Research background for potential Ph.D. students

Matthew Fayers
Queen Mary, University of London, Mile End Road, London E1 4NS, U.K.
m.fayers@qmul.ac.uk

Introduction

Here's some background on the kind of things that I think about in my research, and the kind of things you might think about if you came to do a Ph.D. under my supervision at QMUL. Please email me if you're interested and would like to know more.

For the purposes of this document, I'm assuming you're nearing the end of an undergraduate degree in mathematics, and that you're more inclined towards pure mathematics. Don't worry if you don't understand very much of what's here; it's just supposed to provide a flavour.

My research is at the interface of algebra and combinatorics. You should have some idea of what these are, but different people have different opinions, and there doesn't seem to be a written definition, so I'll give an idea of what I think they are.

Algebra to the layman means using letters to represent numbers, but actually it's the study of structure and symmetry. Broadly, this means that you have an algebraic object with a simple definition, often in terms of binary relations. The quintessential example is here is a group; you know the very straightforward definition of a group, and you should have an idea of how complex groups can be. But the beauty of algebra is that because the definitions are more rigid than in other areas, you can hope to *classify* things, i.e. give a complete list. For example, the classification of finite simple groups is one of the great achievements of mathematics.

So the idea of algebra is to give a simple definition in terms of a list of rules, and see what objects there are satisfying these rules.

Combinatorics is one of the most abused terms in mathematics – it means very different things to different people. Broadly, it's supposed to be about counting; this doesn't necessarily mean counting with numbers, but it suggests working with finite sets of simple objects, and not requiring any deep background knowledge (the unkind definition of a combinatorialist is 'a mathematician who doesn't know any mathematics'). Exactly what these objects are depends on which area of combinatorics you're working in. What I do is *algebraic combinatorics*, which means combinatorial problems that come from algebra; this means that you get a lot of the rigidity and symmetry from algebra, but often it's more straightforward to think about combinatorial objects.

Now I'll describe some topics within these areas that I think about.

Representations of the symmetric group

A lot of my work involves or stems from the representation theory of the symmetric group. Let's start with the symmetric group: there's actually a different symmetric group for each non-negative integer n: the group $\mathfrak{S}_n$ is simply the group of permutations of the set $\{1,\dots,n\}$. This is one of the best-understood groups, and is the focus of a great deal of research. Now let's discuss representations: given a groups G, a representation of G is a homomorphism from G to the group of endomorphisms of some vector space. If you think of endomorphisms of a vector space as matrices, a representation is just a way of assigning a matrix to each element of the group, in such a way that the group operation corresponds to multiplication of these matrices.

The representation theory of the symmetric group is pretty well understood, if we assume that the entries of our matrices are real or complex numbers (i.e. our vector space is a vector space over $\mathbb{R}$ or $\mathbb{C}$). More modern research involves *modular* representation theory, where we take vector spaces over a finite field. (You'll probably have seen some examples of finite fields; for example, the field $\mathbb{F}_3$, which has three elements $0, 1, 2$, with addition and multiplication done modulo 3.) Here the situation is suddenly more complicated. Let's do a small example to see what's going on. Let's take the symmetric group $\mathfrak{S}_3$, and suppose our vector space is three-dimensional, with basis $\{e_1, e_2, e_3\}$. Then we get a representation of $\mathfrak{S}_3$ by letting the group elements act on the basis elements e_1, e_2, e_3 in the same way that they act on the number $1, 2, 3$. For example, the permutation which swaps the numbers 1 and 2 and fixes 3 would map to the matrix

$$M_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Regardless of what our field is, we get a representation of $\mathfrak{S}_3$. And in fact this this has *subrepresentations*: this means subspaces of our vector space which are fixed by all the matrices in our representation. For example, we've got the subspace V spanned by the vector $e_1 + e_2 + e_3$; any permutation will fix this vector. We've also got the subspace W consisting of all vectors $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. If our field is $\mathbb{R}$ or $\mathbb{C}$, then V and W are *complementary* subrepresentations, which means that any vector in our three dimensional space can be written uniquely as the sum of a vector in V and a vector in W. But if our field is $\mathbb{F}_3$, then we actually have $V \subset W$ (because $1+1+1 = 0$ in $\mathbb{F}_3$), so V and W can't possibly be complementary; in fact there is no subrepresentation which is complementary to V or W.

