Generalising T-Designs

Derek Patterson

1. An Introduction to Designs

In the beginning, there were designs and over time certain people - mathematicians - studied them, thereby discovering various pertinent properties. They classified the designs according to these properties, produced suitable definitions and proved further properties. In this way, a branch of combinatorics- 'Design Theory'- came into being. To this day, new discoveries are still being made and design theory has become a vibrant and exciting area of mathematics. So, what is a design?

A combinatorial design consists of:
1. A set X.
2. A collection (multiset)$\mathcal{A}$ of subsets of X.

Given a specific type of design, one can select the subsets in $\mathcal{A}$ such that certain conditions, pertinent to that type, are met. For example, in the case of a 'Latin Square', the n elements of X are arranged in an n x n array such that each element occurs precisely once in each row and once in each column.

Below is a Latin Square of side n, based on the ground-set $X = \{1,2,3\}$

1	2	3
3	1	2
2	3	1

Another type of design is that of a 'One-Factorisation'. In such a design, the elements of X are put into pairs and the pairs, in turn, into classes. Each element appears exactly once in each class and each possible pair of elements exactly once in the design.

Below is a one-factorisation of $X = \{1,2,3,4,5,6\}$

1,2	1,3	1,4	1,5	1,6
3,4	2,5	2,6	2,4	2,3
5,6	4,6	3,5	3,6	4,5

2. Block Designs and their Parameters

In a 'Block Design', the subsets of X that comprise $\mathcal{A}$ are known as **blocks**.

Clearly, when analysing a design, it is helpful to have a standard notation for its parameters. Questions then arise such as, "Can such a design exist with certain parameters?", "If some parameters are fixed, what can be said about the other parameters?" etc.

For the purposes of many types of block design, the following represents a fairly standard notation.

v:= $|X|$
b:= the number of blocks in the design.
k:= the size of the blocks in the design.
r:= for $x \in X$, the number of blocks of which x is a member.
λ:= for $x, y \in X$, the number of blocks containing both x and y.
The elements of X are known as 'points' or 'varieties', for historical reasons. Hence, v = the number of points in a design.

It is implicit in the above definitions that k and r are constants. In some designs, this will not be true and, in such cases one would replace k by K to represent a set of possible values for the size of the blocks in the design, for example.

Also, λ might vary depending upon the choice of x and y. In addition, it might be used to describe the number of blocks containing a given number of points (not just pairs of points).

3. BIBDs and T-Designs

A block is complete if it contains all the points in X. A **BIBD**, or balanced, incomplete, block design is a design in which each block is incomplete and is of a fixed size, k. It is referred to as being balanced because each pair of points is contained in the same number of blocks, λ. In addition, in such a design, each point is contained in the same number of blocks, r.
In fact, a BIBD is a special case of a ***t*-design**. In a t-design, any t-subset of X is contained in λ blocks with k and r being constants. Hence, a BIBD is a 2-design.

A fundamental result is that if a design is a t-design, then it is also an s-design for $s \leq t$ and $s,t \in \mathcal{N}$
Historically, BIBDs and similar designs have been of great importance for the design of experiments. Consequently, there has been a great amount of study in this area.

A cornerstone in Design-Theory is the following result:
Fisher's Inequality: In a BIBD,

$$b \geq v$$

This has provided the basis for much research such as extending the result to t-designs in general.

4. A Generalisation of T-Designs

In a recent paper [1], Peter Cameron introduced the idea of a t-($\mathbf{v}$,$\mathbf{k}$,λ) design, which he defined roughly as follows:

Let k and t be integers with $k > t > 0$ and let $\mathbf{k}$ be $(k_1, \ldots\ldots, k_m)$ a composition of k such that $k_1 + k_2 + \ldots. + k_m = k$.
In addition, $\mathbf{v} = (v_1, \ldots\ldots, v_m)$ where $v_i \geq k_i$ and $v_i = |X_i|$ for all i where the ground set X is divided into pairwise disjoint sets $(X_1, X_2, \ldots, X_m)$

Then, if $\mathcal{B} \subseteq$

$$\binom{X_1}{k_1} X \ldots . X \binom{X_m}{k_m}$$

and $\mathbf{t}$ is defined to be an m-tuple of integers $(t_1, \ldots\ldots, t_m)$ summing to t such that $0 \leq t_i \leq k_i$ for all m, then if $\mathbf{T} = (T_1, T_2, \ldots\ldots, T_m)$ with $T_i = \binom{X_1}{t_1} fori = 1,\ldots,m$

there are precisely λ members
$\mathbf{k} = (k_1, \ldots, k_m) \in B$ for which $T_i \subseteq k_i$ for $i = 1, \ldots m$

What is fascinating about these sorts of designs is that they embrace many well-known and much-studied designs by setting the parameters appropriately.

For example, the Latin Square illustrated earlier is a $2-((3,3,3),(1,1,1),1)$ design.

So, if $X_1 = \{1,2,3\}$, $X_2 = \{4,5,6\}$ and $X_3 = \{7,8,9\}$ then X_1 can be thought of as representing the symbols, X_2 the columns and X_3 the rows. Therefore, a block $\{1,4,7\}$ would mean that symbol 1 is in column 4 and row 7 with this representation.

