Defect 2 spin blocks of symmetric groups and canonical basis coefficients

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2010 Mathematics subject classification: 20C30, 20C25, 17B37, 05E10

Abstract

This paper addresses the decomposition number problem for spin representations of symmetric groups in odd characteristic. Our main aim is to find a combinatorial formula for decomposition numbers in blocks of defect 2, analogous to Richards’s formula for defect 2 blocks of symmetric groups. In fact we find a formula for the corresponding “$q$-decomposition numbers”, i.e. the canonical basis coefficients in the level 1 $q$-deformed Fock space of type $A_{2n}^{(2)}$: a special case of a conjecture of Leclerc and Thibon asserts that these coefficients yield the spin decomposition numbers in characteristic $2n + 1$.

1 Introduction

The most significant outstanding problem in the representation theory of the symmetric group $S_m$ is the determination of the decomposition numbers, describing the composition factors of the reduction of an ordinary irreducible representation modulo a prime. A complete solution to this problem seems to be far out of reach at present, but a wide variety of results are known dealing with special cases. One of these is Richards’s combinatorial formula [R, Theorem 4.4] giving the decomposition numbers for all blocks of symmetric groups of defect 2.

The theory of decomposition numbers for projective representations of $S_m$ (or equivalently, representations of a Schur cover $\hat{S}_m$) is much less advanced. The representations not arising directly from representations of $S_m$ are called spin representations of $\hat{S}_m$. Although the ordinary irreducible spin characters were classified by Schur in 1911 [Sch], the corresponding representations were not constructed until 1990, by Nazarov [N]. As with the case of representations of $S_n$, the combinatorics of partitions plays a central role in the theory. The modular theory of spin representations was initiated in the 1960s by Morris, who conjectured the block structure for spin representations; this conjecture was proved by Humphreys [H]. But a suitable parameterisation of the irreducible modular spin representations was not found until 2002, by Brundan and Kleshchev. Decomposition numbers for spin representations have been computed in degree at most 18 [MY2, BMO, Ma], but very few general results are known. These general results include the Brundan–Kleshchev regularisation theorem [BK3] and Müller’s determination [Mü] of the decomposition numbers for blocks of defect 1. In this paper we address spin blocks of defect 2, in the hope of finding a spin version of Richards’s formula.
In fact most of this paper is concerned with quantum algebra. For every Kac–Moody algebra $g$ of classical affine type, Kashiwara et al. [KMPY] construct a $q$-deformed Fock space of level 1; this is a module for the quantum group $U_q(g)$. In the case where $g$ is of type $A_{2n}^{(2)}$, Leclerc and Thibon [LT] studied this Fock space further, introducing partition combinatorics and drawing a connection with spin representations of $S_n$ in characteristic $2n + 1$; this connection revolves around the fact that the action of the standard generators of $U_q(g)$ corresponds to Morris’s branching rules describing induction and restriction of spin representations between $S_m$ and $S_{m+1}$. The submodule of the Fock space generated by the empty partition is called the basic representation, and possesses an important basis called the canonical basis. The coefficients expressing canonical basis elements in terms of the standard basis for the Fock space are called “$q$-decomposition numbers”, in view of a conjecture by Leclerc and Thibon [LT, Conjecture 6.2] that (after specialising at $q = 1$ and suitable rescaling) these coefficients coincide with decomposition numbers for spin representations of $S_m$ in characteristic $h$, provided $h$ is sufficiently large. The main results of the present paper (Theorems 5.2 and 6.4) are combinatorial formulæ for the $q$-decomposition numbers corresponding to spin blocks of $S_n$ of defect 1 or 2. Our formula for defect 1 (combined with Müller’s results) shows that the Leclerc–Thibon conjecture holds for blocks of defect 1. Our formula for defect 2 is very similar in spirit to Richards’s formula, though there is some “exceptional” behaviour for up to three canonical basis vectors in each block. Even though the Leclerc–Thibon conjecture in its original formulation now seems very unlikely to be true, we expect that it is true for blocks of defect 2, so that our formula specialises to give a formula for the decomposition numbers for spin blocks of defect 2.

The results in this paper appear to be the first significant results on $q$-decomposition numbers in type $A_{2n}^{(2)}$. It is to be expected that similar results will hold in other affine types, and can be proved via the same techniques. At the end of the paper we provide a brief discussion of corresponding results for type $A_{2n+1}^{(2)}$ and their relationship to the results for type $A_{2n}^{(2)}$, which partly explains the exceptional behaviour seen in some of the canonical basis vectors.

Our main technique is to exploit the combinatorics of partitions, reconciling the action of the quantum group on the Fock space with the combinatorial notions underlying the formula, in particular leg lengths and the dominance order. We build on the work of Kessar and Schaps [K, KS] to derive properties of Scopes–Kessar pairs of blocks to enable inductive proofs of our main results. The techniques we develop can be applied to blocks of higher defect, and modified to provide results for other Kac–Moody types.

We now summarise the layout of the paper. In Section 2, we explain the combinatorial background needed, which revolves around strict partitions and their $h$-bar-cores. In Section 3, we summarise the background relating to quantum algebra and the Fock space that we study in this paper. Section 4 introduces Scopes–Kessar pairs. In Section 5 we state and prove our main result for blocks of defect 1. In Section 6 we introduce the combinatorics particular to the case of blocks of defect 2, and state our main result. In Section 7 we give the base case for an induction proof of the main theorem; this involves explicit computation of the canonical basis for blocks with bar-cores of the form $(l, l-1, \ldots, 1)$. In Section 8 we further develop Scopes–Kessar pairs in the particular case of defect 2 blocks. Finally in Section 9 we give the inductive step to complete the proof of our main result. In Section 11 we outline the connection between our results and spin representations, and in Section 12 we discuss the relationship with the Fock space of type $A_{2n+1}^{(2)}$.

The research in this paper would not have been possible without extensive calculations using GAP [GAP].
2 Combinatorial background

Throughout this paper $h$ denotes an odd integer greater than 1, and we write $h = 2n + 1$.

In this section we outline the combinatorial set-up underlying both the Fock space of type $A^{(2)}_{h-1}$ and the modular spin representations of symmetric groups.

2.1 $h$-strict partitions, cores and blocks

A partition is an infinite weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers with finite sum. We write $|\lambda| = \lambda_1 + \lambda_2 + \ldots$, and say that $\lambda$ is a partition of $|\lambda|$. The integers $\lambda_1, \lambda_2, \ldots$ are called the parts of $\lambda$, and the number of positive parts of $\lambda$ is called the length of $\lambda$, written $\text{len}(\lambda)$. We may write $a \in \lambda$ or say that $\lambda$ contains $a$ if $\lambda_r = a$ for some $r$. When writing partitions, we usually group together equal parts with a superscript and omit the trailing zeroes, and we write the unique partition of 0 as $\varnothing$. We say that $\lambda$ is strict if $\lambda_r > \lambda_{r+1}$ for all $r < \text{len}(\lambda)$.

The Young diagram of a partition $\lambda$ is the set

$$[\lambda] = \{(r, c) \in \mathbb{N}^2 \mid c \leq \lambda_r\}$$

whose elements we call the nodes of $\lambda$. In general, a node means an element of $\mathbb{N}^2$. We draw Young diagrams as arrays of boxes using the English convention, in which $r$ increases down the page and $c$ increases from left to right.

The dominance order is a partial order $\preceq$ defined on the set of partitions of a given size by

$$\lambda \preceq \mu \iff \lambda_1 + \cdots + \lambda_r \leq \mu_1 + \cdots + \mu_r \text{ for all } r \geq 0.$$

We will also need two total orders on partitions. Given partitions $\lambda, \mu$, we write $\lambda \leq_{\text{lex}} \mu$ if there is $r$ such that $\lambda_r < \mu_r$ while $\lambda_s = \mu_s$ for all $s < r$. We say that $\lambda <_{\text{colex}} \mu$ if there is $r$ such that $\lambda_r > \mu_r$ while $\lambda_s = \mu_s$ for all $s > r$. Then $\leq_{\text{lex}}$ and $\leq_{\text{colex}}$ are total orders on $\mathcal{P}_h$ (called the lexicographic and colexicographic orders) which both refine the dominance order.

Finally we introduce some natural set-theoretic notation. Suppose $\lambda$ and $\mu$ are partitions.

- If $a$ is a natural number, we write $a \in \lambda$ to mean that there is some $r$ with $\lambda_r = a$.
- We write $\lambda \sqcup \mu$ for the partition obtained by combining the parts of $\lambda$ and $\mu$ and arranging them into decreasing order.
- We write $\lambda \cap \mu$ for partition in which the number of parts equal to $a$ is the smaller of the number of parts of $\lambda$ equal to $a$ and the number of parts of $\mu$ equal to $a$, for each $a$.
- If $\mu$ is strict and $\mu_r \in \lambda$ for each $r$, we define $\lambda \setminus \mu$ to be the partition obtained by deleting one copy of $\mu_r$ from $\lambda_r$ for each $r$.

Now we introduce the odd integer $h$ into the combinatorics. A partition $\lambda$ is $h$-strict if for every $r < \text{len}(\lambda)$ either $\lambda_r > \lambda_{r+1}$ or $\lambda_r \equiv 0 \pmod{h}$. An $h$-strict partition is restricted if for every $r$ either $\lambda_{r+1} > \lambda_r - h$ or $\lambda_{r+1} = \lambda_r - h \not\equiv 0 \pmod{h}$. Throughout this paper we write $\mathcal{P}_h$ for the set of all $h$-strict partitions.
For example, the 3-strict partitions of 8 are
\[(8), (7, 1), (6, 2), (5, 3), (5, 2, 1), (4, 3, 1), (3^2, 2),\]
and of these only the last three are restricted.

The residue of a node \((r, c)\) is the smaller of the residues of \(c - 1\) and \(-c\) modulo \(h\). A node of residue \(i\) is called an \(i\)-node.

For example, if we take \(h = 5\) and \(\lambda = (11, 8, 6, 5^2)\), the residues of the nodes of \(\lambda\) are given in the following diagram.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 1 & 2 \\
0 & 1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 & 0 \\
\end{array}
\]

Given \(\lambda \in \mathcal{P}_h\) and \(0 \leq i \leq n\), let \(\mu\) be the smallest \(h\)-strict partition such that \([\mu] \subseteq [\lambda]\) and \([\lambda] \setminus [\mu]\) consists entirely of \(i\)-nodes. These nodes are called the removable \(i\)-nodes of \(\lambda\). Note that in the case \(i = 0\), a removable \(i\)-node of \(\lambda\) might not be a removable node in the conventional sense. For example, referring to the diagram above, we see that when \(h = 5\) the removable 0-nodes of \((11, 8, 6, 5^2)\) are \((1, 10), (1, 11), (3, 6)\) and \((5, 5)\).

Similarly, the addable \(i\)-nodes of \(\lambda\) are the \(i\)-nodes that can be added to \(\lambda\) (possibly together with other \(i\)-nodes) to create a larger \(h\)-strict partition. For example, when \(h = 5\) the addable 1-nodes of the partition \(\lambda = (11, 8, 6, 5^2)\) above are \((1, 12), (2, 9)\) and \((3, 7)\).

The \(h\)-content of a partition \(\lambda\) is the multiset of residues of the nodes of \(\lambda\). For example, when \(h = 5\) the partition \((11, 8, 6, 5^2)\) has fifteen 0-nodes, thirteen 1-nodes and seven 2-nodes, so we write its 5-content as \([0^{15}, 1^{13}, 2^7]\).

Now we introduce \(h\)-bar-cores and blocks. Suppose \(\lambda \in \mathcal{P}_h\). Removing an \(h\)-bar from \(\lambda\) means constructing a smaller \(h\)-strict partition by doing one of two things:

- replacing a part \(\lambda_r \geq h\) with \(\lambda_r - h\) and reordering the parts into decreasing order;
- removing two parts which sum to \(h\).

\(\lambda\) is called an \(h\)-bar-core if it is not possible to remove an \(h\)-bar from \(\lambda\). In general, the \(h\)-bar-core of \(\lambda \in \mathcal{P}_h\) is the \(h\)-bar-core obtained by repeatedly removing \(h\)-bars until it is not possible to remove any more. It is an easy exercise to show that the \(h\)-bar-core of \(\lambda\) is well-defined, and the \(h\)-bar-weight is the number of \(h\)-bars removed to reach the \(h\)-bar-core.

For example, suppose \(\lambda = (9, 6, 3, 1)\). Then the 5-bar-core of \(\lambda\) is \((3, 1)\), and its 5-bar-weight is 3, as we see from the following diagrams.

![Diagrams showing the process of removing 5-bars from \(\lambda\) to reach its 5-bar-core.]

Later we will need the following result.

**Proposition 2.1 [MY1, Theorem 5].** Suppose \(\lambda, \mu \in \mathcal{P}_h\) with \(|\lambda| = |\mu|\). Then \(\lambda\) and \(\mu\) have the same \(h\)-bar-core if and only if they have the same \(h\)-content.
Suppose $\tau$ is an $h$-bar-core and $w \geq 0$. We define $P_{\tau,w}$ to be the set of all $h$-strict partitions with $h$-bar-core $\tau$ and $h$-bar-weight $w$. We call $P_{\tau,w}$ the combinatorial block with bar-core $\tau$ and bar-weight $w$.

We end this section with some more notation which we shall use repeatedly: if $\tau$ is a strict partition and $x < y$ are integers, we write $\{x, y\}_\tau$ for the number of parts of $\tau$ lying strictly between $x$ and $y$. As a special case of this, we set $\Gamma(\tau) = \{0, h\}_\tau$.

2.2 The abacus

Abacus notation for partitions was introduced by James, and has proved to be a valuable tool in the combinatorial modular representation theory of symmetric groups. A different abacus notation for strict partitions was introduced by Bessenrodt, Morris and Olsson to play the analogous role in the theory of spin representations. Here we introduce an alternative abacus notation which appears not to have been used before.

We take an abacus with $h$ vertical runners numbered $-n, 1 - n, \ldots, n-1, n$ from left to right. On runner $i$ we mark positions labelled with the integers in $i + h\mathbb{Z}$ increasing down the runner, so that (if $a \not\equiv n \pmod{h}$) position $a + 1$ appears directly to the right of position $a$. For example, if $h = 9$, the abacus is drawn as follows.

```
-4 -3 -2 -1  0  1  2  3  4
```

Now given a strict partition $\lambda$ of length $l$, the abacus display for $\lambda$ is obtained by placing black beads in positions $\lambda_1, \ldots, \lambda_l$ and in all negative positions except $-\lambda_1, \ldots, -\lambda_l$. We place a white bead in position 0.

For example, if $h = 9$ and $\lambda = (15, 11, 5, 4, 2, 1)$, the abacus display for $\lambda$ is as follows.

![Abacus Display](image)

Now given a strict partition $\lambda$ of length $l$, the abacus display for $\lambda$ is obtained by placing black beads in positions $\lambda_1, \ldots, \lambda_l$ and in all negative positions except $-\lambda_1, \ldots, -\lambda_l$. We place a white bead in position 0.

For example, if $h = 9$ and $\lambda = (15, 11, 5, 4, 2, 1)$, the abacus display for $\lambda$ is as follows.

![Abacus Display](image)

Here we have used a convention we shall apply throughout the paper: whenever we show an abacus display (or just a portion of an abacus display consisting of certain chosen runners), all positions above those shown are occupied, and all positions below those shown are unoccupied.

We shall also occasionally consider the abacus display of a partition which is $h$-strict but not strict. In this case, if a part $ah$ occurs $t$ times in $\lambda$, we regard the abacus as having $t$ beads at
position $ah$, and $t$ empty spaces at position $-ah$; we depict this by labelling the bead at position $ah$ and the empty space at position $-ah$ with the integer $t$.

The effect of adding a node to an $h$-strict partition is easy to see on the abacus. The following lemma follows from the definitions.

**Lemma 2.2.** Suppose $\lambda, \mu \in \mathcal{P}_h$, and that $\mu$ is obtained from $\lambda$ by adding an $i$-node.

1. If $1 \leq i \leq n$, then the abacus display for $\mu$ is obtained from the abacus display for $\lambda$ by moving a bead from position $b$ on runner $i$ to runner position $b + 1$, and simultaneously moving a bead from position $-b - 1$ to position $-b$.

2. If $i = 0$, then the abacus display for $\mu$ is obtained from the abacus display for $\lambda$ either by moving a bead from position $b$ on runner $0$ to position $b + 1$ and simultaneously moving a bead from position $-b - 1$ to position $-b$, or moving a bead from position $-1$ to position $1$.