Our aims in the representation theory of the symmetric group are as follows.

- Understand the *irreducible* representations of $\mathfrak{S}_n$ (i.e. those which don't have any subrepresentations) over any field.
- Understanding how these irreducibles 'fit together' to form larger representations.
- Understand *induction* and *restriction* between different symmetric groups: $\mathfrak{S}_n$ actually sits inside $\mathfrak{S}_{n+1}$, and this gives us a natural way to turn a representation of one of these symmetric groups into a representation of the other. We'd like to understand exactly what happens to certain important representations (such as irreducible representations) when we apply these procedures.

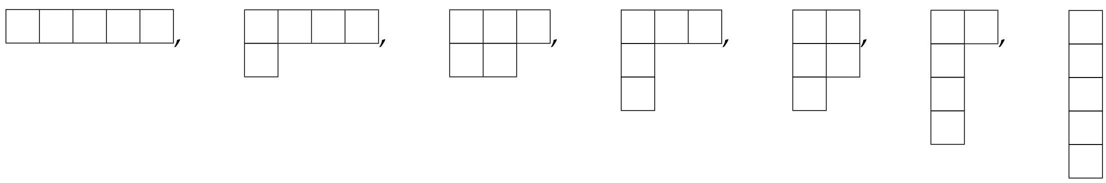
- Understand what happens to representations over $\mathbb{C}$ when we 'reduce modulo p'.

This last part is what we did above with our three-dimensional representation: we defined it over $\mathbb{C}$, and then said 'what happens if we actually regard these matrix entries as being elements of the field $\mathbb{F}_3$?'. So we did reduction modulo 3. One thing that happened when we did this was that the irreducible representation W suddenly became reducible. One of my achievements in representation theory was to answer the question 'which irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$ become reducible when we reduce modulo p?'. A related problem which I'd like a student to look at is the same question for the alternating groups: you may recall that there is a natural way to regard any permutation as *even* or *odd*, and that even permutations form a subgroup of $\mathfrak{S}_n$. This subgroup is called the *alternating group* $\mathfrak{A}_n$, and is very important because it is a simple group. Because there is a close relationship between $\mathfrak{S}_n$ and $\mathfrak{A}_n$, you can deduce a lot about the representation theory of $\mathfrak{A}_n$ from the representation theory of $\mathfrak{S}_n$.

Partitions

Now we come to our first encounter with combinatorics. Let n be a non-negative integer. A *partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of positive integers such that $\lambda_1 \geqslant \dots \geqslant \lambda_l$ and $\lambda_1 + \dots + \lambda_l = n$.

We usually visualise a partition by drawing a *Young diagram*; this is an array of boxes, where there are λ_1 boxes in the first row, λ_2 in the second row, and so on. For example, there are seven partitions of 5, and their Young diagrams are as follows.



Partitions are interesting in their own right, and there are some nice results one can prove. (Here's an example which you should try as an exercise: prove that for any n the number of partitions in which $\lambda_1, \dots, \lambda_l$ are all different is the same as the number of partitions in which $\lambda_1, \dots, \lambda_l$ are all odd.) But we're interested in partitions because they come up in algebra. In particular, the conjugacy classes of $\mathfrak{S}_n$ are labelled by the partitions of n, and this means that the irreducible representations of $\mathfrak{S}_n$ are labelled by partitions of n. This means that one can phrase various results and questions about representations of $\mathfrak{S}_n$ in terms of partitions, and transfer algebraic questions to a combinatorial setting, where one is asking questions about partitions.