Furthermore, the one-factorisation from section 1, can be thought of as a $2-((6,5),(2,1),1)$ design.

So, if $X_1 = \{1,2,3,4,5,6\}$, $X_2 = \{7,8,9,10,11\}$ then X_1 can be thought of as representing the symbols, X_2 as the labels for the classes.

Normal t-designs are achieved by letting $\mathbf{k} = (k)$

So, these designs generalise t-designs to some extent. What about their properties?

5. Generalising Fisher's Inequality

A fairly obvious starting point for finding properties of t-($\mathbf{v}$,$\mathbf{k}$,λ) designs, is to try to obtain the result analogous to Fisher's inequality for BIBDs.

A conjecture due to Leonard Soicher is that:

in a 2-($\mathbf{v}$,$\mathbf{k}$,λ) design,

$$b \geq v - m + 1$$

A proof of this can be seen by using an incidence matrix I for the design, the columns indexed by the blocks and the rows indexed by the points. Then $A_{i,j} = 1$ if point i is in block j and 0 otherwise.

The next step is to obtain a new matrix $J = I.I^T$. Now, the entry $J_{i,j}$ will give the number of blocks in the design that contain both points i and j. By simple row and column operations this matrix can be manipulated to obtain a lower triangular matrix, which must have the same rank as J because the operations preserve rank. It can be shown that under certain circumstances, there will $m-1$ zeros on the diagonal but never more.

Therefore the rank of this matrix is at least $v - m + 1$.

This implies that the rank of I and the rank of I^T must be at least this big as well (as $J = I.I^T$). Therefore, as the columns have been indexed by the blocks, the result follows:

$$b \geq v - m + 1$$

Furthermore, some designs (not merely for $\lambda = 1$) have the property that $b = v - m + 1$ (thereby achieving equality). These include designs with $k = (1, \ldots, 1)$ and $\lambda = 1$, corresponding to orthogonal arrays of order n where m = n+1. Another example is given by a $2-((2,2,2,2,2,2,),(1,1,1,1,1,1),2)$ design. Research into such designs is currently in progress.

6. Showing Existence

One area of research is that of finding parameters for which t-($\mathbf{v}$,$\mathbf{k}$,λ) designs exist and paramaters that ensure non-existence.

Result: A $2-((2n+1,2n),(2,1),2)$ design exists for all positive integers n.

This can be shown by constructing an equivalent design: using $2n$ colours to produce a proper colouring of a graph on $2n+1$ vertices so that
1. every vertex appears twice in each colour class and
2. there are two edges between every pair of vertices.

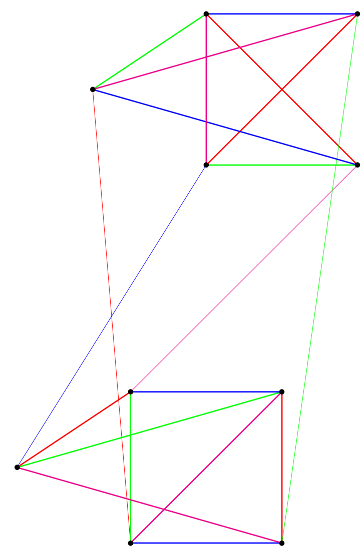
The colour classes correspond to X_2 and the vertices to X_1.

(1) ensures that $\lambda = 2$ for pairs of varieties in which one variety is from X_1 and the other from X_2.

(2) ensures that $\lambda = 2$ for pairs of varieties in which both are from X_1.

In fact, the colouring can be seen more easily if an attempt is made to colour the simple graph K_{2n+1} with $2n$ colours. The best that can be done is that at each vertex, precisely one colour is missing and one edge is not coloured - this edge is removed. Then a second copy of K_{2n+1} is made and that is coloured and amended in similar fashion. The vertices that are missing the same colour in each part can then be joined with an additional edge coloured with the shared missing colour.

The final result looks something like this- the faint edges are the ones added at the end of the process:



Another result due to Leonard Soicher is that if $2 \leq t \leq k$ then
$\mathbf{k}, \in \{(k), (k-1,1), (1,k-1), (1,\ldots.,1), \mathbf{v}\}$

7. Generalised T-Designs

At the moment, the exact definition of a 'Generalised T-Design' has not yet been decided. Although a t-($\mathbf{v}$,$\mathbf{k}$,λ) design is a generalisation of a t-design, there are reasons for believing that it might be preferable to generalise further so that λ is not necessarily a constant in the way that has been set out above. A proposed idea is that it should be a constant for certain 'types' of the design.

The reason for this is that it is considered desirable that as many standard results concerning t-designs should have a corresponding result concerning 'generalised t-designs'. One such standard result is that a t-design is also an s-design for $s < t$. However, it does not follow that a t-($\mathbf{v}$,$\mathbf{k}$,λ) design is also an s-($\mathbf{v}$,$\mathbf{k}$,λ) design for $s < t$. For this reason, amongst others, careful consideration is being given to the exact definition of a 'generalised t-design'.

References

1 P.J. Cameron, A generalisation of t-designs, Discrete Mathematics (2008)