The abacus display also makes it easy to visualise removal of $h$-bars and construction of the $h$-bar-core of an $h$-strict partition. Suppose $\lambda$ is an $h$-strict partition from which we can remove an $h$-bar. There are three ways we do this, and we consider the effect on the abacus display in each case.

- We can replace a part $a + h$ with $a$, where $a \geq 1$. In this case on the abacus we move the bead at position $a + h$ to position $a$, and we move the bead at position $-a$ to position $-a - h$.

- We can delete two parts $a$ and $h - a$, where $1 \leq a < h$. In this case we move the beads at positions $a, h - a$ to positions $a - h, -a$.

- We can delete the part $h$. In this case we move the bead at position $h$ to position $-h$.

We see that in each case, removing an $h$-bar involves moving beads up their runners into unoccupied positions. As a consequence, we find that the abacus display for the $h$-bar-core of $\lambda$ may be obtained by moving all beads up their runners as far as they will go. In particular, we have the following lemma.

**Lemma 2.3.** Suppose $\tau \in \mathcal{P}_h$. Then $\tau$ is an $h$-bar-core if and only if every bead in the abacus display for $\tau$ has a black bead immediately above it.

Returning to the partition in the above example, we obtain the abacus display for the 9-bar-core $(11, 6, 2, 1)$.

![Abacus Display](image)

### 2.3 Abacus notation for $h$-strict partitions of $h$-bar-weight 2

This paper is mostly concerned with $h$-strict partitions of $h$-bar-weight 2, and here we define two further items of notation which apply for $h$-bar-weight 2.

Suppose $\lambda$ has $h$-bar-weight 2, and let $\tau$ be the $h$-bar-core of $\lambda$. First we define two integers which we call the *bar positions* of $\lambda$. When we construct $\tau$ from $\lambda$ by removing two $h$-bars, we record an integer for each $h$-bar:
• if we replace an integer $a$ with $a - h$, we record the integer $a$;
• if we remove the integers $a$ and $h - a$ for $1 \leq a \leq n$, we record the integer $h - a$.

We let $a_\lambda$ and $b_\lambda$ be the two integers thus recorded, with $a_\lambda \leq b_\lambda$.

We also define a notation for $\lambda$ based on the abacus (on the assumption that $\tau$ is understood). Let $a_\lambda, b_\lambda$ be the bar positions of $\lambda$, and suppose these positions lie on runners $i$ and $j$ of the abacus respectively. Then one of the following occurs.

• $i \neq j$. In this case the abacus notation for $\lambda$ is $\langle |i|, |j| \rangle$ (or $\langle |j|, |i| \rangle$).
• $b_\lambda = a_\lambda + h$. In this case, if $a_\lambda \notin \lambda$, the abacus notation for $\lambda$ is
  \[
  \begin{cases}
    \langle i \rangle & \text{if } a_\lambda \notin \lambda \\
    \langle -i \rangle & \text{if } a_\lambda \in \lambda.
  \end{cases}
  \]
• $a_\lambda = b_\lambda = 0$. In this case, the abacus notation for $\lambda$ is $\langle 0, 0 \rangle$.
• $a_\lambda < b_\lambda$, with $a_\lambda + b_\lambda = 2h$. In this case the abacus notation for $\lambda$ is
  \[
  \begin{cases}
    \langle i \rangle & \text{if } h - a_\lambda \in \lambda \\
    \langle -i \rangle & \text{if } h - a_\lambda \notin \lambda.
  \end{cases}
  \]

**Example.** Take $h = 7$. Then the partitions $\lambda = (15, 9, 2)$ and $\mu = (15, 9, 8, 2)$ satisfy $a_\lambda = a_\mu = 8$ and $b_\lambda = b_\mu = 15$. The abacus notations for $\lambda$ and $\mu$ are $\langle 1 \rangle$ and $\langle -1 \rangle$ respectively.

$$
\begin{align*}
\lambda & = (15, 9, 2) \\
\mu & = (15, 9, 8, 2)
\end{align*}
$$

## 3 The $q$-Fock space

Now we introduce the background we shall need from quantum algebra. This is essentially taken from the paper [LT] by Leclerc and Thibon; note, however, that the residues used there are the opposite of ours: a node of residue $i$ in this paper has residue $n - i$ in [LT].

### 3.1 The quantum algebra and the Fock space

We consider the quantum group $\mathcal{U} = U_q(A^{(2)}_{n-1})$ associated to the generalised Cartan matrix of type $A^{(2)}_{n-1}$. This comes with the usual Kac–Moody set-up of simple roots $\alpha_0, \ldots, \alpha_n$ and fundamental weights $\Lambda_0, \ldots, \Lambda_n$. We refer to the book by Hong and Kang [HK] for the background on quantised Kac–Moody algebras.

$\mathcal{U}$ has standard generators $e_i, f_i, t_i$ for $i = 0, \ldots, n$. We define

\[
q_i = \begin{cases} 
q & (i = 0) \\
q^2 & (0 < i < n) \\
q^n & (i = n)
\end{cases}
\]
and then set (for $k \geq 0$)

$$[k]_i = \frac{q^k_i - q^{-k}_i}{q^1_i - q^{-1}_i}, \quad [k]_i^m = [1][2]_i \cdots [k]_i, \quad \epsilon_i^{(k)} = \frac{e_i^k}{[k]_i^m}, \quad f_i^{(k)} = f_i^k/[k]_i^m.$$  

The $q$-deformed Fock space $\mathcal{F}$ (of level 1) is a vector space over $\mathbb{C}(q)$ with $\mathcal{P}_h$ as a basis. This space is naturally a module for $\mathcal{U}$, and we can give combinatorial rules for the action of the divided powers $\epsilon_i^{(k)}$ and $f_i^{(k)}$. (Note that Leclerc and Thibon only give the actions of $e_i$ and $f_i$, and they express them in terms of straightening rules; but our rules can be easily deduced from the rules in [LT].)

First we consider $f_i^{(k)}$. Suppose $\lambda, \mu \in \mathcal{P}_h$ and that $[\mu] \supseteq [\lambda]$ and $[\mu] \setminus [\lambda]$ consists of $a$ nodes of residue $i$; then we write $\lambda \xrightarrow{a_i} \mu$, and we define a coefficient $N(\lambda, \mu)$ as follows: let $s$ be the sum, over all nodes $(r, c)$ of $[\mu] \setminus [\lambda]$, of the number of addable $i$-nodes of $\mu$ to the left of $(r, c)$ minus the number of removable $i$-nodes of $\lambda$ to the left of $(r, c)$. Further, if $i = 0$, let $M$ be the set of integers $m \geq 1$ such that there is a node of $[\mu] \setminus [\lambda]$ in column $mh + 1$ but not in column $mh$; for each $m \in M$, let $b_m$ be the number of times $mh$ occurs as a part of $\lambda$, and set $f_m = 1 - (-q^2)^{b_m}$.

Now define

$$N(\lambda, \mu) = \begin{cases} q_i^k \prod_{m \in M} f_m & (i = 0) \\ q_i^k & (i \neq 0). \end{cases}$$

Then we have

$$f_i^{(a)} \lambda = \sum_{\lambda \rightarrow \mu} N(\lambda, \mu) \mu.$$  

**Example.** Take $h = 5$ and $\lambda = (5, 4)$. Let us calculate $f_i^{(k)} \lambda$ for each $k$. $\lambda$ has three addable 0-nodes, namely $(1, 6)$, $(2, 5)$ and $(3, 1)$. Applying the formula above, we obtain

$$f_0 \lambda = (5, 4, 1) + q(5^2) + (q^4 + q^2)(6, 4),$$

$$f_0^{(2)} \lambda = (5^2, 1) + (q^3 + q)(6, 4, 1) + q^2(6, 5),$$

$$f_0^{(3)} \lambda = (6, 4, 1),$$

$$f_0^{(k)} \lambda = 0 \text{ for all } k \geq 4.$$  

The rule for the action of $e_i$ is similar. We define a coefficient $N(\lambda, \mu)$ whenever $\lambda \xrightarrow{a_i} \mu$ as follows: let $s$ be the sum, over all nodes $(r, c)$ of $[\lambda] \setminus [\mu]$, of the number of removable $i$-nodes of $\mu$ to the right of $(r, c)$ minus the number of addable $i$-nodes of $\lambda$ to the right of $(r, c)$. Further, if $i = 0$, let $M$ be the set of integers $m \geq 1$ such that there is a node of $[\lambda] \setminus [\mu]$ in column $mh$ but not in column $mh + 1$; let $b_m$ be the number of times $mh$ occurs as a part of $\lambda$, and set $f_m = 1 - (-q^2)^{b_m}$.

Now define

$$N(\lambda, \mu) = \begin{cases} q_i^k \prod_{m \in M} f_m & (i = 0) \\ q_i^k & (i \neq 0). \end{cases}$$

Then we have

$$e_i^{(a)} \lambda = \sum_{\lambda \rightarrow \mu} N(\lambda, \mu) \mu.$$
\( \mathcal{F} \) has a weight space decomposition \( \mathcal{F} = \bigoplus_{\alpha} \mathcal{F}_{\lambda - \alpha} \), with \( \alpha \) ranging over positive roots. If \( \alpha \) is the root \( \sum_{i=0}^{n} a_i \alpha_i \) with each \( a_i \) non-negative, then the weight space \( \mathcal{F}_{\lambda - \alpha} \) is spanned by the \( h \)-strict partitions \( \lambda \) having exactly \( a_i \) \( i \)-nodes, for each \( i \). By Proposition 2.1 these partitions form a combinatorial block, with a corresponding bar-core \( \tau \) and bar-weight \( w \), and we abuse terminology by referring to the weight space spanned by these partitions as the block with bar-core \( \tau \) and bar-weight \( w \). This helps us to avoid over-taxing the word “weight”, and keeps in mind the connection with blocks in the sense of modular representation theory. For any \( h \)-bar-core \( \tau \) and non-negative integer \( w \) there is a block with bar-core \( \tau \) and bar-weight \( w \), and we write this as \( B_{\tau, w} \). The aim of this paper is to study blocks with small bar-weight.

### 3.2 The canonical basis

Now let \( V_0 \) denote the submodule of \( \mathcal{F} \) generated by the empty partition \( \emptyset \). Then \( V_0 \) is isomorphic to the irreducible highest-weight \( U \)-module \( \Lambda(\emptyset) \). \( V_0 \) possesses a canonical basis, defined as follows. The bar involution is the \( C(q + q^{-1}) \)-linear involution \( v \rightarrow \bar{v} \) on \( V_0 \) defined by \( \emptyset = \emptyset, \overline{\emptyset} = \emptyset \) and \( f_i \overline{v} = \overline{f_i v} \). We say that a vector \( v \in V_0 \) is bar-invariant if \( v = \overline{v} \); this means that \( v \) can be written as a linear combination, with coefficients lying in \( C(q + q^{-1}) \), of vectors of the form \( f_{i_1} \ldots f_{i_k} \emptyset \). For each restricted \( h \)-strict partition \( \mu \), there is a unique vector \( G(\mu) \in \mathcal{F} \) with the following properties.

(CB1) \( G(\mu) \) is bar-invariant.

(CB2) When we write \( G(\mu) = \sum_{\lambda} d_{\lambda \mu} \lambda \), the coefficient \( d_{\lambda \mu} \) equals 1 while all the other coefficients \( d_{\lambda \mu} \) are polynomials divisible by \( q \).

\( G(\mu) \) is called the canonical basis vector corresponding to \( \mu \), and the set

\[ \{ G(\mu) \mid \mu \in \mathcal{P}_h \} \]

is the canonical basis of \( V_0 \). Every bar-invariant vector in \( V_0 \) is a linear combination, with coefficients in \( C(q + q^{-1}) \), of canonical basis vectors. The canonical basis can be computed recursively via the LT algorithm [LT, Section 4].

The coefficients \( d_{\lambda \mu} \) are called canonical basis coefficients or \( q \)-decomposition numbers, and they satisfy the following additional property:

(CB3) If \( d_{\lambda \mu} \neq 0 \), then \( \mu \preceq \lambda \), and \( \lambda \) and \( \mu \) have the same \( h \)-content.

This means in particular that canonical basis vectors are weight vectors, so we can consider the canonical basis for a given block.

**Example.** Take \( h = 5 \). Using the rules above, we can compute

\[ f_1 f_2 f_1 f_0^{(3)} f_1 f_2 f_1 f_0 \emptyset = (6, 4) + q^2(7, 3) + q^2(8, 2) + q^4(9, 1), \]

so this is the canonical basis vector \( G(6, 4) \). We can also compute

\[ f_1 f_2 f_1 f_0^{(2)} f_1 f_2 f_1 f_0 \emptyset = (5, 3, 2) + q^2(5, 4, 1) + (q^2 + 1)(6, 4) + (q^4 + q^2)(7, 3) + q^2(8, 2) + q^4(9, 1). \]

Subtracting \( G(6, 4) \), we obtain

\[ (5, 3, 2) + q^2(5, 4, 1) + q^2(6, 4) + q^4(7, 3), \]

and this is the canonical basis vector \( G(5, 3, 2) \).
One approach to understanding canonical basis coefficients is to begin with blocks of small bar-weight. The case of bar-weight 0 is straightforward: given an $h$-bar-core $\tau$, the block $B_{\tau,0}$ is 1-dimensional, spanned by the vector $G(\tau) = \tau$. In the remainder of this paper we will compute the canonical basis coefficients for blocks of bar-weight 1 and 2.

4 Scopes–Kessar pairs

Our main tool for computing canonical bases for blocks of a given bar-weight $w$ will be $r$: $k$-pairs. These are pairs of blocks $B_{\sigma,w}, B_{\tau,w}$ such that $f^k_i$ gives a vector space isomorphism $B_{\sigma,w} \to B_{\tau,w}$ for some $i$; this means that we can deduce the canonical basis coefficients for $B_{\tau,w}$ from those for $B_{\sigma,w}$, and these canonical basis coefficients will be very similar (in some cases identical). The genesis of this theory is the fundamental work by Scopes [Sco] for blocks of symmetric groups; a version for double covers of symmetric groups was developed by Kessar [K].

First we define a family of involutions $\psi_0, \ldots, \psi_n$ on $P_h$. Take $\lambda \in P_h$ and $i \in \{0, \ldots, n\}$. Define the $i$-signature of $\lambda$ by working along the edge of $\lambda$ from left to right, writing a $+$ for each addable $i$-node and a $-$ for each removable $i$-node. Now construct the reduced $i$-signature by repeatedly deleting adjacent pairs $+$ in the $i$-signature. The removable $i$-nodes corresponding to the $-$ signs in the reduced $i$-signature are called normal $i$-nodes of $\lambda$, while the addable $i$-nodes corresponding to the $+$ signs are called the conormal nodes of $\lambda$.

Suppose $\lambda$ has $r$ normal $i$-nodes and $s$ conormal $i$-nodes. Define $\psi_i(\lambda)$ by:

- adding the leftmost $s - r$ conormal $i$-nodes, if $s \geq r$;
- removing the rightmost $r - s$ normal $i$-nodes, if $r \geq s$.

It is an easy exercise to check that $\psi_i(\lambda)$ is also an $h$-strict partition.

Remark. The involution $\psi_i$ derives from the crystal structure of $\mathcal{P}$: the theory of crystal bases (see [HK] for an introduction) defines a directed graph (the crystal of $\mathcal{P}$) with vertex set $P_h$ and edges labelled by residues $i \in \{0, \ldots, n\}$; the subgraph formed by the edges labelled $i$ is just a disjoint union of directed paths, and the effect of $\psi_i$ is to reverse each of these paths.

Example. Suppose $h = 3, \lambda = (5,4,2,1)$ and $i = 0$. The addable and removable 0-nodes of $\lambda$ are indicated in the following diagram.

```
+ +
+ +
+ +
```

We see that the 0-signature of $\lambda$ is $- + - + +$, so that the reduced 0-signature is $- + +$; $\lambda$ has one normal 0-node $(4,1)$, and two conormal 0-nodes $(1,6)$ and $(1,7)$. So $\psi_0(\lambda) = (6,4,2,1)$.

We remark on two very easy special cases.

Lemma 4.1. Suppose $\lambda \in P_h$ and $i \in \{0, \ldots, n\}$.

1. If $\lambda$ has no removable $i$-nodes, then $\psi_i(\lambda)$ is obtained by adding all the addable $i$-nodes to $\lambda$. 