Coxeter groups and Hecke algebras

One of the reasons why the symmetric group $\mathfrak{S}_n$ is so interesting is that it is a *reflection group*. To see what this means, let's go back to the three-dimensional representation of the symmetric group $\mathfrak{S}_3$ above. You may recognise the matrix M_{12} as the matrix of the reflection in the plane $x = y$. Similarly, the group element which swaps the numbers 2 and 3 is represented by the reflection in the plane $y = z$. These two elements generate $\mathfrak{S}_3$, and so our group of matrices is a reflection group, i.e. a group of endomorphisms of a vector space which is generated by the reflections it contains. There are other finite reflection groups (for example, the group of symmetries of a regular polygon), but actually there aren't that many. One important thing about reflection groups is that they are *Coxeter groups*. I won't say here precisely what this means, but it says that any one of these groups has a very simple presentation (you may not be familiar with a presentation for a group; it's a way of defining a group by giving a generating set and a list of relations between the generators). This has lots of consequences for the structure of the group, and the study of Coxeter groups is a very active area of research. I recommend to anyone reading some of the book by Humphreys [Hum] which gives a very easy-to-read introduction to Coxeter groups.

There's an important algebraic object connected to a Coxeter group, which is a *Hecke algebra*. This is an algebra (which just means a ring containing a field) whose definition depends on the choice of a Coxeter group, a field $\mathbb{F}$, and an element q of that field. This ring is remarkable because it arises in diverse areas of maths, and because its representation theory (i.e. the study of modules for this ring) is very similar to the representation theory of the symmetric group that we looked at above. And so again, the study of the representation theory is connected with combinatorics, although here the combinatorial objects involved are no longer partitions, but other similar objects.

One special case of this which should be fun for a student to work on is the special case where our chosen element q is zero. This case works differently in that the irreducible representations all have dimension 1 (i.e. all the matrices are 1×1 matrices), so you work with really low-dimensional vector spaces and small matrices (normally with the representation theory of the symmetric group, the dimensions of representations get very large very quickly, so it becomes impractical to try to think about explicit matrices).

Quantum groups, crystal bases

Another area that I'm interested in is quantum groups. These are algebras derived from Lie algebras. Don't worry if you don't know what a Lie algebra is – there are some good introductory-level books. A quantum group is an algebra that comes from a Lie algebra that has importance for mathematical physics (really don't worry about this – I don't know any physics at all, so I don't go at all in this direction). There are also some important connections between certain quantum groups and the representation theory of symmetric groups and Hecke algebras. The way this works is through crystal bases and canonical bases. Crystal bases are one of the most important areas in algebraic combinatorics. The idea is that we've got a quantum group and we've got some representation. We've got the same problem as above in terms of understanding what the representation looks like, because the dimension is really large (and therefore we can't just write down matrices), but here it's much worse because representations of quantum groups are *infinite-dimensional*. So we need to have some way of getting hold of a representation of a quantum group and seeing what it looks like, and one way to do this is via a crystal basis. Basically, this means you have a basis for your representation (this will be infinite, since the representation is infinite-dimensional, but we hope it will be in some way easy to comprehend), and then you have 'crystal operators' which give you simple rules that tell you roughly how the generators of your quantum group act on the crystal basis. It's difficult to go into more detail here without giving huge amounts of background, but the idea is that you get a combinatorial way of interpreting a representation: your crystal basis gives you a crystal graph, which is a directed labelled graph in which the vertices are the elements of the crystal basis, and the arrows tell you what the crystal operators do.

There's lots of interest in trying to understand what these crystal graphs look like. What this means is that we take an important representation of a quantum group, and try to *model* its crystal graph, i.e. find a set of nice combinatorial objects such as partitions to represent the vertices, with a simple combinatorial rule to describe the vertices. One example is probably the simplest example of an infinite crystal graph, which has vertices labelled by a certain class of partitions called n-regular partitions (where n is some chosen positive integer). Here the crystal graph is really important because (if n is a prime) it also describes how induction and restriction of irreducible representations of the symmetric group in characteristic n work. For example, here's part of the crystal graph for the case $p = 3$. (Don't worry about what it all means – just marvel at the beauty, or something.)

