Finite simple groups are the prime numbers of group theory. Just as prime numbers are the basic building blocks of integers, finite simple groups are the building blocks of finite groups.

Here we consider the following problem related to finite simple groups:

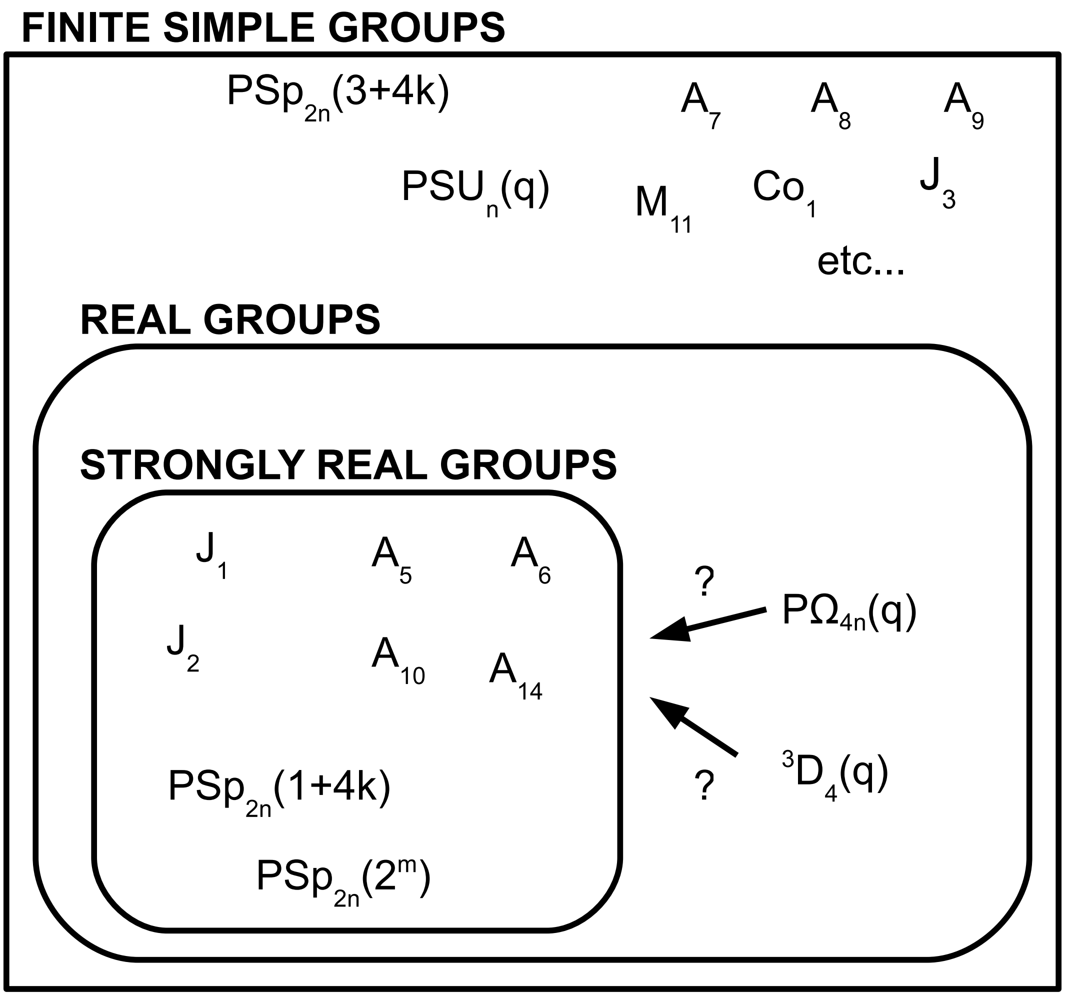
> *In which finite simple groups is every element strongly real?*

Suppose that G is a group and g is an element of G.

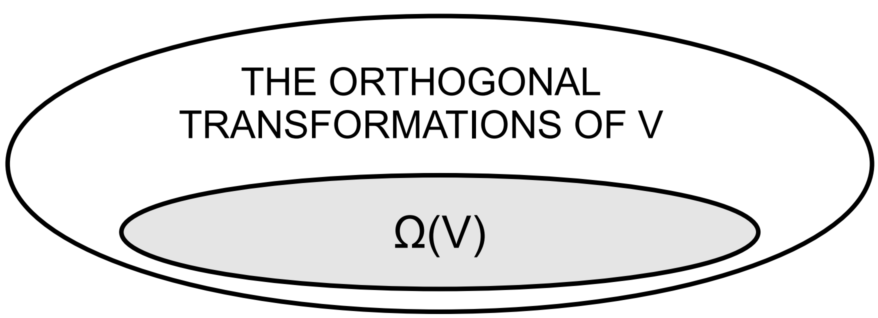
- The group G is *simple* if it has exactly two normal subgroups, $\{1\}$ and G.
- The element g is *real* if there is $x \in G$ which inverts g, that is, $x^{-1}gx = g^{-1}$.
- The element g is *strongly real* if there is an involution $x \in G$ such that $x^{-1}gx = g^{-1}$.
- An element x is an *involution* if $x^2 = 1$ and $x \neq 1$.