2. If \( \lambda \) has no addable \( i \)-nodes, then \( \psi_i(\lambda) \) is obtained by removing all the removable \( i \)-nodes from \( \lambda \).

We also note some properties of the functions \( \psi_i \), which are well known and easy to prove. (The first of these comes from the following simple observation: if \( \lambda, \mu \in \mathcal{P}_h \) and \( \mu \) is obtained by adding an \( i \)-node \((r, c)\) to \( \lambda \), then \((r, c)\) is the rightmost normal \( i \)-node of \( \mu \) if and only if it is the leftmost conormal \( i \)-node of \( \lambda \).)

**Lemma 4.2.** Suppose \( \lambda \in \mathcal{P}_h \) and \( i \in \{0, \ldots, n\} \).

1. \( \psi_i^2(\lambda) = \lambda \).
2. \( \psi_i(\lambda) \) is restricted if and only if \( \lambda \) is restricted.

Now we consider the particular case of \( h \)-bar-cores.

**Proposition 4.3.** Suppose \( \sigma \) is an \( h \)-bar-core and \( 0 \leq i \leq n \). Then:

1. \( \sigma \) cannot have both addable and removable \( i \)-nodes;
2. \( \psi_i(\sigma) \) is an \( h \)-bar-core;
3. if \( \tau \) is an \( h \)-bar-core obtained by adding some \( i \)-nodes to \( \sigma \), then \( \tau = \psi_i(\sigma) \).

**Proof.** We use the abacus, in particular Lemmas 2.2 and 2.3.

First suppose \( 1 \leq i < n \). If \( \sigma \) has addable \( i \)-nodes, then there is at least one bead on runner \( i \) of the abacus display for \( \sigma \) with an empty space immediately to its right. But by Lemma 2.3 the assumption that \( \sigma \) is an \( h \)-bar-core means that every bead in the abacus display has a bead immediately above it. So runners \( i \) and \( i + 1 \) of the abacus display have the following form.

![Abacus Display](image)

As a consequence, we see that \( \sigma \) has no removable \( i \)-nodes, proving (1). Moreover, we see that the abacus display for \( \psi_i(\sigma) \) is obtained by simply switching runners \( i \) and \( i + 1 \) (and also runners \(-i - 1\) and \(-i\)) so every bead in the abacus display for \( \psi_i(\sigma) \) has a bead immediately above it. So by Lemma 2.3 \( \psi_i(\sigma) \) is an \( h \)-bar-core, so (2) holds. For (3), apply (1) to \( \tau \): since by assumption \( \tau \) has removable \( i \)-nodes, it cannot have addable \( i \)-nodes, so in order to obtain \( \tau \) from \( \sigma \), all the addable \( i \)-nodes must have been added, and therefore \( \tau = \psi_i(\sigma) \).

The case \( i = n \) is very similar, except that here there is only one pair of runners to consider, namely \( n \) and \(-n\), and the phrase “to the right of” must be reinterpreted appropriately.

The case \( i = 0 \) is also similar, except that here there are three runners to consider, namely runners \(-1\), 0 and 1. Here the fact that every bead in the abacus display for \( \sigma \) has a bead immediately above it and the assumption that \( \sigma \) has at least one addable \( 0 \)-node, together with the symmetry of the abacus, means that the configuration on runners \(-1\), 0 and 1 of the abacus...
display for $\sigma$ is as follows.

\[ \text{Diagram of abacus display for } \sigma. \]

Now the abacus display for $\psi_0(\sigma)$ is obtained by switching runners $-1$ and $1$, and the proof works as for $i \geq 1$.

As a by-product of the above proof, we see that if an $h$-bar-core $\sigma$ has any addable 0-nodes, it must have an odd number of them. Now we return to arbitrary $h$-strict partitions.

**Lemma 4.4.** Suppose $0 \leq i \leq n$ and $\lambda \in P_h$, and that $\lambda$ has $h$-bar-core $\sigma$ and $h$-bar-weight $w$. Then $\psi_i(\lambda)$ has $h$-bar-core $\psi_i(\sigma)$ and $h$-bar-weight $w$.

**Proof.** We assume $1 \leq i < n$, with the cases $i = 0$ and $i = n$ being similar. Let $\delta_i(\lambda)$ equal the number of addable $i$-nodes of $\lambda$ minus the number of removable $i$-nodes of $\lambda$. First we claim that $\delta_i(\lambda) = \delta_i(\sigma)$. To see this, we use the abacus display for $\lambda$. Lemma 2.2 shows that, if we take $N \gg 0$, then $\delta_i(\lambda)$ equals the number of beads on runner $i$ of the abacus display after position $i - Nh$ minus the number of beads on runner $i + 1$ after position $i + 1 - Nh$. Constructing the abacus display for $\sigma$ involves moving beads up their runners, so does not affect these numbers of beads; so $\delta_i(\sigma) = \delta_i(\lambda)$.

The way the reduced $i$-signature is constructed means that $\delta_i(\lambda)$ is also the number of conormal $i$-nodes of $\lambda$ minus the number of normal $i$-nodes. So $\psi_i(\lambda)$ is obtained from $\lambda$ by adding $\delta_i(\lambda)$ $i$-nodes if $\delta_i(\lambda) \geq 0$, or removing $-\delta_i(\lambda)$ $i$-nodes if $\delta_i(\lambda) < 0$. Since $\delta_i(\lambda) = \delta_i(\sigma)$, the same applies for $\sigma$ and $\psi_i(\sigma)$.

We can easily compare the $h$-contents of $\lambda$ and $\sigma$: removing an $h$-bar entails removing one $n$-node and two nodes of each residue $i \in \{0, \ldots, n-1\}$; so the $h$-content of $\sigma$ is obtained from the $h$-content of $\lambda$ by removing $w$ copies of $n$ and $2w$ copies of $i$ for each $0 \leq i < n$. The previous paragraph implies that the same relationship holds between the $h$-contents of $\psi_i(\lambda)$ and $\psi_i(\sigma)$. So $\psi_i(\lambda)$ has the same $h$-content as an $h$-strict partition with $h$-bar-core $\psi_i(\sigma)$ and $h$-bar-weight $w$, and so by Proposition 2.1 $\psi_i(\lambda)$ has $h$-bar-core $\psi_i(\sigma)$ and $h$-bar-weight $w$. \qed

Now we can introduce Scopes–Kessar pairs. Suppose $\sigma$ is an $h$-bar-core with $k$ addable $i$-nodes, where $k \geq 1$. Let $\tau = \psi_i(\sigma)$. We say that $B_{\tau,w}$ and $B_{\tau,w}$ form a $[w:k]$-pair with residue $i$. It follows from Lemma 4.4 that $\psi_i$ restricts to a bijection between $P_{\tau,w}$ and $P_{\tau,w}$.

We define a partition $\lambda \in P_{\tau,w}$ to be unexceptional (for the pair $(B_{\tau,w}, B_{\tau,w})$) if it has no removable $i$-nodes, and we define a partition $\mu \in P_{\tau,w}$ to be unexceptional if it has no addable $i$-nodes. We consider the effect of the operator $f_i^{(k)}$.

**Proposition 4.5.** Suppose $\sigma, \tau, k, i$ are as above.

1. If $\lambda \in P_{\tau,w}$ is unexceptional, then $\lambda$ has exactly $k$ addable $i$-nodes, and $f_i^{(k)} \lambda = \psi_i(\lambda)$.

2. If $\mu \in P_{\tau,w}$ is restricted and $G(\mu)$ is a linear combination of unexceptional partitions, then (extending $\psi_i$ linearly)

   \[ G(\psi_i(\mu)) = \psi_i(G(\mu)). \]
Proof.
1. For the first statement we use the fact (shown in the proof of Lemma 4.4) that the number of addable $i$-nodes minus the number of removable $i$-nodes is the same for $\lambda$ as it is for $\sigma$; by Proposition 4.3 $\sigma$ has $k$ addable $i$-nodes and no removable $i$-nodes, so the same is true for $\lambda$.

Now the second statement follows from the formula for the action of $f_{i}^{(k)}$.

2. Write $G(\mu) = \sum_{\lambda} d_{\lambda \mu} \lambda$, with each $\lambda$ unexceptional. Then by (1)

$$f_{i}^{(k)} G(\mu) = \sum_{\lambda} d_{\lambda \mu} \psi_{i}(\lambda).$$

Since this vector is bar-invariant and each coefficient is divisible by $q$ except for the coefficient of $\psi_{i}(\mu)$ which equals 1, this vector must equal $G(\psi_{i}(\mu))$. 

This result provides a result analogous to the Scopes–Kessar equivalences for blocks of symmetric groups and their double covers; it says that if every partition in $B_{\sigma, w}$ is unexceptional, then $B_{\sigma, w}$ and $B_{\tau, w}$ have the same canonical basis (up to relabelling of basis elements); we say that $B_{\sigma, w}$ and $B_{\tau, w}$ are Scopes–Kessar equivalent in this case. (This is essentially the same as saying that $(\sigma, \tau)$ is a $w$-compatible pair, as defined by Kessar and Schaps [KS, Definition 3.1].) We now give some conditions on $i, k$ where this holds. (In many cases these conditions are weaker than the conditions given by Kessar.)

4.1 $[1 : k]$-pairs

Proposition 4.6. Suppose $\sigma$ and $\tau$ are $h$-bar-cores, and that $B_{\sigma, i}$ and $B_{\tau, 1}$ form a $[1 : k]$-pair of residue $i$, where either $i \neq 0$ or $k \geq 3$. Then $B_{\sigma, i}$ and $B_{\tau, 1}$ are Scopes–Kessar equivalent.

Proof. We need to show that a partition $\lambda \in \mathcal{P}_{\sigma, i}$ cannot have a removable $i$-node. We begin with the case $1 \leq i < n$, and examine runners $i$ and $i + 1$ of the abacus. In the abacus display for $\sigma$, there are $k$ beads on runner $i$ with an empty space to the right.

The abacus display for $\lambda$ is obtained by moving the lowest beads on runners $j, -j$ down one space each, for some $j$ (or moving the lowest black bead down two spaces, if $j = 0$). So the possible configurations of runners $i$ and $i + 1$ in the abacus display for $\lambda$ are as follows, and we see that $\lambda$ does not have a removable $i$-node.

The situation when $i = n$ is similar, though now there are $2k$ positions on runner $n$ immediately followed by an empty space on runner $-n$. Now there are two possible configurations of these two runners; in these diagrams we place runner $-n$ to the right of runner $n$, adjusting vertically.
so that position \( ah + n + 1 \) is directly to the right of position \( ah + n \).

![Diagram](image)

Again, we see that \( \lambda \) has no removable \( n \)-nodes.

In the case where \( i = 0 \) and \( k \geq 3 \), we write \( k = 2l + 1 \). Now we examine runners \(-1, 0, 1\) of the abacus. In the abacus display for \( \sigma \) there are \( k \) positions \( b \) on runner \(-1\) with a bead at position \( b \) but no bead at position \( b + 2 \); these are at positions \(-1 - lh, -1 - (l - 1)h, \ldots, -1 + lh\). Again, we see that for each possible configuration \( \lambda \) has no removable \( i \)-nodes.

![Diagram](image)

Note that Proposition 4.6 is not true in the case where \( i = 0 \) and \( k = 1 \); in fact, blocks forming a \([1 : 1]\)-pair of residue 0 have different canonical bases (though these canonical bases become equal under the specialisation \( q \to 1 \)).

### 4.2 \([2 : k]\)-pairs with residue \( n\)

Now we come to blocks of bar-weight 2. The next result is essentially equivalent to [KS, Lemma 4.1].

**Proposition 4.7.** Suppose \( \sigma \) and \( \tau \) are \( h \)-bar-cores, and that \( B_{\sigma, 2} \) and \( B_{\tau, 2} \) form a \([2 : k]\)-pair of residue \( n \). Then \( B_{\sigma, 2} \) and \( B_{\tau, 2} \) are Scopes–Kessar equivalent.

**Proof.** In the abacus display for \( \sigma \) there are \( 2k \) positions \( b \) on runner \( n \) such that there is a bead at position \( b \) but not at position \( b + 1 \). By considering the possible configurations of runners \( n \) and \(-n\) in the abacus display for \( \lambda \in \mathcal{P}_{\sigma, 2} \), we see that \( \lambda \) cannot have a removable \( n \)-node. In the following diagrams, we illustrate runners \( n \) and \(-n\) in the same way as in the proof of Proposition 4.6. In addition to the two configurations shown there, we have the following

![Diagram](image)
Defect 2 spin blocks and canonical basis coefficients

three possibilities.

\begin{align*}
2k - 2 & \quad (k = 1) \\
2k - 4 & \quad (k \geq 2)
\end{align*}

4.3 \([2 : k]\)-pairs with \(k \geq 2\)

The next result is the most closely related to Scopes’s original equivalence for the symmetric groups; the argument using the abacus is essentially the argument in [Sco, Lemma 2.1].

**Proposition 4.8.** Suppose \(\sigma\) and \(\tau\) are \(h\)-bar-cores, and that \(B_{\sigma,2}\) and \(B_{\tau,2}\) form a \([2 : k]\)-pair of residue \(i\), where \(1 \leq i < n\) and \(k \geq 2\). Then \(B_{\sigma,2}\) and \(B_{\tau,2}\) are Scopes–Kessar equivalent.

**Proof.** In the abacus display for \(\sigma\), there are \(k\) beads on runner \(i\) with no bead to their immediate right. Now consider the abacus display for a partition \(\lambda \in \mathcal{P}_{\sigma,2}\): this is obtained by moving beads down their runners, with at most two of these moves taking place on runners \(i\) and \(i + 1\). So it is not possible for there to be a bead on runner \(i + 1\) with an empty space to its immediate left, and hence \(\lambda\) cannot have a removable \(i\)-node. \(\square\)

4.4 \([2 : k]\)-pairs with residue 0

**Proposition 4.9.** Suppose \(\sigma\) and \(\tau\) are \(h\)-bar-cores, and that \(B_{\sigma,2}\) and \(B_{\tau,2}\) form a \([2 : k]\)-pair of residue 0, where \(k \geq 5\). Then \(B_{\sigma,2}\) and \(B_{\tau,2}\) are Scopes–Kessar equivalent.

**Proof.** Similarly to the proof of Proposition 4.7, we write \(k = 2l + 1\), and examine the possible configurations of runners \(-1, 0, 1\) in the abacus display for a partition \(\lambda \in \mathcal{P}_{\sigma,2}\), seeing in each case that \(\lambda\) has no removable 0-nodes. In addition to the three configurations in the proof of Proposition 4.6, we find the following.

\begin{align*}
l - 1 & \quad l - 2 \\
l - 1 & \quad l - 2 \\
l - 1 & \quad l - 2 \\
l - 1 & \quad l - 2
\end{align*}

We will study \([2 : 3]\)-pairs of residue 0 below. \([2 : 1]\)-pairs of residue 0 are rather more complex, and we avoid these in this paper.
4.5 \([w : k]\)-pairs and orders on partitions

We end this section by considering how the various orderings defined in Section 2.1 change as we pass through a \([w : k]\)-pair. Given \(\lambda, \mu \in \mathcal{P}_h\), we write \(\lambda \leftrightarrow \hat{\lambda}\) if \(\hat{\lambda}\) can be obtained from \(\lambda\) by adding and/or removing \(i\)-nodes. Observe that if \(B_{\sigma,w}\) and \(B_{\tau,w}\) form a \([w : k]\)-pair of residue \(i\) and \(\lambda \in \mathcal{P}_{\sigma,w}\), \(\hat{\lambda} \in \mathcal{P}_{\tau,w}\) with either \(\lambda\) or \(\hat{\lambda}\) unexceptional, then \(\lambda \leftrightarrow \hat{\lambda}\) if and only if 

\[\hat{\lambda} = \psi_i(\lambda)\]

Lemma 4.10. Suppose \(B_{\sigma,w}\) and \(B_{\tau,w}\) form a \([w : k]\)-pair of residue \(i\). Suppose \(\lambda, \mu \in \mathcal{P}_{\sigma,w}\) and \(\hat{\lambda}, \hat{\mu} \in \mathcal{P}_{\tau,w}\) with \(\lambda \leftrightarrow \hat{\lambda}\) and \(\mu \leftrightarrow \hat{\mu}\), and that \(\lambda\) is unexceptional.

1. If \(\lambda \gg \text{lex} \mu\), then \(\hat{\lambda} \gg \text{lex} \hat{\mu}\).
2. If \(\lambda < \text{colex} \mu\), then \(\hat{\lambda} < \text{colex} \hat{\mu}\).

