a compact subset of \mathbb{R}^3 , and so on up to Q^n . Finally, K is a closed subset of the compact set Q^n and hence is compact by Lemma 7.5. \square

Theorem 7.9. *The image of a compact set under a continuous map is compact.*

Proof. Let K be a compact set and f be a continuous map. Let (y_n) be an arbitrary sequence of elements of $f(K)$. Then $y_n = f(x_n)$ where $x_n \in K$. Since K is compact, the sequence (x_n) has a subsequence (x_{n_k}) which converges to a limit $\alpha \in K$. Then by Theorem 5.3 the subsequence $y_{n_k} = f(x_{n_k})$ converges to the limit $f(\alpha) \in f(K)$. This proves that $f(K)$ is compact. \square

Lemma 7.10. *If a Cauchy sequence has a convergent subsequence, then the sequence itself is convergent with the same limit.*

Proof. Let (x_n) be a Cauchy sequence and (x_{n_k}) be a subsequence converging to α . Then for any $\varepsilon > 0$, since (x_n) is Cauchy, there exists N_ε such that $\varrho(x_n, x_m) < \varepsilon/2$ for all $n, m \geq N_\varepsilon$. Also, since (x_{n_k}) converges to α , there exists K_ε such that $\varrho(x_{n_k}, \alpha) < \varepsilon/2$, for all $k \geq K_\varepsilon$. Choose k so that $k \geq K_\varepsilon$ and $n_k \geq N_\varepsilon$. Then applying the triangle inequality we obtain

$$\varrho(x_n, \alpha) \leq \varrho(x_n, x_{n_k}) + \varrho(x_{n_k}, \alpha) < \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \text{for all } n > N_\varepsilon.$$

Since ε is an arbitrary positive number, this implies that $x_n \rightarrow \alpha$. \square

Definition 7.11. A metric space (X, ϱ) is said to be compact if the set X is compact.

Theorem 7.12. *A compact metric space (X, ϱ) is complete.*

Proof. Let (x_n) be an arbitrary Cauchy sequence in (X, ϱ) . Since X is compact, this sequence has a subsequence that converges to a limit in X . By Lemma 7.10, the whole sequence (x_n) converges to the same limit. \square

Corollary 7.13. *Suppose (X, ϱ) is a compact metric space. Any continuous function $f : (X, \varrho) \rightarrow \mathbb{R}$ is bounded and furthermore attains its upper and lower bounds.*

Proof. By Theorem 7.9, $f(X)$ is compact, and hence bounded by Theorem 7.4.

Denote by $\beta \in \mathbb{R}$ the supremum of the set $f(X)$. Let (y_n) be a sequence in $f(X)$ converging to β . Since $f(X)$ is compact, it is closed, and (y_n) tends to a limit in $f(X)$. Thus $\beta \in f(X)$ and necessarily $\beta = f(\alpha)$ for some $\alpha \in X$. Similarly for infimum. \square

Definition 7.14. We say that a real-valued function f defined on a metric space (X, ϱ) is *uniformly continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $\varrho(x, y) < \delta$.

Obviously, a uniformly continuous function is continuous.

Theorem 7.15. *Suppose $f : (X, \varrho) \rightarrow (Y, \sigma)$ is a continuous function and that (X, ϱ) is compact. Then f is uniformly continuous.*

Proof. With a view to obtaining a contradiction, assume that f is continuous but not uniformly continuous. Then there exists some $\varepsilon > 0$ such that, for every $\delta > 0$, there is some pair of points $x, y \in X$ such that $\varrho(x, y) < \delta$ and $\sigma(f(x), f(y)) \geq \varepsilon$. (This is just the formal negation of the condition for f to be uniformly continuous.)

