MTH6126 Metric Spaces, Autumn 2010: Sheet 9

Please send comments and corrections to `m.jerrum@qmul.ac.uk`.

Please drop solutions to **Questions 2 and 3** in the orange box on level 2 by 17:30 on Friday 17th December.

1. Let $f : [1, \infty) \to [1, \infty)$ be given by $f(x) = x + 1/x$. Show that $|f(x) - f(y)| < |x - y|$ for all distinct x, y, but that f has no fixed point. Which condition in the statement of the Contraction Mapping Theorem fails to hold in this example? What happens to the sequence defined by the recurrence $x_0 = 1$ and $x_n = f(x_{n-1})$ for $n > 1$, and why?

2. Which of the following are compact subsets of $\mathbb{R}^2$ with the Euclidean metric? Justify your answers.

   (a) $[0, 1] \times [0, 1]$,

   (b) $\{(x, y) : x^2 + y^2 < 1\}$,

   (c) $\{(x, 0) : \sin x \geq 0\}$,

   (d) $(\mathbb{Q} \cap [0, 1]) \times [0, 1]$,

   (e) $\{(x^3y - z^2, xyz + y^3z) : 0 \leq x, y, z \leq 1\}$.

3. Let $A \subseteq \mathbb{R}$ be a subset of the real line with the usual metric. Three properties that A may have are: (i) A is closed, (ii) A is bounded, and (iii) A is compact. These are eight potential combinations, but not all of them are possible. Which of the eight are possible? Give an example of a set A that realises each possible case. Explain why the remaining cases are impossible.

4. Let K, L be compact subsets of a metric space. Prove that $K \cup L$ and $K \cap L$ are compact. Are arbitrary unions and intersections of compact sets compact?

5. Prove that if the set $A \subseteq \mathbb{R}$ is not compact then there exists a continuous real function on A that is unbounded. (Hint: Consider two cases.)