Proof. We prove the first part only, as the proof of the second part is very similar.

Since \(\lambda > \text{lex} \mu\), there is \(r \geq 1\) such that \(\lambda_r > \mu_r\) while \(\lambda_s = \mu_s\) for all \(s < r\). So for any \(s < r\), \(\lambda\) has an addable \(i\)-node in row \(s\) if and only if \(\mu\) does. Since \(\lambda\) is unexceptional, \(\hat{\lambda}\) is obtained from \(\lambda\) by adding all the addable \(i\)-nodes, so we get \(\hat{\lambda}_s \geq \hat{\mu}_s\) for all \(s < r\). If \(\hat{\lambda}_s > \hat{\mu}_s\) for any \(s \leq r\) then we are done, so assume \(\hat{\lambda}_s = \hat{\mu}_s\) for \(s < r\), and \(\hat{\lambda}_r \leq \hat{\mu}_r\). This means that \(\mu\) has at least one addable \(i\)-node in row \(r\), and that the node \((r, \lambda_r)\) is an \(i\)-node. This node cannot be a removable \(i\)-node of \(\lambda\) because \(\lambda\) is unexceptional, which means that \(\lambda_{r+1} = \lambda_r - 1\) and \(\lambda_r \not\equiv 1 \pmod{h}\). Furthermore, we have \(\hat{\lambda}_r = \hat{\mu}_r\) (the only way this could in theory fail is if \(i = 0\) and \(\hat{\mu}_r = \mu_r + 2 = \lambda_r + 1\), but then the node \((r, \mu_r + 2)\) would also be an addable node of \(\lambda\), so that \(\hat{\lambda}_r = \hat{\mu}_r\)).

The deduction that \(\lambda_r \not\equiv 1 \pmod{h}\) means that \(\mu_r \not\equiv 0 \pmod{h}\), and hence \(\mu_{r+1} < \mu_r\). Hence \(\hat{\mu}_{r+1} < \mu_r \leq \lambda_r - 1 = \lambda_{r+1} \leq \hat{\lambda}_{r+1}\), and so we have \(\hat{\lambda} > \text{lex} \hat{\mu}\).

We end this section with a result which will be important in the induction proofs of our main theorems.

Lemma 4.11. Suppose \(\tau\) is an \(h\)-bar-core of length \(l\). Then one of the following occurs.

- \(\tau = (l, l - 1, \ldots, 1)\) and \(l \leq n\).
- \(\tau\) has a removable \(i\)-node, for some \(i \neq 0\).
- \(\tau\) has at least three removable \(0\)-nodes.

Proof. If \(\tau = (l, l - 1, \ldots, 1)\), then necessarily \(l \leq n\), since otherwise \(\tau\) would include the parts \(n\) and \(n + 1\), so would not be an \(h\)-bar-core. So assume that \(\tau \neq (l, l - 1, \ldots, 1)\). Assume also that \(\tau\) has no removable nodes of any non-zero residue. By assumption \(l > 0\), so the node \((l, \tau_l)\) is removable, and is therefore a removable \(0\)-node. This is particular means that \(\tau_l = 1\) (since otherwise it would be possible to remove an \(h\)-bar from \(\tau\)), and now the assumption \(\tau \neq (l, l - 1, \ldots, 1)\) means that there is \(r < l\) such that \(\tau_r = \tau_{r+1} \geq 2\). So the node \((r, \tau_r)\) is also a removable \(0\)-node. So \(\tau\) has at least two removable \(0\)-nodes, and hence at least three, by the remark following the proof of Proposition 4.3. \(\square\)
5 Blocks of bar-weight 1

In this section we determine the $q$-decomposition numbers for blocks of bar-weight 1. The decomposition numbers for spin blocks of symmetric groups of defect 1 are known, thanks to Müller [Mü]; the results in the present section provide a $q$-analogue of Müller’s results.

Throughout this section we fix an $h$-bar-core $\tau$. Recall that we define $\Gamma(\tau)$ to be the number of parts of $\tau$ less than $h$.

**Proposition 5.1.** There are exactly $n + 1$ $h$-strict partitions in $P_{\tau,1}$. These are totally ordered by the dominance order, with all except the most dominant one being restricted.

**Proof.** There are three different ways to add an $h$-bar to $\tau$ to obtain an $h$-strict partition.

**type $\oplus$:** Given $a \in \tau$ such that $a + h \notin \tau$, define $\tau[a + h] = \tau \cup (a + h) \setminus (a)$. This partition is restricted unless $a = \tau_1$.

**type 0:** Define $\tau[h] = \tau \cup (h)$. This partition is restricted unless $\tau = \emptyset$.

**type $\ominus$:** Given $b \in \{n + 1, \ldots, h - 1\}$ such that neither $b$ nor $h - b$ lies in $\tau$, define $\tau[b] = \tau \cup (b, h - b)$. This partition is restricted.

Now observe that if $\tau[i]$ and $\tau[j]$ are two partitions constructed in this way, then $\tau[i] \triangleleft \tau[j]$ if and only if $i < j$. Moreover, the number of partitions $\tau[a]$ of type $\oplus$ equals $\Gamma(\tau)$, and (because $\tau$ cannot contain integers $a$ and $h - a$) the number of partitions $\tau[b]$ of type $\ominus$ equals $n - \Gamma(\tau)$. So there are $n + 1$ partitions in $P_{\tau,1}$ altogether, and only the most dominant fails to be restricted. \hfill $\Box$

In view of Proposition 5.1, we can label the partitions in $P_{\tau,1}$ as $\tau(0) \triangleleft \cdots \triangleleft \tau(n)$. Now we can give the main result of this section.

**Theorem 5.2.** Suppose $\tau$ is an $h$-bar-core, and let $\tau(0) \triangleleft \cdots \triangleleft \tau(n)$ be the partitions in $P_{\tau,1}$. Then

$$d_{\tau(r)\tau(s)} = \begin{cases} 
1 & (r = s) \\
q & (r = s + 1, h \in \tau(r)) \\
q^2 & (r = s + 1, h \notin \tau(r)) \\
0 & \text{(otherwise)}.
\end{cases}$$

For example, with $h = 7$ and $\tau = (4,2)$, the $q$-decomposition numbers are given by the following matrix.

<table>
<thead>
<tr>
<th></th>
<th>(6,4,2,1)</th>
<th>(7,4,2)</th>
<th>(9,4)</th>
<th>(11,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6,4,2,1)</td>
<td>1</td>
<td>q</td>
<td>q^2</td>
<td>q^2</td>
</tr>
<tr>
<td>(7,4,2)</td>
<td>q</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(9,4)</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(11,2)</td>
<td></td>
<td></td>
<td></td>
<td>q^2</td>
</tr>
</tbody>
</table>

We prove Theorem 5.2 by induction. For the initial cases of the induction, we assume that $\tau$ has the form $(l, l - 1, \ldots, 1)$ for some $0 \leq l \leq n$. In this situation, Theorem 5.2 can be re-cast as follows.
Proposition 5.3. Suppose \( \tau = (l, l-1, \ldots, 1) \) for \( 0 \leq l \leq n \), and suppose \( \mu \in \mathcal{P}_{\tau,1} \) is restricted. Then one of the following occurs.

1. \( \mu = \tau \sqcup (h-b,b-1) \) for \( l+2 \leq b \leq n \). In this case
   \[
   G(\mu) = \mu + q^2(\tau \sqcup (h-b+1,b-1)).
   \]

2. \( \mu = \tau \sqcup (h-l-1,l+1) \) and \( l \leq n-1 \). In this case
   \[
   G(\mu) = \mu + q(\tau \sqcup (h)).
   \]

3. \( \mu = \tau \sqcup (h) \) and \( l \geq 1 \). In this case
   \[
   G(\mu) = \mu + q^2(\tau \sqcup (h+1) \setminus (1)).
   \]

4. \( \mu = \tau \sqcup (a+h) \setminus (a) \) with \( 1 \leq a \leq l-1 \). In this case
   \[
   G(\mu) = \mu + q^2(\tau \sqcup (a+h+1) \setminus (a+1)).
   \]

Proof. It is clear that \( \mu \) satisfies one (and only one) of the given conditions, so we just need to calculate \( G(\mu) \). In each case we construct an \( h \)-bar-core \( \xi \), and show that the given vector can be constructed from \( G(\xi) = \xi \) by applying the operators \( f_i \). The defining properties of the canonical basis then guarantee that this vector must equal \( G(\mu) \).

1. Let \( \xi = \tau \sqcup (h-b,b-1) \). Then
   \[
   f_{b-1} G(\xi) = \mu + q^2(\tau \sqcup (h-b+1,b-1)).
   \]

2. Let \( \xi = \tau \sqcup (h-l-1) \). Then
   \[
   f_0 f_1 \ldots f_l G(\xi) = \mu + q(\tau \sqcup (h)).
   \]

3. Let \( \xi = \tau \sqcup (h-1) \setminus (1) \). Then
   \[
   f_0(2) G(\xi) = \mu + q^2(\tau \sqcup (h+1) \setminus (1)).
   \]

4. Let \( \xi = \tau \sqcup (a+h) \setminus (a+1) \). Then
   \[
   f_a G(\xi) = \mu + q^2(\tau \sqcup (a+h+1) \setminus (a+1)).
   \]

Proof of Theorem 5.2. We proceed by induction on \( |\tau| \). Let \( l = \text{len}(\tau) \), and consider the three possibilities in Lemma 4.11. If \( \tau = (l, l-1, \ldots, 1) \) with \( l \leq n \), then Proposition 5.3 gives the result. Alternatively, there is a residue \( i \) such that either \( i \neq 0 \) and \( \tau \) has a removable \( i \)-node, or \( i = 0 \) and \( \tau \) has at least three removable \( i \)-nodes. So define the \( h \)-bar-core \( \xi \) by removing all the removable \( i \)-nodes from \( \tau \). Now \( B_{\xi,1} \) and \( B_{\tau,1} \) are Scopes–Kessar equivalent, by Proposition 4.6. Since the lexicographic order refines the dominance order, we have
   \[
   \xi(0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} \xi(n), \quad \tau(0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} \tau(n),
   \]
   and so by Lemma 4.10 \( \psi_0(\xi(r)) = \tau(r) \) for all \( r \). So by Proposition 4.5 \( d_{\xi(r)} \xi(s) = d_{\tau(s)} \tau(r) \) for all \( r, s \). Since \( i \neq 0 \) we also have \( h \in \xi(r) \) if and only if \( h \in \tau(r) \), and the result follows from the inductive hypothesis.
6 Blocks of bar-weight 2

Now we come to the main object of study in this paper: blocks of bar-weight 2. In this section we further develop the combinatorics of blocks of bar-weight 2 and state our main theorem.

We continue to work with a fixed $h$-bar-core $\tau$.

6.1 The dominance order

First we consider the dominance order in $P_{\tau,2}$, which will be central to our formula for the canonical basis coefficients. Given $\lambda, \mu \in P_{\tau,2}$, we give a simple sufficient criterion for $\lambda \preceq \mu$ in terms of the bar positions $a_\lambda, b_\lambda$ introduced in Section 2.2. The following is an analogue of (half of) [R, Lemma 4.4].

Proposition 6.1. Suppose $\lambda, \mu \in P_{\tau,2}$. If $a_\lambda \leq a_\mu$ and $b_\lambda \leq b_\mu$, then $\lambda \preceq \mu$.

Proof. We use the equivalent characterisation of the dominance order in terms of conjugate partitions. Letting $\lambda_1$ denote the partition conjugate to $\lambda$ (defined by $\lambda_1^r = \max \{ c \geq 1 | \lambda_c \geq r \}$), we have [JK, Lemma 1.4.11] $\lambda \preceq \mu$ if and only if $\lambda' \geq \mu'$. Now note that replacing an integer $a$ with $a + h$ in a strict partition entails adding one node to each of columns $a + 1, \ldots, a + h$; similarly, inserting the parts $a, h - a$ with $a \leq n$ entails adding two nodes to each of columns $1, \ldots, a$, and one node to each of columns $a + 1, \ldots, h - a$.

So if $a_\lambda \leq a_\mu$ and $b_\lambda \leq b_\mu$, then when we construct $\lambda$ from $\tau$, we add nodes in earlier columns than when we construct $\mu$ from $\tau$; so $\lambda \preceq \mu$.

6.2 Leg lengths

Richards’s formula for decomposition numbers of defect 2 blocks of symmetric groups involves leg lengths of rim hooks. Here we introduce the corresponding combinatorics for our situation.

Suppose $\lambda \in P_{\tau,2}$, and let $a_\lambda, b_\lambda$ be bar positions for $\lambda$, as defined above. Recall that we write $(x, y)^{\lambda, \tau}$ for the number of integers strictly between $x$ and $y$ that lie in both $\lambda$ and $\tau$. Take $c \in \{a_\lambda, b_\lambda\}$, and define the leg length corresponding to $c$ as follows:

- if $c \geq h$, define the corresponding leg length to be $(c - h, c)^{\lambda, \tau}$;
- if $n < c < h$, define the corresponding leg length to be $h - c + (h - c, c)^{\lambda, \tau}$.

We define $\partial \lambda$ to be the absolute difference between the leg lengths of $\lambda$.

Remark. Leg lengths for $h$-bars are also defined by Hoffman and Humphreys in [HH, p.185]. However, our definition differs slightly from theirs. For example, take $h = 7$ and $\lambda = (10, 5, 4)$. Then $a_\lambda = 4$ and $b_\lambda = 10$, giving leg lengths 1 and 3, and hence $\partial \lambda = 2$. Using the notion of leg length from [HH], the leg length corresponding to $c = 4$ would be 2, giving $\partial \lambda = 1$.

The key to Richards’s combinatorial formula for decomposition numbers is the interplay between the $\partial$ function and the dominance order. In our setting, the following result will be important.

Proposition 6.2. Suppose $\lambda, \mu \in P_{\tau,2}$. If $\lambda$ and $\mu$ are incomparable in the dominance order, then $|\partial \lambda - \partial \mu| \geq 2$. 
Proof. By Proposition 6.1, we may assume (interchanging \( \lambda \) and \( \mu \) if necessary) that \( b_\lambda > b_\mu \) and \( a_\lambda < a_\mu \). Then we claim that \( \partial \lambda \geq \partial \mu + 2 \).

There are several cases to check, depending on the relative order of \( b_\lambda, b_\mu, a_\mu, a_\lambda, b_\lambda - h, b_\mu - h, a_\mu - h, a_\lambda - h \). We show the calculations in three example cases, leaving the reader to check the other cases.

1. First suppose \( b_\lambda < h \). Then the leg lengths of \( \lambda \) are (in decreasing order)
   \[ h - a_\lambda + \langle h - a_\lambda, a_\lambda \rangle, \quad h - b_\lambda + \langle h - b_\lambda, b_\lambda \rangle \]
   and similarly for \( \mu \), giving
   \[ \partial \lambda - \partial \mu = b_\lambda - b_\mu + a_\mu - a_\lambda - \langle h - b_\lambda, h - b_\mu \rangle - \langle h - a_\mu, h - a_\lambda \rangle - \langle a_\lambda, a_\mu \rangle - \langle b_\mu, b_\lambda \rangle. \]
   Now the fact that \( \tau \) does not contain two parts summing to \( h \) implies that
   \[ \langle h - b_\lambda, h - b_\mu \rangle < b_\lambda - b_\mu \]
   \[ \langle h - a_\mu, h - a_\lambda \rangle + \langle a_\lambda, a_\mu \rangle < a_\mu - a_\lambda \]
   so that \( \partial \lambda - \partial \mu \geq 2 \).