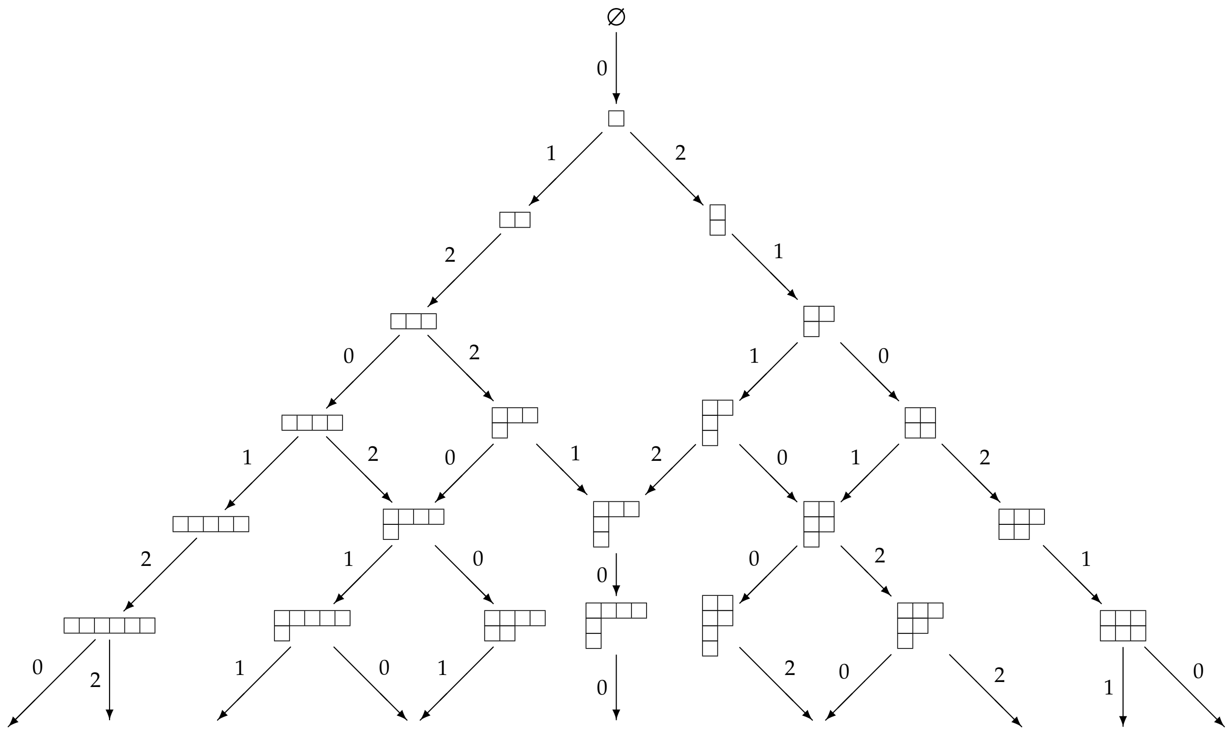


Figure 1

Background reading

I've picked the books below because (I hope) they're really easy to read. I wouldn't suggest that you buy all of them and read them cover-to-cover, but if you're keen you should go to the library and have a look; if you came to do a Ph.D. with me, you'd certainly be reading some of these books in detail.

[J] G. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics **682**, Springer, 1978.

[JK] G. James & A. Kerber, *The representation theory of the symmetric group*, Encyclopædia of Mathematics and its Applications **16**, Addison–Wesley, 1981.

[Hum] J. Humphreys, *Coxeter groups and reflection groups*, Cambridge studies in advanced mathematics **29**, Cambridge Univ. Press, 1992.

[M] A. Mathas, *Iwahori–Hecke algebras and Schur algebras of the symmetric group*, University lecture series **15**, American Mathematical Society, 1999.

[HK] J. Hong & S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate studies in mathematics **42**, American mathematical society, 2002.