For each $n \in \mathbb{N}$, let $x_n, y_n \in X$ be points such that $\varrho(x_n, y_n) < 1/n$ and $\sigma(f(x_n), f(y_n)) \geq \varepsilon$. Since X is compact, there is subsequence (x_{n_k}) of (x_n) that converges in (X, ϱ) to α , say. It is easy to see that the corresponding subsequence (y_{n_k}) of y_n also converges to α . (Given $\eta > 0$, we may choose N such that $\varrho(y_{n_k}, x_{n_k}) < \eta/2$ and $\varrho(x_{n_k}, \alpha) < \eta/2$ for all $k \geq N$. Then, by the triangle inequality, $\varrho(y_{n_k}, \alpha) \leq \varrho(y_{n_k}, x_{n_k}) + \varrho(x_{n_k}, \alpha) < \eta$, for all $k \geq N$.)

Since f is continuous, the sequences $f(x_{n_k})$ and $f(y_{n_k})$ must both converge in (Y, σ) to $f(\alpha)$. So for some sufficiently large k we have $\sigma(f(x_{n_k}), f(\alpha)) < \varepsilon/2$ and $\sigma(f(y_{n_k}), f(\alpha)) < \varepsilon/2$. Thus $\sigma(f(x_{n_k}), f(y_{n_k})) \leq \sigma(f(x_{n_k}), f(\alpha)) + \sigma(f(y_{n_k}), f(\alpha)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. But this contradicts the construction of the sequences (x_n) and (y_n) . \square

There is a second definition of “compact”, perhaps a more common one. Suppose (X, ϱ) is a metric space, K a subset of X , and $\{A_\omega : \omega \in \Omega\}$ a collection of subsets of X . We say that $\{A_\omega\}$ is a *cover* of K iff $K \subseteq \bigcup_{\omega \in \Omega} A_\omega$; it is an *open cover* if every subset A_ω is open. If $\Omega_0 \subseteq \Omega$ then

$\{A_\omega : \omega \in \Omega_0\}$ is a *subcover* if $K \subseteq \bigcup_{\omega \in \Omega_0} A_\omega$; it is an *finite subcover* if Ω_0 is finite.

Definition 7.16. A subset K of a metric space (X, ρ) is said to be *compact* if any open cover of K has a finite subcover.

Theorem 7.17. *A set is compact if and only if it is sequentially compact.*

Proof. The proof is beyond the scope of the course. \square

We have seen that a compact metric space is complete and bounded. You may be ready to conjecture that the converse also holds but this is not the case.

Example 7.18. Recall that $C[0, 1]$ is the space of continuous functions $[0, 1] \rightarrow \mathbb{R}$ equipped with the sup metric d_∞ ; recall also that $C[0, 1]$ is complete. Consider the closed ball $B_1[0]$ of radius 1 in this space, centered at the constant function 0. This ball contains precisely the continuous functions $[0, 1] \rightarrow [-1, 1]$. Consider $B_1[0]$ as a subspace of $C[0, 1]$. $B_1[0]$ is a closed subset of a complete metric space, and hence itself complete. It is certainly bounded.

However, $B_1[0]$ is not compact. Consider the functions $f_n : [0, 1] \rightarrow [-1, 1]$, for $n \in \mathbb{N}^+$, defined by

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \leq x < 2^{-(n+1)}; \\ 2^{n+1}x - 1 & \text{for } 2^{-(n+1)} \leq x < 2^{-n}; \\ 2 - 2^n x & \text{for } 2^{-n} \leq x < 2^{-(n-1)}, \text{ and} \\ 0 & \text{for } 2^{-(n-1)} \leq x \leq 1. \end{cases}$$

You may verify that $d_\infty(f_n, f_m) = 1$ whenever $n \neq m$. (Consider the values of the two functions at the point 2^{-n} .) So f_n contains no Cauchy subsequences and hence no convergent subsequences.

Remark 7.19. There is a natural strengthening of the condition of boundedness to *total boundedness*, under which space is compact iff it is complete and totally bounded. This is beyond the scope of the course.

Finally, there do exist natural compact function spaces.

Example 7.20. Consider the space of functions $f : [0, 1] \rightarrow [-1, 1]$ that satisfy the Lipschitz condition $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0, 1]$. With the sup metric, this becomes a compact metric space, but this claim is not completely straightforward to verify.