2. For our next case, suppose \( a_\mu = b_\mu = h \). If \( b_\lambda - h \geq h - a_\lambda \), then we can obtain \( \mu \) from \( \lambda \) by moving nodes from columns \( b_\lambda - h + 1, \ldots, b_\lambda - h + a_\lambda \) to columns \( h - a_\lambda + 1, \ldots, h \) and from columns \( b_\lambda - h + a_\lambda + 1, \ldots, b_\lambda \) to columns \( a_\lambda + 1, \ldots, h \); each moved node moves to an earlier column, which gives \( \lambda \geq \mu \), contrary to assumption. So instead we must have \( b_\lambda - h < h - a_\lambda < a_\lambda < h \). Now the leg lengths for \( \lambda \) are
   \[ h - a_\lambda + \langle h - a_\lambda, a_\lambda \rangle, \quad \langle b_\lambda - h, b_\lambda \rangle \]
   while obviously \( \partial \mu = 0 \), so that
   \[ \partial \lambda - \partial \mu = h - a_\lambda - \langle b_\lambda - h, h - a_\lambda \rangle - \langle a_\lambda, h \rangle + \langle h, b_\lambda \rangle. \]
   Because \( \tau \) is an \( h \)-bar-core, \( \langle h, b_\lambda \rangle \leq \langle 0, b_\lambda - h \rangle \), so that
   \[ \partial \lambda - \partial \mu \geq h - a_\lambda - \langle 0, b_\lambda - h \rangle - \langle b_\lambda - h, h - a_\lambda \rangle - \langle a_\lambda, h \rangle = h - a_\lambda - \langle 0, h - a_\lambda \rangle + 1 \]
   with the 1 arising because \( b_\lambda - h \in \tau \). But now the fact that \( \tau \) does not contain two parts summing to \( h \) gives \( \langle 0, h - a_\lambda \rangle + \langle a_\lambda, h \rangle < h - a_\lambda \), so that \( \partial \lambda - \partial \mu \geq 2 \).

3. For another example case, suppose \( a_\lambda > h \) and \( b_\lambda < a_\lambda + h \). Then the leg lengths for \( \lambda \) are
   \[ \langle a_\lambda - h, a_\lambda \rangle - 1, \quad \langle b_\lambda - h, b_\lambda \rangle \]
   with the \(-1\) occurring because \( b_\lambda - h \) lies between \( a_\lambda - h \) and \( a_\lambda \) and belongs to \( \tau \) but not \( \lambda \), so does not get counted in the leg length. A similar statement applies for \( \mu \), so that
   \[ \partial \lambda - \partial \mu = \langle a_\lambda - h, a_\lambda \rangle - \langle a_\mu - h, a_\mu \rangle + \langle b_\mu - h, b_\mu \rangle - \langle b_\lambda - h, b_\lambda \rangle = \langle a_\lambda - h, a_\mu - h \rangle + \langle b_\mu - h, b_\lambda \rangle + \langle a_\lambda, a_\mu \rangle + \langle b_\mu, b_\lambda \rangle + 2 \]
   with the 2 arising from counting the occurrences of \( a_\mu - h, b_\mu - h, b_\lambda - h \), which all belong to \( \tau \). Since \( \tau \) is an \( h \)-bar-core we have \( \langle a_\lambda - h, a_\mu - h \rangle, \langle a_\lambda, a_\mu \rangle \) and \( \langle b_\mu - h, b_\lambda - h \rangle \), so that \( \partial \lambda - \partial \mu \geq 2 \). \( \square \)
Now (continuing the analogy with Richards’s work) we define the colour of a partition \( \lambda \in \mathcal{P}_{\tau,2} \). The way we define this depends on the value of \( \partial \lambda \).

\( \partial \lambda = 0 \): Suppose first that \( \lambda \) has a 2h-bar, i.e. either \( a_\lambda = b_\lambda - h \) or \( a_\lambda + b_\lambda = 2h \). We define the leg length of this 2h-bar analogously to the leg length of an \( h \)-bar: if \( b_\lambda \geq 2h \) then we define the leg length as \( l = |\lambda \cap \tau \cap \{b_\lambda - 2h + 1, \ldots, b_\lambda - 1\}| \), while if \( b_\lambda < 2h \) we define the leg length to be \( l = 2h - b_\lambda + |\lambda \cap \tau \cap \{2h - b_\lambda + 1, \ldots, b_\lambda - 1\}| \). Now we say \( \lambda \) is black if \( l + 2\Gamma(\tau) \) is congruent to 0 or 3 modulo 4, and white otherwise.

Alternatively, suppose \( \lambda \) has two \( h \)-bars, with common leg length \( l \). Say that \( \lambda \) is black if \( l + \Gamma(\tau) \) is odd, and white otherwise.

\( \partial \lambda = 1 \): If \( b_\lambda > h \), we say that \( \lambda \) is grey. If \( b_\lambda \leq h \), then let \( l, l + 1 \) be the two leg lengths of \( \lambda \). We say that \( \lambda \) is black if \( l + \Gamma(\tau) \) is odd, and white otherwise.

\( \partial \lambda \geq 2 \): In this case, we say \( \lambda \) is grey.

In the rest of the paper, we write \( \text{cl}\lambda \) for the colour of \( \lambda \). We end this subsection with a lemma we shall need later.

**Lemma 6.3.** Suppose \( \lambda \in \mathcal{P}_{\tau,2} \) with \( \partial \lambda = 0 \), and \( \lambda \) has a 2h-bar. Then \( a_\lambda \geq h \).

**Proof.** If \( a_\lambda < h \), there are two possibilities.

\( b_\lambda = a_\lambda + h \): In this case the leg lengths of \( \lambda \) are

\[
h - a_\lambda + \langle h - a_\lambda, a_\lambda \rangle_\tau, \quad \langle a_\lambda, a_\lambda + h \rangle_\tau,
\]

and we claim that these cannot be equal, contradicting the assumption that \( \partial \lambda = 0 \). Since \( \tau \) is an \( h \)-bar-core, we have

\[
\langle h - a_\lambda, a_\lambda \rangle_\tau \geq \langle 2h - a_\lambda, a_\lambda + h \rangle_\tau
\]

so in order for the two leg lengths to be equal we would have to have \( \langle a_\lambda, 2h - a_\lambda \rangle_\tau \geq h - a_\lambda \). But \( \tau \) cannot contain integers \( h + i \) and \( h - i \) for any \( 0 \leq i < h - a_\lambda \), so \( \langle a_\lambda, 2h - a_\lambda \rangle_\tau < h - a_\lambda \).

\( b_\lambda = 2h - a_\lambda \): In this case the leg lengths of \( \lambda \) are

\[
h - a_\lambda + \langle h - a_\lambda, a_\lambda \rangle_\tau, \quad \langle h - a_\lambda, 2h - a_\lambda \rangle_\tau,
\]

and if these are equal then

\[
h - a_\lambda = \langle a_\lambda, 2h - a_\lambda \rangle_\tau.
\]

But, as in the previous case, this cannot happen. \( \square \)

### 6.3 Special partitions

Our analogue of Richards’s formula breaks down for a few canonical basis vectors in \( B_{\tau,2} \), and our main theorem will deal with these separately. In order to deal with these, we single out some partitions which we call the *special* partitions in \( \mathcal{P}_{\tau,2} \). There are up to six of these, though some may be undefined, depending on the value of \( \Gamma(\tau) \).
6.4 The main theorem

Now we can give our main theorem for blocks of bar-weight 2.

Theorem 6.4. Suppose τ is an h-bar-core, and suppose λ, µ ∈ P_{τ,2} with µ restricted.

1. If µ ≠ hτ, hτ, hτ, then there is a partition µ⁺ in P_{τ,2} such that µ⁺ ▷ µ, △µ⁺ = △µ and cl µ = cl µ⁺. If we let µ⁺ be the least dominant such partition, and define

\[ q_{λµ} = \begin{cases} 
q & \text{if } \{h,2h\} \cap \lambda \neq \emptyset = \{h,2h\} \cap \mu \\
1 & \text{otherwise,}
\end{cases} \]

then

\[ q_{λµ}d_{λµ} = \begin{cases} 
1 & (λ = µ) \\
q^2 & (µ < λ < µ⁺, |△λ - △µ| = 1) \\
q^4 & (λ = µ⁺) \\
0 & \text{(otherwise).}
\end{cases} \]

2. If µ = hτ, then

\[ d_{λµ} = \begin{cases} 
1 & (λ = hτ) \\
q^3 + q & (hτ < λ < hτ, △λ = 1) \\
q^2 & (λ = hτ) \\
q^2 & (hτ < λ < hτ, △λ = 1) \\
q^4 & (λ = hτ) \\
0 & \text{(otherwise).}
\end{cases} \]


3. If $\mu = \Psi_\tau$, then

$$d_{\lambda \mu} = \begin{cases} 1 & (\lambda = \Psi_\tau) \\ q & (\lambda = \Delta_\tau) \\ q^2 & (\Psi_\tau < \lambda < \Delta_\tau, \partial \lambda = 2) \\ q^4 + q^2 & (\lambda = \Delta_\tau) \\ q^2 & (\Delta_\tau < \lambda < \Psi_\tau, \partial \lambda = 1) \\ q^3 & (\lambda = \Psi_\tau) \\ 0 & (\text{otherwise}). \end{cases}$$

4. If $\mu = \Psi_\tau$, then

$$d_{\lambda \mu} = \begin{cases} 1 & (\lambda = \Psi_\tau) \\ q & (\Psi_\tau < \lambda < \Psi_\tau, \partial \lambda = 2) \\ q & (\lambda = \Psi_\tau) \\ q^2 & (\lambda = \Delta_\tau) \\ q^3 + q & (\Psi_\tau < \lambda < \Delta_\tau, \partial \lambda = 2) \\ q^5 + q^3 & (\lambda = \Delta_\tau) \\ 0 & (\text{otherwise}). \end{cases}$$

7 The base case

As with the proof of Theorem 5.2, we prove Theorem 6.4 by induction, taking as base cases the blocks whose bar-core has at most one removable node, and no removable nodes of non-zero residue.

**Assumptions and notation in force for Section 7**

$\tau$ is the $h$-bar-core $(l, l-1, \ldots, 1)$ for $0 \leq l \leq n$. Given integers $x_1, \ldots, x_s > l$ and $l \geq y_1, \ldots, y_t \geq 0$, we will write $(x_1, \ldots, x_s | y_1, \ldots, y_t)$ to mean the partition $\tau \sqcup (x_1, \ldots, x_s) \setminus (y_1, \ldots, y_t)$.

In the following table, we classify the different types of partition in $B_{\tau,2}$, calculate their $\partial$-values and give the conditions under which they are restricted.

<table>
<thead>
<tr>
<th>partition</th>
<th>conditions</th>
<th>$\partial$</th>
<th>restricted</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(h-a, h-b, b, a \mid \cdot)$</td>
<td>$l &lt; a &lt; b \leq n$</td>
<td>$b - a$</td>
<td>always</td>
</tr>
<tr>
<td>$(h, h \mid \cdot)$</td>
<td>—</td>
<td>0</td>
<td>if $l &gt; 0$</td>
</tr>
<tr>
<td>$(h+a, h-b, b \mid a)$</td>
<td>$0 \leq a \leq l &lt; b \leq n$</td>
<td>$a + b - l$</td>
<td>always</td>
</tr>
<tr>
<td>$(h+a, h-a \mid \cdot)$</td>
<td>$1 \leq a \leq n$</td>
<td>$a$</td>
<td>always</td>
</tr>
<tr>
<td>$(h+b, h+a \mid b, a)$</td>
<td>$0 \leq a &lt; b \leq l$</td>
<td>$b - a - 1$</td>
<td>if $a \leq l - 2$</td>
</tr>
<tr>
<td>$(2h-a, a \mid \cdot)$</td>
<td>$l &lt; a \leq n$</td>
<td>$a$</td>
<td>never</td>
</tr>
<tr>
<td>$(2h+a \mid a)$</td>
<td>$0 \leq a \leq l$</td>
<td>$l - a$</td>
<td>never</td>
</tr>
</tbody>
</table>

Given this, we can find all the partitions in $P_{\tau,2}$ with $\partial$-value equal to 0 or 1, and compute their colours.
Lemma 7.1. The partitions \( \lambda \in P_{\tau,2} \) with \( \partial \lambda = 0 \) are given by the following table.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>conditions</th>
<th>cl ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (h, h</td>
<td>\cdot) )</td>
<td>—</td>
</tr>
<tr>
<td>( (h+a+1, h+a</td>
<td>a+1, a) ) ( 0 \leq a &lt; l )</td>
<td>{ black if ( a ) is even } { white if ( a ) is odd }</td>
</tr>
<tr>
<td>( (2h+1</td>
<td>1) ) ( 0 \leq l )</td>
<td>{ black if ( l ) is even } { white if ( l ) is odd }</td>
</tr>
</tbody>
</table>

Lemma 7.2. The partitions \( \lambda \in P_{\tau,2} \) with \( \partial \lambda = 1 \) are given by the following table.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>conditions</th>
<th>cl ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (h-a, h-a-1, a+1, a</td>
<td>\cdot) ) ( l &lt; a &lt; n )</td>
<td>{ black if ( a+l ) is odd } { white if ( a+l ) is even }</td>
</tr>
<tr>
<td>( (h, h-l-1, l+1</td>
<td>\cdot) ) ( l &lt; n )</td>
<td>white</td>
</tr>
<tr>
<td>( (h+1, h-1</td>
<td>\cdot) )</td>
<td>—</td>
</tr>
<tr>
<td>( (h+a+2, h+a</td>
<td>a+2, a) ) ( 0 \leq a \leq l-2 )</td>
<td>grey</td>
</tr>
<tr>
<td>( (2h-1, 1</td>
<td>\cdot) ) ( 0 = l )</td>
<td>grey</td>
</tr>
<tr>
<td>( (2h+1</td>
<td>l-1) ) ( 1 \leq l )</td>
<td>grey</td>
</tr>
</tbody>
</table>

Next we need to consider the special partitions in \( P_{\tau,2} \). The following result comes directly from the definitions in Section 6.3.

Lemma 7.3.
1. If \( l \leq n-2 \), the partition \( \wp_{\tau} \) equals \( (h-l-1, h-l-2, l+2, l+1 | \cdot) \), and is restricted.
2. If \( l \leq n-1 \), the partition \( \wp_{\tau} \) equals \( (h, h-l-1, l+1 | \cdot) \), and is restricted.
3. The partition \( \wp_{\tau} \) equals \( (h, h | \cdot) \), and is restricted if and only if \( l \geq 1 \).
4. If \( l \leq n-1 \), the partition \( \wp_{\tau} \) equals \( (h+1, h-1 | \cdot) \).
5. The partition \( \wp_{\tau} \) equals \( (h+1, h | 1) \) if \( l \geq 1 \), or \( (2h | \cdot) \) if \( l = 0 \).
6. If \( l \geq 2 \), the partition \( \wp_{\tau} \) equals \( (h+2, h+1 | 2, 1) \); if \( l = 1 \), then \( \wp_{\tau} \) equals \( (2h+1 | 1) \).

Now that we have a list of restricted partitions in \( P_{\tau,2} \), we can calculate the canonical basis vectors. For each restricted \( \mu \in P_{\tau,2} \) we compute \( G(\mu) \) by applying a suitable combination of operators \( f_i \) to a known canonical basis vector \( G(\xi) \). This partition \( \xi \) will have \( h \)-bar-weight \( 0 \) (in which case \( G(\xi) = \xi \)) or \( 1 \) (so that \( G(\xi) \) is known from Theorem 5.2), or will lie in the block \( B_{(l-1,...,1),2} \) (in which case \( G(\xi) \) is known by induction on \( l \)). The results are given in the following table. In each case we give the vector \( G(\mu) \) in the form of two columns (containing coefficients and partitions), and we give the \( \partial \)-value of each partition involved and show how \( G(\mu) \) is obtained. In each case it is easy to verify that the coefficients \( d_{\lambda,\mu} \) satisfy Theorem 6.4.

We begin with the restricted partitions \( \mu \neq \wp_{\tau}, \wp_{\tau,1}, \tau \).