STRONGLY REAL FINITE SIMPLE GROUPS

Ann Cook Prize entry
Johanna Rämö
May 2008



- We know which of the finite simple groups are real, so we only need to consider those groups. [4]
- We also already know that some of the finite simple groups are strongly real. (For more information, see [3].)
- The only groups that are left are
  - the orthogonal groups PΩ(V), where V is a vector space of dimension $4n$
  - the exceptional groups ${}^3D_4(q)$
- What can we say about PΩ(V)? It is a quotient group of the group $\Omega(V)$. If we can show that $\Omega(V)$ is strongly real, then PΩ(V) will be strongly real too.
- Elements of $\Omega(V)$ are orthogonal transformations, so we start by considering them.



ORTHOGONAL TRANSFORMATIONS

Suppose that

- V is a finite dimensional vector space over a finite field F
- $Q: V \to F$ is a nondegenerate quadratic form of V
- S is an orthogonal transformation of V, that is, $Q(vS) = Q(v)$ for all $v \in V$

Theorem. ([1], [2], [5]) *There is an orthogonal involution H that inverts S.*

How is this involution found? First the space V is written as a sum $V = \bigoplus V_i$ of certain S-invariant subspaces.

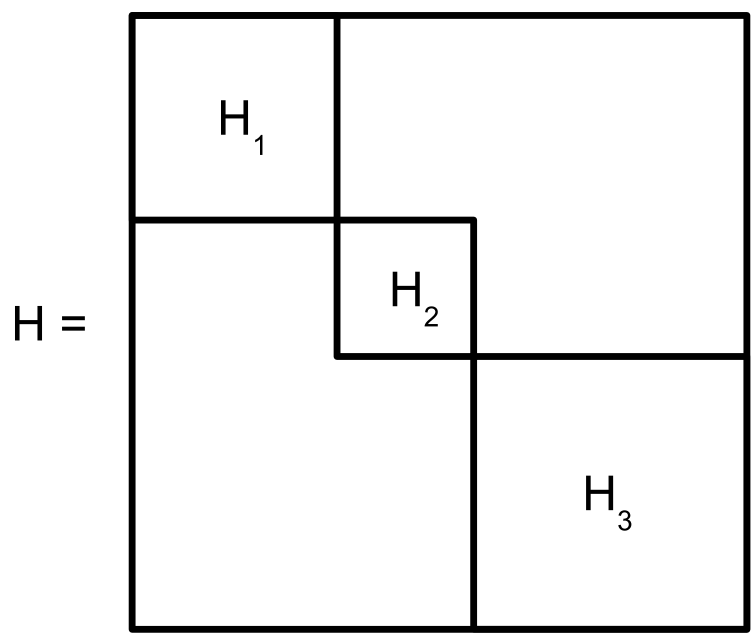
It can be shown that there is an orthogonal involution H_i of V_i that inverts $S|_{V_i}$. Now the trasformation $H = \bigoplus H_i$ is the desired involution. If H is represented as a matrix, then the transformations H_i form blocks of H.

THE GROUPS $\Omega(V)$

Can we show that for an element of $\Omega(V)$ the inverting involution H is in $\Omega(V)$? The task is easier if we assume that the characteristic of the field is even. Then it is known that

$$H \in \Omega(V) \Leftrightarrow \text{Rank } (\text{id}_V - H) \text{ is even.}$$

We can now calculate the rank corresponding to each block H_i, and conclude that H is in $\Omega(V)$ if the sum of the ranks is even.



CONCLUSION

Since $\Omega(V)$ is not real if the dimension of V is not divisible by four, we can assume that 4|dim(V). Now we can only have certain kinds of combinations of the blocks H_i, and it is easier to calculate the ranks.

For most combinations, we can prove that the involution H is in $\Omega(V)$. If this can be shown to be true in all cases, the problem is solved for PΩ(V) in even characteristic.

References

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