<table>
<thead>
<tr>
<th>( d_{\lambda,\mu} )</th>
<th>( \lambda )</th>
<th>( \partial \lambda )</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( (h-l-2, h-l-3, l+3, l+2</td>
<td>\cdot) )</td>
<td>1 ( (h-l-1, h-l-3, l+3, l+1</td>
<td>\cdot) )</td>
</tr>
<tr>
<td>n</td>
<td>(d_{k,l} \lambda )</td>
<td>(\tilde{c} \lambda )</td>
<td>conditions</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>2</td>
<td>((h-l-1, h-a, a, l+1</td>
<td>\cdot))</td>
<td>(a-l-1)</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h-l-1, h-a+1, a-1, l+1</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h, h-a, a</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^4)</td>
<td>((h, h-a+1, a-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_{a-1}(h-l-1, h-a, a-1, l+1</td>
<td>\cdot) + q(h, h-a, a-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td>3</td>
<td>((h-a, h-b, b, a</td>
<td>\cdot))</td>
<td>(b-a)</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h-a, h-b+1, b-1, a</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h-a+1, h-b, b-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^4)</td>
<td>((h-a+1, h-b+1, b-1, a-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_{a-1}f_{b-1}(h-a, h-b, b-1, a-1</td>
<td>\cdot))</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>((h-a, h-a-1, a+1, a</td>
<td>\cdot))</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h-a+1, h-a-1, a+1, a-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h-a+2, h-a, a-2</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^4)</td>
<td>((h-a+2, h-a+1, a-1, a-2</td>
<td>\cdot))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_{a-1}(h-a, h-a-1, a+1, a-1</td>
<td>\cdot) + q^2(h-a+2, h-a, a-1, a-2</td>
<td>\cdot))</td>
</tr>
<tr>
<td>5</td>
<td>((h+a, h-b, b</td>
<td>a))</td>
<td>(b+a-l)</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h+a, h-b+1, b-1</td>
<td>a))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h+a+1, h-b, b</td>
<td>a+1))</td>
</tr>
<tr>
<td></td>
<td>(q^4)</td>
<td>((h+a+1, h-b+1, b-1</td>
<td>a+1))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_{b-1}(h+a, h-b, b-1</td>
<td>a) + q^2(h+a+1, h-b, b-1</td>
<td>a+1))</td>
</tr>
<tr>
<td>6</td>
<td>((h+a, h-l-1, l+1</td>
<td>a))</td>
<td>(a+1)</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h+a, h-a</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h+a+1, h-l-1, l+1</td>
<td>a+1))</td>
</tr>
<tr>
<td></td>
<td>(q^4)</td>
<td>((h+a+1, h-a-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_a(h+a, h-l-1, l+1</td>
<td>a+1) + q^2(h+a, h-a-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td>7</td>
<td>((h+l, h-b, b</td>
<td>l))</td>
<td>(b)</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h+l, h-b+1, b-1</td>
<td>l))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h+b-1, h-b+1</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^4)</td>
<td>((h+b, h-b</td>
<td>\cdot))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_{b-1}(h+l, h-b, b-1</td>
<td>l) + q^2(h+b-1, h-b</td>
<td>\cdot))</td>
</tr>
<tr>
<td>8</td>
<td>((h+l, h-l-1, l+1</td>
<td>l))</td>
<td>(l+1)</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h+l, h-l</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h+l+1, h-l-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_l(h+l, h-l-1</td>
<td>\cdot))</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>((h+l, h-l</td>
<td>\cdot))</td>
<td>(l)</td>
</tr>
<tr>
<td></td>
<td>(q^2)</td>
<td>((h+l, h</td>
<td>l))</td>
</tr>
<tr>
<td></td>
<td>(q^3)</td>
<td>((h+l+1, h-l-1</td>
<td>\cdot))</td>
</tr>
<tr>
<td></td>
<td>(q^4)</td>
<td>((2h-l-1, l+1</td>
<td>\cdot))</td>
</tr>
<tr>
<td>G(μ)</td>
<td>(f_0 f_1 \ldots f_{l-1}(h+l, h-l</td>
<td>l) + q^2(h+l+1, h-l-1</td>
<td>l) + q^2(2h-l-1, l+1</td>
</tr>
</tbody>
</table>
partitions appearing, and we can complete the verification of Theorem 6.4 in the base cases.

<table>
<thead>
<tr>
<th>(d_{i\mu})</th>
<th>(\lambda)</th>
<th>(\bar{\lambda})</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.</td>
<td>((h+a, h-a, \cdot))</td>
<td>(a)</td>
<td>(1 \leq a \leq l - 1)</td>
</tr>
<tr>
<td>(q)</td>
<td>((h+a, h, \cdot))</td>
<td>(a - 1)</td>
<td></td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+a+1, h-a-1, \cdot))</td>
<td>(a + 1)</td>
<td></td>
</tr>
<tr>
<td>(q^3)</td>
<td>((h+a+1, h, a+1))</td>
<td>(a)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_a((h+a, h-a-1, \cdot) + q(h+a, h, a+1)))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.</td>
<td>((h+a, h-a, \cdot))</td>
<td>(a)</td>
<td>(l + 1 \leq a \leq n - 1)</td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+a+1, h-a-1, \cdot))</td>
<td>(a + 1)</td>
<td></td>
</tr>
<tr>
<td>(q^2)</td>
<td>((2h-a-1, a+1, \cdot))</td>
<td>(a + 1)</td>
<td></td>
</tr>
<tr>
<td>(q^4)</td>
<td>((2h-a, a, \cdot))</td>
<td>(a)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_a((h+a, h-a-1, \cdot) + q^2(2h-a-1, a, \cdot)))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.</td>
<td>((h+n, n+1, \cdot))</td>
<td>(n)</td>
<td>(l \leq n - 1)</td>
</tr>
<tr>
<td>(q^4)</td>
<td>((h+n+1, n, \cdot))</td>
<td>(n)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_n(h+n, n, \cdot))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>((h+n, n+1, \cdot))</td>
<td>(n)</td>
<td>(l = n)</td>
</tr>
<tr>
<td>(q)</td>
<td>((h+n, h, n))</td>
<td>(n - 1)</td>
<td></td>
</tr>
<tr>
<td>(q^3)</td>
<td>((2h, \cdot))</td>
<td>(n)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_0 f_1 \ldots f_n(h+n, n, \cdot))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14.</td>
<td>((h+l, h+a, l, a))</td>
<td>(l - a - 1)</td>
<td>(0 \leq a \leq l - 2)</td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+l, h+a+1, l, a+1))</td>
<td>(l - a - 2)</td>
<td></td>
</tr>
<tr>
<td>(q^2)</td>
<td>((2h+a, a))</td>
<td>(l - a)</td>
<td></td>
</tr>
<tr>
<td>(q^4)</td>
<td>((2h+a+1, a+1))</td>
<td>(l - a - 1)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_0^{(2)}((h+l, h-1, l, 1) + q^2(2h-1, 1)))</td>
<td>((a = 0))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_a((h+l, h+a, l, a+1) + q^2(2h+a, a+1)))</td>
<td>((a \geq 1))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.</td>
<td>((h+l-1, h+l-2, l-1, l-2))</td>
<td>(0)</td>
<td>(l \geq 2)</td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+l, h+l-2, l, l-2))</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td>(q^2)</td>
<td>((2h+l-1, l-1))</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td>(q^4)</td>
<td>((2h+l, l))</td>
<td>(0)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_{l-1}((h+l-1, h+l-2, l, l-2) + q^2(2h+l-1, l)))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>((h+b, h+a, b, a))</td>
<td>(b - a - 1)</td>
<td>(0 \leq a \leq b - 2 \leq l - 3)</td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+b, h+a+1, b, a+1))</td>
<td>(b - a - 2)</td>
<td></td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+b+1, h+a, b+1, a))</td>
<td>(b - a)</td>
<td></td>
</tr>
<tr>
<td>(q^4)</td>
<td>((h+b+1, h+a+1, b+1, a+1))</td>
<td>(b - a - 1)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_b((h+b, h+a, b+1, a) + q^2(h+b, h+a+1, b+1, a+1)))</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17.</td>
<td>((h+a+1, h+a, a+1, a))</td>
<td>(0)</td>
<td>(0 \leq a \leq l - 3)</td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+a+2, h+a, a+2, a))</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td>(q^2)</td>
<td>((h+a+3, h+a+1, a+3, a+1))</td>
<td>(1)</td>
<td></td>
</tr>
<tr>
<td>(q^4)</td>
<td>((h+a+3, h+a+2, a+3, a+2))</td>
<td>(0)</td>
<td></td>
</tr>
<tr>
<td>(G(\mu) = f_{a+1}((h+a+1, h+a, a+2, a) + q^2(h+a+3, h+a+1, a+3, a+2)))</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now we do the same for \(G(\Theta_\tau), G(\Theta_\tau')\) and \(G(\Delta_\tau)\). In each case we identify the special partitions appearing, and we can complete the verification of Theorem 6.4 in the base cases.
### Defect 2 spin blocks and canonical basis coefficients

<table>
<thead>
<tr>
<th>(d_{\lambda \mu} )</th>
<th>(\lambda)</th>
<th>(\tilde{c}_{\lambda})</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>18. 1</td>
<td>((h, h - 1</td>
<td>\cdot))</td>
<td>(\Delta_{\tau})</td>
</tr>
<tr>
<td>19.</td>
<td>((h, h - 1</td>
<td>\cdot))</td>
<td>(\hat{\Psi}_{\tau})</td>
</tr>
<tr>
<td>20.</td>
<td>((2h - 2</td>
<td>\cdot))</td>
<td>(\hat{\Psi}_{\tau})</td>
</tr>
<tr>
<td>21.</td>
<td>((h, h - 1</td>
<td>\cdot))</td>
<td>(\hat{\Psi}_{\tau})</td>
</tr>
<tr>
<td>22.</td>
<td>((h - l - 1, h - l - 2</td>
<td>\cdot))</td>
<td>(\hat{\Psi}_{\tau})</td>
</tr>
<tr>
<td>23.</td>
<td>((h - 1, h - 2, 2</td>
<td>\cdot))</td>
<td>(\hat{\Psi}_{\tau})</td>
</tr>
</tbody>
</table>

\(G(\mu) = f_0^{(2)}((h, h - 1|1) + q^2(h + 2, h - 1|2, 1))\)

\(G(\mu) = f_0^{(2)}((h, h - 1|1) + q^2(2h - 1|2, 1))\)

\(G(\mu) = f_0^{(2)}((h - 1, h - l - 1, l + 1|1) + q(h, h - 1|1))\)

\(G(\mu) = f_0((h, h - 1|\cdot) + q^2(2h - 1|\cdot))\)

\(G(\mu) = f_0 \cdots f_1((h - l - 1, h - l - 2, l + 2|\cdot) + q(h, h - l - 1|\cdot))\)

\(G(\mu) = f_0((h - 1, h - 2, 2|\cdot) + q(h, h - 1|\cdot))\)
8 More on $[2 : k]$-pairs

Now we come to the inductive step in the proof of Theorem 6.4. For this, we need to study $[2 : k]$-pairs in more detail.

8.1 $[2 : 1]$-pairs of non-zero residue

Throughout Section 8.1 we assume that $\sigma$ and $\tau$ are $h$-bar-cores, and $B_{\sigma, 2}$ and $B_{\tau, 2}$ form a $[2 : 1]$-pair with residue $i$, where $1 \leq i < n$.

Our aim is to compare the canonical bases of $B_{\sigma, 2}$ and $B_{\tau, 2}$. We start by finding the exceptional partitions in $P_{\sigma, 2}$ and $P_{\tau, 2}$.

Consider runners $-i - 1, -i, i, i + 1$ in the abacus display for $\sigma$; let $b_{-i-1}, b_{-i}, b_i, b_{i+1}$ be the positions of the lowest beads on each of these runners. Then (because $\sigma$ is an $h$-bar-core) $b_{-i-1} + b_{i+1} = b_i + b_{i+1} - h$, and (because $\sigma$ has exactly one addable $i$-node) $b_i = b_{i+1} + h - 1$.

Now we consider three possibilities, depending on the relative order of $b_{-i-1}, b_{-i}, b_i, b_{i+1}$:

Type A The addable $i$-node of $\sigma$ lies in column $ah + i + 1$ for some $a \geq 1$, in which case $b_{-i} < b_{-i-1} < b_{i+1} < b_i$;

Type B The addable $i$-node of $\sigma$ lies in column $i + 1$, in which case $b_{-i} < b_{i+1} < b_{-i-1} < b_i$;

Type C The addable $i$-node of $\sigma$ lies in column $ah - i$ for some $a \geq 1$, in which case $b_{i+1} < b_i < b_{-i-1}$.

For example, take $h = 7$. We illustrate $[2 : 1]$-pairs of the three possible types, with $i = 1$ in each case.

<table>
<thead>
<tr>
<th>type A</th>
<th>type B</th>
<th>type C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = (8, 2, 1)$</td>
<td>$\sigma = (3, 1)$</td>
<td>$\sigma = (5, 4)$</td>
</tr>
<tr>
<td>$\tau = (9, 2, 1)$</td>
<td>$\tau = (3, 2)$</td>
<td>$\tau = (6, 4)$</td>
</tr>
</tbody>
</table>

Now we can classify and study the exceptional partitions in $B_{\sigma, 2}$ and $B_{\tau, 2}$. We use the abacus notation for partitions introduced in Section 2.2.

**Proposition 8.1.** Suppose $B_{\sigma, 2}$ and $B_{\tau, 2}$ form a $[2 : 1]$-pair with residue $i \geq 1$.

1. There are three exceptional partitions $P_{\tau, 2}$, which can be written $\alpha < \beta < \gamma$, and there are three exceptional partitions in $P_{\tau, 2}$, which can be written $\hat{\alpha} < \hat{\beta} < \hat{\gamma}$. These partitions have abacus notation given by the following table.

<table>
<thead>
<tr>
<th>Type A</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\hat{\alpha}$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle -i \rangle$</td>
<td>$\langle i, i + 1 \rangle$</td>
<td>$\langle i + 1 \rangle$</td>
<td>$\langle i \rangle$</td>
<td>$\langle i, i + 1 \rangle$</td>
<td>$\langle -i - 1 \rangle$</td>
<td></td>
</tr>
<tr>
<td>Type B</td>
<td>$\langle i + 1 \rangle$</td>
<td>$\langle i \rangle$</td>
<td>$\langle -i - 1 \rangle$</td>
<td>$\langle i, i + 1 \rangle$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type C</td>
<td>$\langle i \rangle$</td>
<td>$\langle -i \rangle$</td>
<td>$\langle -i - 1 \rangle$</td>
<td>$\langle i, i + 1 \rangle$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. \( \psi_i(\alpha) = \hat{\alpha}, \quad \psi_i(\beta) = \hat{\gamma}, \quad \psi_i(\gamma) = \hat{\beta}. \)

3. In the Fock space,
\[ f_i\alpha = q^{-2}\hat{\alpha} + \hat{\beta}, \quad f_i\beta = \hat{\alpha} + \hat{\gamma}, \quad f_i\gamma = \hat{\beta} + q^2\hat{\gamma}. \]

4. \( \alpha \) is restricted, with \( G(\alpha) = \alpha + q^2\beta + q^4\gamma. \)

5. \( \hat{\alpha} \) is restricted, with \( G(\hat{\alpha}) = \hat{\alpha} + q^2\hat{\beta} + q^4\hat{\gamma}. \)

6. If \( \mu \in \mathcal{P}_{\sigma,2} \) is restricted and the values \( d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu} \) are given by one of the rows of Table 1 below, then the values \( d_{\hat{\alpha}\Phi(\mu)}, d_{\hat{\beta}\Phi(\mu)}, d_{\hat{\gamma}\Phi(\mu)} \) are given in the same row, while \( d_{\lambda\mu} = d_{\psi(\lambda)\psi(\mu)} \) for all unexceptional \( \lambda \in \mathcal{P}_{\sigma,2}. \)

<table>
<thead>
<tr>
<th>( d_{\alpha\mu} )</th>
<th>( d_{\beta\mu} )</th>
<th>( d_{\gamma\mu} )</th>
<th>( d_{\hat{\alpha}\Phi(\mu)} )</th>
<th>( d_{\hat{\beta}\Phi(\mu)} )</th>
<th>( d_{\hat{\gamma}\Phi(\mu)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>q^2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>q^2</td>
<td>0</td>
<td>1</td>
<td>q</td>
</tr>
<tr>
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<td>q^2</td>
<td>0</td>
<td>q^2</td>
<td>2</td>
<td>q</td>
</tr>
<tr>
<td>1</td>
<td>q^2</td>
<td>q^4</td>
<td>1</td>
<td>q^4</td>
<td>q</td>
</tr>
<tr>
<td>q^2</td>
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<td>q^2</td>
<td>0</td>
<td>q^2</td>
<td>2</td>
</tr>
<tr>
<td>q^2</td>
<td>q^4</td>
<td>0</td>
<td>q^4</td>
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</tr>
<tr>
<td>q^4</td>
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<td>q^2</td>
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<td>q^2</td>
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</tr>
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<td>q^3 + q</td>
<td>0</td>
<td>q^3 + q</td>
</tr>
<tr>
<td>q^2</td>
<td>q^4 + q^2</td>
<td>0</td>
<td>q^4 + q^2</td>
<td>0</td>
<td>q^2</td>
</tr>
<tr>
<td>q^3 + q</td>
<td>q^5 + q^3</td>
<td>0</td>
<td>q^5 + q^3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>q^3 + q</td>
<td>0</td>
<td>q^3 + q</td>
<td>0</td>
<td>q^3 + q</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1

7. If \( \mu \) is a restricted partition in \( \mathcal{P}_{\sigma,2} \), the triple \( (d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu}) \) cannot equal \( (0, q, 0), (q, 0, q), (q^2, q^3, 0) \) or \( (q^3 + q, q^2, q^2) \).

8. Suppose \( \mu \) is a restricted partition in \( \mathcal{P}_{\sigma,2} \), and \( d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu} \) satisfy the following conditions.

(i) \( d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu} \in \{0, 1, q^2, q^4\}. \)

(ii) \( d_{\alpha\mu} \) and \( d_{\beta\mu} \) are not both equal to \( q^2 \).

(iii) If \( d_{\alpha\mu} = q^4 \), then \( d_{\beta\mu} = d_{\gamma\mu} = 0. \)

Then the triple \( (d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu}) \) is one of those given in Table 1.
Proof.
1. By examining runners \(i, i+1\) of the abacus, we find that the exceptional partitions in \(\mathcal{P}_{\tau,2}\) are precisely \(\langle-i\rangle, \langle i,i+1\rangle\) and \(\langle i+1\rangle\), and similarly for \(\mathcal{P}_{\tau,2}\).

\[
\begin{array}{ccc}
\mathcal{P}_{\tau,2} & \mathcal{P}_{\tau,2} \\
\langle-i\rangle & \langle i,i+1\rangle & \langle i+1\rangle \\
\langle i\rangle & \langle i,i+1\rangle & \langle -i-1\rangle
\end{array}
\]

The dominance ordering on these triples of partitions is easily checked from the abacus.

2,3. These statements follow by considering the arrangement of addable and removable \(i\)-nodes for \(\alpha, \beta, \gamma\). For example, \(\gamma\) has two addable \(i\)-nodes and one removable \(i\)-node, with the removable \(i\)-node to the right of the addable \(i\)-nodes. Adding the leftmost addable \(i\)-node yields \(\hat{\beta}\), while adding the other addable \(i\)-node yields \(\hat{\gamma}\). Hence \(\psi_i(\gamma) = \hat{\beta}\), and \(f_i \gamma = \hat{\beta} + q^2 \hat{\gamma}\).

4. Consider the \(h\)-strict partition \(\delta\) obtained by removing the unique removable \(i\)-node from any of \(\alpha, \beta, \gamma\). \(\delta\) is an \(h\)-bar-core, so \(G(\delta) = \delta\). So the vector

\[
f_i \delta = \alpha + q^2 \beta + q^4 \gamma
\]

is bar-invariant, and therefore equals \(G(\alpha)\) (and \(\alpha\) is necessarily restricted).

5. From parts (3) and (4) we can calculate

\[
f_i G(\alpha) = (q^{-2} + q^2)(\hat{\alpha} + q^2 \hat{\beta} + q^4 \hat{\gamma}).
\]

Hence the vector \(\hat{\alpha} + q^2 \hat{\beta} + q^4 \hat{\gamma}\) is bar-invariant, and so must equal \(G(\hat{\alpha})\).

6. In each case, one can compute the bar-invariant vector \(f_i G(\mu)\) using Proposition 4.5(1) and part (3) of the present proposition, and then reduce this using the known vector \(G(\hat{\alpha})\) to obtain a canonical basis vector in \(B_{\tau,2}\), which turns out to be \(G(\psi_i(\mu))\). We give two examples of this calculation.

Suppose we are in the third case in Table 1, with

\[
G(\mu) = \beta + q^2 \gamma + \sum_{\lambda \text{ unexceptional}} d_{\lambda \mu} \lambda,
\]

so that in particular \(\mu = \beta\). Then

\[
f_i G(\mu) = \hat{\alpha} + q^2 \hat{\beta} + (q^4 + 1) \hat{\gamma} + \sum_{\lambda \text{ unexceptional}} d_{\lambda \mu} \psi_i(\lambda).
\]

Subtracting \(G(\hat{\alpha})\), we obtain

\[
\hat{\gamma} + \sum_{\lambda \text{ unexceptional}} d_{\lambda \mu} \psi_i(\lambda)
\]

which therefore equals \(G(\hat{\gamma}) = G(\psi_i(\mu))\).
Now suppose we are in the penultimate case in Table 1, with
\[ G(\mu) = (q^3 + q)\lambda + (q^5 + q^3)\beta + \sum_{\lambda \text{ unexceptional}} d_{\lambda \mu} \lambda. \]

Then in particular \( \mu \) is unexceptional. We obtain a bar-invariant vector
\[ f_i G(\mu) = (q^5 + q^3 + q + q^{-1})\hat{\alpha} + (q^3 + q)\hat{\beta} + (q^5 + q^3)\hat{\gamma} + \sum_{\lambda \text{ unexceptional}} d_{\lambda \mu} \psi_i(\lambda). \]

Subtracting the bar-invariant vector \((q + q^{-1}) G(\hat{\alpha})\), we obtain
\[ (q^5 + q^3)\hat{\alpha} + \sum_{\lambda \text{ unexceptional}} d_{\lambda \mu} \psi_i(\lambda) \]
which must therefore equal \( G(\psi_i(\mu))\).

7.8. Let \( \delta \) be the \( h \)-bar-core from the proof of part (4) above. In the Fock space we have
\[ e_i\lambda = q^{-4}\delta, \quad e_i\beta = q^{-2}\delta, \quad e_i\gamma = \delta \]
and \( e_i\lambda = 0 \) for all unexceptional \( \lambda \in P_{\sigma,2} \). Since \( G(\delta) = \delta \), we obtain
\[ e_i G(\mu) = \left( q^{-4}d_{\lambda \mu} + q^{-2}d_{\beta \mu} + d_{\gamma \mu} \right) G(\delta). \]

This vector is bar-invariant, so the Laurent polynomial \( q^{-4}d_{\lambda \mu} + q^{-2}d_{\beta \mu} + d_{\gamma \mu} \) must be symmetric in \( q \) and \( q^{-1} \). One can check that this is not the case for the triples in (7). For part (8), we note one more condition which follows from the basic properties of canonical basis vectors.

(iv) If \( d_{\gamma \mu} = 1 \), then \( d_{\lambda \mu} = d_{\beta \mu} = 0 \).

If \( q^{-4}d_{\lambda \mu} + q^{-2}d_{\beta \mu} + d_{\gamma \mu} \) is symmetric in \( q \) and \( q^{-1} \) and conditions (i–iv) also hold, then only possibility for the triple \((d_{\lambda \mu}, d_{\beta \mu}, d_{\gamma \mu})\) not given by the table is \((1, 0, q^4)\). But this possibility cannot happen by (4). \( \square \)

8.2 \([2 : 3]-\text{pairs of residue 0}\)

Now we give results corresponding to those of Section 8.1 for \([2 : 3]-\text{pairs of residue 0}\).

Throughout Section 8.2 we assume that \( \sigma \) and \( \tau \) are \( h \)-bar-cores, and that \( B_{\sigma,2} \) and \( B_{\tau,2} \) form a \([2 : 3]-\text{pair with residue 0}\).

**Proposition 8.2.**

1. The exceptional partitions in \( P_{\sigma,2} \) can be written \( \alpha \prec \beta \prec \gamma \), where
\[ \alpha = \langle 1 \rangle, \quad \beta = \langle 0, 1 \rangle, \quad \gamma = \langle 0 \rangle. \]

The exceptional partitions in \( P_{\tau,2} \) can be written \( \hat{\alpha} \prec \hat{\beta} \prec \hat{\gamma} \), where
\[ \hat{\alpha} = \langle 0 \rangle, \quad \hat{\beta} = \langle 0, 1 \rangle, \quad \hat{\gamma} = \langle -1 \rangle. \]
2. \[ \psi_0(\alpha) = \hat{\alpha}, \quad \psi_0(\beta) = \hat{\gamma}, \quad \psi_0(\gamma) = \hat{\beta}. \]

3. In the Fock space,
\[
\begin{align*}
 f^{(3)}_0(\alpha) & = q^{-3} \hat{\alpha} + q^{-1} \hat{\beta} \\
 f^{(3)}_0(\beta) & = (1 + q^{-2}) \hat{\alpha} + \hat{\beta} + (q^2 + 1) \hat{\gamma} \\
 f^{(3)}_0(\gamma) & = \hat{\alpha} + (q^2 + 1) \hat{\beta} + (q^4 + q^2) \hat{\gamma}.
\end{align*}
\]

4. \( \alpha \) is restricted, with \( G(\alpha) = \alpha + q\beta + q^3\gamma \).

5. \( \hat{\alpha} \) is restricted, with \( G(\hat{\alpha}) = \hat{\alpha} + q^2 \hat{\beta} + q^4 \hat{\gamma} \).

6. If \( \mu \in \mathcal{P}_{\sigma,2} \) is restricted and the values \( d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu} \) are given by one of the rows of Table 2, then the values \( d_{\overline{\alpha}\phi(\mu)}, d_{\overline{\beta}\phi(\mu)}, d_{\overline{\gamma}\phi(\mu)} \) are given in the same row, while \( d_{\lambda\mu} = d_{\phi(\lambda)\phi(\mu)} \) for all unexceptional \( \lambda \in \mathcal{P}_{\sigma,2} \).

<table>
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<th>( d_{\alpha\mu} )</th>
<th>( d_{\beta\mu} )</th>
<th>( d_{\gamma\mu} )</th>
<th>( d_{\overline{\alpha}\phi(\mu)} )</th>
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</tr>
</tbody>
</table>

Table 2

7. If \( \mu \in \mathcal{P}_{\sigma,2} \) is restricted, the triple \( (d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu}) \) cannot equal \((q^2, q^4 + q^2, 0), (q^3 + q, 0, q^3 + q)\) or \((q^5 + q, q^5 + q^3, 0)\).

8. Suppose \( \mu \in \mathcal{P}_{\sigma,2} \) is restricted and \( d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu} \) satisfy the following conditions.

(i) \( d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu} \in \{0, 1, q, q^2, q^3, q^4\} \).
(ii) \( d_{\alpha\mu} \) and \( d_{\beta\mu} \) are not both equal to \( q^2 \).
(iii) \( d_{\beta\mu} \) and \( d_{\gamma\mu} \) are not both equal to \( q \).
(iv) If \( d_{\alpha\mu} = q^4 \), then \( d_{\beta\mu} = d_{\gamma\mu} = 0 \).

Then the triple \( (d_{\alpha\mu}, d_{\beta\mu}, d_{\gamma\mu}) \) is one of those given in Table 2.
Proof.
1. This is a matter of checking possible abacus configurations. Runners \(-1, 0, 1\) of the abacus displays for the exceptional partitions are shown below.

2,3. As in Proposition 8.1, these statements follow by considering the configuration of addable and removable 0-nodes for each of \(\alpha, \beta\) and \(\gamma\).

As an example, we show how to compute the coefficient of \(\hat{\alpha}\) in \(f^{(3)}_0\beta\). As we can see from the abacus display, the addable and removable 0-nodes of \(\beta = \langle 0, 1 \rangle\) consist of (from left to right):

- an addable node in column 1;
- a removable node in column \(h\);
- addable nodes in columns \(h - 1, 2h\) and \(2h + 1\).

There are three possible ways to add three of the addable 0-nodes, yielding the partitions \(\hat{\alpha}, \hat{\beta}\) and \(\hat{\gamma}\). In particular, \(\hat{\alpha}\) is obtained by adding the addable nodes in columns 1, \(h + 1\) and \(2h\). Since \(h\) occurs exactly once as a part of \(\beta\), the definition of the action of \(f^{(3)}_0\) gives a coefficient of \(q^{-2} \times (1 + q^2) = (1 + q^{-2})\).

4. This is proved as in Proposition 8.1, using the \(h\)-bar-core \(\delta = \sigma \cup (2h - 1)\). Since \(G(\delta) = \delta\), we have a bar-invariant vector

\[ f_0\delta = \alpha + q\beta + q^3\gamma \]

which must be \(G(\alpha)\).

5. From parts (3) and (4) we can calculate

\[ f^{(3)}_0 G(\alpha) = (q^{-3} + q^{-1} + q + q^3)(\hat{\alpha} + q^2\hat{\beta} + q^4\hat{\gamma}). \]

Hence the vector \(\hat{\alpha} + q^2\hat{\beta} + q^4\hat{\gamma}\) is bar-invariant, and so must equal \(G(\hat{\alpha})\).

6. This is proved in the same way as Proposition 8.1(6), using (3) and (5).

7,8. These parts are proved in the same way as in Proposition 8.1. In this case (taking \(\delta\) as in the proof of (4)) we have

\[ e_0\alpha = q^{-4}\delta \]
\[ e_0\beta = (q^{-1} + q^{-3})\delta \]
\[ e_0\gamma = (q + q^{-1})\delta \]

and \(e_0\lambda = 0\) for all unexceptional \(\lambda\). Since \(e_0 G(\mu)\) must equal \(f\delta\) with \(f\) symmetric in \(q\) and \(q^{-1}\), the result is easily obtained.
8.3 [2 : k]-pairs, ∂-values and dominance

Now we consider how the dominance orders and the ∂-values differ for corresponding $h$-strict partitions in two blocks forming a [2 : k]-pair.

Throughout Section 8.3 we assume that $\sigma$ and $\tau$ are $h$-bar-cores, and that $B_{r,2}$ and $B_{r,2}$ form a [2 : k]-pair with residue $i$, where either $i \geq 1$ or $k \geq 3$.

First we consider ∂-values and colours. We start with unexceptional partitions.

**Proposition 8.3.** Suppose $\lambda \in \mathcal{P}_{c,2}$ is unexceptional. Then $\partial \phi_{i}(\lambda) = \partial \lambda$.

**Proof.** For this proof, we write $\mu = \phi_{i}(\lambda)$. To prove that $\partial \lambda = \partial \mu$, we prove the stronger statement that $\mu$ has the same leg lengths as $\lambda$. Let $a_{\lambda} < b_{\lambda}$ be the bar-positions for $\lambda$. Define $\overline{B}_{\lambda}$ to be the set of occupied positions in the abacus display for $\lambda$, and set

$$B_{\lambda} = \overline{B}_{\lambda} \backslash \{\pm a_{\lambda}, \pm (a_{\lambda} - h), \pm b_{\lambda}, \pm (b_{\lambda} - h)\} \cup \{0\}.$$ define $a_{\mu}$, $b_{\mu}$ and $B_{\mu}$ similarly.

Now observe that the leg lengths of $\lambda$ are simply

$$|B_{\lambda} \cap (a_{\lambda} - h, a_{\lambda})|, \quad |B_{\lambda} \cap (b_{\lambda} - h, b_{\lambda})|$$

(where we use the usual notation $(x, y)$ for the open interval between $x$ and $y$). The leg lengths of $\mu$ can be expressed similarly, so we need to compare the sets $B_{\lambda} \cap (a_{\lambda} - h, a_{\lambda})$ and $B_{\mu} \cap (a_{\mu} - h, a_{\mu})$ (and the corresponding sets for $b_{\lambda}$ and $b_{\mu}$).

We consider the cases $i \neq 0$ and $i = 0$ separately. First suppose $i \neq 0$. In this case, we let $\phi_{i}$ denote the bijection from $\mathbb{Z}$ to $\mathbb{Z}$ given by

$$\phi_{i}(c) = \begin{cases} 
    c + 1 & (c \equiv i \text{ or } -i - 1 \text{ (mod $h$)}) \\
    c - 1 & (c \equiv i + 1 \text{ or } -i \text{ (mod $h$)}) \\
    c & \text{ (otherwise).}
\end{cases}$$

Then $a_{\mu} = \phi_{i}(a_{\lambda})$ unless either

(a) $a_{\lambda} = n + 1$ and $i = n$, or

(b) $a_{\lambda} = b_{\lambda} - 1 \equiv i \text{ or } -i - 1 \text{ (mod $h$)}.$

In either of these cases $a_{\mu} = a_{\lambda}$. Similarly $b_{\mu} = \phi_{i}(b_{\lambda})$ except in case (b) above, in which case $b_{\mu} = b_{\lambda}$.

In any case, $B_{\mu} = \phi_{i}(B_{\lambda})$; so since $\phi_{i}$ is a bijection on $\mathbb{Z}$, to compare the sizes of the sets $B_{\lambda} \cap (a_{\lambda} - h, a_{\lambda})$ and $B_{\mu} \cap (a_{\mu} - h, a_{\mu})$ we just need to consider the possible integers $c$ such that $c \in B_{\lambda} \cap (a_{\lambda} - h, a_{\lambda})$ and $\phi_{i}(c) \notin (a_{\mu} - h, a_{\mu})$, or $c \in B_{\mu} \cap (a_{\mu} - h, a_{\mu})$ and $\phi_{i}(c) \notin (a_{\lambda} - h, a_{\lambda})$.

Consider first the case $a_{\mu} = a_{\lambda}$. The only way an integer $c$ can satisfy $c \in (a_{\lambda} - h, a_{\lambda}) \notin \phi_{i}(c)$ is if

- $c = a_{\lambda} - 1$ and $a_{\lambda} \equiv i + 1 \text{ or } -i \text{ (mod $h$)}$, or
- $c = a_{\lambda} - h + 1$ and $a_{\lambda} \equiv i \text{ or } -i - 1 \text{ (mod $h$)}$. 


But if \( a_\lambda \equiv i + 1 \) or \(-i \) (mod \( h \)) then \( \phi_i(a_\lambda) = a_\lambda - 1 \), so the only way we can have \( a_\mu = a_\lambda \) in case (a) above; that is, \( a_\lambda = n + 1 = i + 1 \). But in that case \( a_\lambda - 1 = h - a_\lambda \not\in B_\lambda \cup B_\mu \).

Similarly if \( a_\lambda \equiv i \) or \(-i -1 \) (mod \( h \)) then \( \phi_i(a_\lambda) = a_\lambda + 1 \), so the only way we can have \( a_\mu = a_\lambda \) is in case (b) above. But then \( a_\lambda - h + 1 = b_\lambda - h \not\in B_\lambda \cup B_\mu \).

So we see that \( \phi_i \) maps \( B_\lambda \cap (a_\lambda - h, a_\lambda) \) bijectively to \( B_\mu \cap (a_\mu - h, a_\mu) \).

Now consider the case \( a_\mu \neq a_\lambda \). This can happen in either of two ways.

\[
a_\lambda \equiv i \text{ or } -i -1 \text{ (mod } h) \text{ and } a_\mu = a_\lambda + 1.
\]

Observe that if \( c \in (a_\lambda - h, a_\lambda) \), then \( \phi_i(c) \in (a_\lambda - h + 1, a_\lambda + 1) \) unless \( c = a_\lambda - h + 1 \). Similarly, if \( \phi_i(c) \in (a_\lambda - h + 1, a_\lambda + 1) \), then \( c \in (a_\lambda - h, a_\lambda) \) unless \( c = a_\lambda \). So to show that \( |B_\lambda \cap (a_\lambda - h, a_\lambda)| = |B_\mu \cap (a_\mu - h, a_\mu)| \), we just need to show that \( a_\lambda - h + 1 \in B_\lambda \) if and only if \( a_\lambda \in B_\mu \); that is, \( a_\lambda - h + 1 \in B_\lambda \) if and only if \( a_\lambda + 1 \in B_\lambda \). If \( a_\lambda - h + 1 \not\in B_\lambda \equiv a_\lambda + 1 \), then \( b_\lambda = a_\lambda + 1 \), so that \( a_\mu = a_\lambda \), contrary to assumption. On the other hand, if \( a_\lambda - h + 1 \in B_\lambda \neq a_\lambda + 1 \), then one of the following must occur on runners \( i, i+1 \) of the abacus display for \( \lambda \).

But then \( \lambda \) is exceptional, contrary to assumption.

\[
a_\lambda \equiv i + 1 \text{ or } -i \text{ (mod } h) \text{ and } a_\mu = a_\lambda - 1.
\]

This case is similar to the previous one; here we must show that \( a_\lambda - h - 1 \in B_\lambda \) if and only if \( a_\lambda - 1 \in B_\mu \), and this is done in a similar way.

So in all cases, \( |B_\lambda \cap (a_\lambda - h, a_\lambda)| = |B_\mu \cap (a_\mu - h, a_\mu)| \).

Now we consider the case \( i = 0 \), where we take the same approach. We define the bijection \( \phi_0 : \mathbb{Z} \rightarrow \mathbb{Z} \) by

\[
\phi_0(c) = \begin{cases} 
    c + 2 & (c \equiv -1 \text{ (mod } h)) \\
    c - 2 & (c \equiv 1 \text{ (mod } h)) \\
    c & \text{ (otherwise)}. 
\end{cases}
\]

Suppose first that \( a_\lambda \neq \pm 1 \text{ (mod } h) \). Then \( a_\mu = a_\lambda \), and the only way we can have an integer \( c \) with \( c \in (a_\lambda - h, a_\lambda) \) \( \neq \phi_0(c) \) is if \( a_\lambda \equiv 0 \text{ (mod } h) \) and \( c = a_\lambda - 1 \) or \( a_\lambda - h + 1 \).

So if \( a_\lambda \neq 0 \text{ (mod } h) \), then \( \phi_0 \) maps \( B_\lambda \cap (a_\lambda - h, a_\lambda) \) bijectively to \( B_\mu \cap (a_\lambda - h, a_\lambda) \). If \( a_\lambda \equiv 0 \text{ (mod } h) \), then the assumption that \( \lambda \) is exceptional means that \( a_\lambda - 1, a_\lambda - h - 1 \in B_\lambda \) while \( a_\lambda + 1, a_\lambda - h + 1 \not\in B_\lambda \). So

\[
B_\lambda \cap (a_\lambda - h, a_\lambda) = B_\lambda \cap (a_\lambda - h + 1, a_\lambda - 1) \cup \{a_\lambda - 1\}
\]

\[
B_\mu \cap (a_\lambda - h, a_\lambda) = B_\mu \cap (a_\lambda - h + 1, a_\lambda - 1) \cup \{a_\lambda - h + 1\}
\]

and hence \( |B_\lambda \cap (a_\lambda - h, a_\lambda)| = |B_\mu \cap (a_\lambda - h, a_\lambda)| \).

Now suppose \( a_\lambda \equiv -1 \text{ (mod } h) \). Then the assumption that \( \lambda \) is exceptional means that \( b_\lambda \neq a_\lambda + 1, a_\lambda + 2 \), so \( a_\mu = \phi_0(a_\lambda) \). If \( c \in (a_\lambda - h, a_\lambda) \), then \( \phi_0(c) \in (a_\lambda - h + 2, a_\lambda + 2) \) unless \( c = a_\lambda - h + 1 \) or \( a_\lambda - h + 2 \), while if \( c \in (a_\lambda - h + 2, a_\lambda + 2) \) then \( \phi_0(c) \in (a_\lambda - h, a_\lambda) \) unless \( c = a_\lambda \) or \( a_\lambda + 2 \). So in order to show \( |B_\lambda \cap (a_\lambda - h, a_\lambda)| = |B_\mu \cap (a_\lambda - h, a_\lambda)| \), we must show that

\[
|B_\lambda \cap (a_\lambda - h + 1, a_\lambda - h + 2)| = |B_\mu \cap \{a_\lambda, a_\lambda + 1\}|,
\]
that is
\[ |\mathcal{B}_\lambda \cap \{a_\lambda - h + 1, a_\lambda - h + 2\}| = |\mathcal{B}_\lambda \cap \{a_\lambda + 1, a_\lambda + 2\}|. \]

This follows from the fact that \( \lambda \) is unexceptional by considering possible abacus displays.

So in all cases (for \( i \neq 0 \) and \( i = 0 \)) \( |\mathcal{B}_\lambda \cap (a_\lambda - h, a_\lambda)| = |\mathcal{B}_\mu \cap (a_\mu - h, a_\mu)|. \) In the same way we can show that \( |\mathcal{B}_\lambda \cap (b_\lambda - h, b_\lambda)| = |\mathcal{B}_\mu \cap (b_\mu - h, b_\mu)|. \)

Now we do the same for colours.

**Proposition 8.4.** Suppose \( \lambda \in \mathcal{P}_{c,2} \) is unexceptional. Then \( \text{cl } \psi_i(\lambda) = \text{cl } \lambda \).

**Proof.** Again, we write \( \mu = \psi_i(\lambda) \). Recall that \( \Gamma(\sigma) \) denotes the number of parts of \( \sigma \) which are less than \( h \). Observe that we have \( \Gamma(\sigma) = \Gamma(\tau) \): for \( i \geq 3 \) this is completely clear, while if \( i = 0 \) then the assumption that \( k \geq 3 \) means that \( h - 1 \in \sigma \), and the set \( \tau \cap \{1, \ldots, h - 1\} \) is obtained from \( \sigma \cap \{1, \ldots, h - 1\} \) by replacing \( h - 1 \) with 1.

By Proposition 8.3 \( \varnothing = \hat{\psi} \), so if \( \varnothing \geq 2 \) the result is trivial. We consider the two remaining cases.

\( \varnothing = 0 \): Let \( l \) be the (repeated) leg length of \( \lambda \) (and of \( \mu, \) from the proof of Proposition 8.3).

Suppose \( \lambda \) has a 2h-bar. by Lemma 6.3 this means that \( h \leq a_\lambda = b_\lambda - h \). The definition of colour means that \( \lambda \) is black if and only if either

- \( a_\lambda \notin \sigma \) and \( l + \Gamma(\sigma) \) is even, or
- \( a_\lambda \in \sigma \) and \( l + \Gamma(\sigma) \) is odd.

Now \( \phi_i(a_\lambda) \) and \( \phi_i(b_\lambda) \) are the bar positions of \( \mu \), and \( \phi_i(a_\lambda) \in \tau \) if and only if \( a_\lambda \in \sigma \). Now the fact that \( \Gamma(\tau) = \Gamma(\sigma) \) means that \( \lambda \) is black if and only if \( \mu \) is.

Alternatively, suppose \( \lambda \) has two h-bars. Then \( \lambda \) is black if and only if \( l + \Gamma(\sigma) \) is odd. Clearly also \( \mu \) has two h-bars, and so \( \mu \) is black if and only if \( l + \Gamma(\tau) = l + \Gamma(\sigma) \) is odd.

\( \varnothing = 1 \): In this case let \( l, l + 1 \) be the leg lengths of \( \lambda \) and of \( \mu \).

If \( b_\lambda > h \), then we claim that \( b_\lambda > h \): the only way this could potentially fail is if \( i = 0 \) and \( b_\lambda = h + 1 \); but then \( \lambda \) would be exceptional. So \( \lambda \) and \( \mu \) are both grey if \( b_\lambda > h \).

Similarly, if \( b_\lambda \leq h \) then \( b_\mu \leq h \). In this case \( \lambda \) and \( \mu \) are both black if \( l + \Gamma(\sigma) = l + \Gamma(\tau) \) is odd, and both white otherwise.

Next we compare the dominance orders in \( \mathcal{P}_{c,2} \) and \( \mathcal{P}_{\tau,2} \).

**Proposition 8.5.** Suppose \( \lambda, \mu \in \mathcal{P}_{c,2} \) are both unexceptional and \( |\varnothing \lambda - \varnothing \mu| \leq 1 \). Then \( \lambda \leq \mu \) if and only if \( \psi_i(\lambda) \leq \psi_i(\mu) \).

**Proof.** By Proposition 6.2 \( \lambda \) and \( \mu \) are comparable in the dominance order. By Proposition 8.3 \( |\varnothing \psi_i(\lambda) - \varnothing \psi_i(\mu)| \leq 1 \), so \( \psi_i(\lambda) \) and \( \psi_i(\mu) \) are also comparable in the dominance order. Hence

\[ \lambda \leq \mu \iff \lambda \leq_{\text{lex}} \mu \iff \psi_i(\lambda) \leq_{\text{lex}} \psi_i(\mu) \iff \psi_i(\lambda) \leq \psi_i(\mu), \]

with the middle implication following from Lemma 4.10(1).
Remark. In fact, Proposition 8.5 is true even without the hypothesis that $|\partial \lambda - \partial \mu| \leq 1$, but this is harder to prove, and we do not need it.

Now we come to the exceptional partitions.

**Proposition 8.6.** Suppose that either $i = 0$ and $k = 3$, or $i \geq 1$ and $k = 1$. Let $\alpha \lessdot \beta \lessdot \gamma$ be the exceptional partitions in $P_{\sigma,2}$, and $\hat{\alpha} \lessdot \hat{\beta} \lessdot \hat{\gamma}$ the exceptional partitions in $P_{\tau,2}$. Then there is an integer $d \geq 1$ such that the following hold.

1. $\partial \alpha = \partial \gamma = \partial \hat{\beta} = d$ and $\text{cl} \alpha = \text{cl} \hat{\beta} = \text{cl} \gamma$.
2. $\partial \hat{\alpha} = \partial \beta = \partial \hat{\gamma} = d - 1$ and $\text{cl} \hat{\alpha} = \text{cl} \beta = \text{cl} \hat{\gamma}$.
3. If $\lambda \in P_{\tau,2}$ is unexceptional with $|\partial \lambda - d| \leq 1$, then either
   - $\lambda \triangleright \gamma$ and $\psi_i(\lambda) \triangleright \hat{\beta}$, or
   - $\lambda \lessdot \alpha$ and $\psi_i(\lambda) \lessdot \hat{\beta}$.
4. If $\lambda \in P_{\tau,2}$ is unexceptional with $|\partial \lambda - (d - 1)| \leq 1$, then either
   - $\lambda \triangleright \beta$ and $\psi_i(\lambda) \triangleright \hat{\gamma}$, or
   - $\lambda \lessdot \beta$ and $\psi_i(\lambda) \lessdot \hat{\alpha}$.

**Proof.** To prove parts (1) and (2), we define two integers $l$ and $m$ depending on the abacus configuration of $\sigma$. We continue to use the notation $\langle x, y \rangle_\sigma$ for the number of parts of $\sigma$ lying strictly between $x$ and $y$.

- Suppose $i \geq 1$ and the addable $i$-node of $\sigma$ lies in column $i + 1$. Then define
  $$l = i + \langle i + 1, h - i - 1 \rangle_\sigma, \quad m = \langle i + 1, h + i \rangle_\sigma.$$

- Suppose $i \geq 1$ and the addable $i$-node of $\sigma$ lies in column $h - i$. Then define
  $$l = i + \langle i + 1, h - i - 1 \rangle_\sigma, \quad m = \langle h - i, 2h - i - 1 \rangle_\sigma.$$

- Suppose $i \geq 1$ and the addable $i$-node of $\sigma$ lies in column $ah + i + 1$ for $a \geq 1$. Then define
  $$l = \langle (a - 1)h + i + 1, ah + i - 1 \rangle_\sigma, \quad m = \langle ah + i + 1, (a + 1)h + i - 1 \rangle_\sigma.$$

- Suppose $i \geq 1$ and the addable $i$-node of $\sigma$ lies in column $ah + h - i$ for $a \geq 2$. Then define
  $$l = \langle ah - i, (a + 1)h - i - 2 \rangle_\sigma, \quad m = \langle (a + 1)h - i, (a + 2)h - i - 2 \rangle_\sigma.$$

- Suppose $i = 0$. Define
  $$l = \langle 1, h - 1 \rangle_\sigma, \quad m = \langle h + 1, 2h - 1 \rangle_\sigma.$$
Now by checking the partitions \( \alpha, \beta, \gamma, \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) in each of the four cases, we find that

\begin{align*}
\alpha & \text{ has leg lengths } l + 1, m, & \hat{\alpha} & \text{ has leg lengths } l + 1, m + 1 \\
\beta & \text{ has leg lengths } l, m & \hat{\beta} & \text{ has leg lengths } l + 1, m \\
\gamma & \text{ has leg lengths } l + 1, m & \hat{\gamma} & \text{ has leg lengths } l, m,
\end{align*}

so that, taking \( d = l - m + 1 \), the values of \( \partial \) satisfy the relations in parts (1) and (2). Now we consider colours: if \( d = 1 \), then clearly \( \alpha, \gamma \) and \( \hat{\beta} \) are all grey; similarly if \( d = 2 \) then \( \hat{\alpha}, \hat{\gamma} \) and \( \beta \) are all grey. It remains to consider the case where \( d = 1 \), so that \( \partial \hat{\alpha} = \partial \hat{\gamma} = \partial \beta = 0 \). This means that the values \( l \) and \( m \) defined above coincide. Note first that we cannot be in case 8.3, because then \( \sigma \) would have \( i \) parts in the range \( \{h - i + 1, \ldots, h + i - 1\} \), which cannot happen if \( \sigma \) is an \( h \)-bar-core. In the other three cases, we find that

\begin{align*}
\hat{\alpha} & \text{ has a } 2h \text{-bar with leg length } 2l + 2 \\
\hat{\gamma} & \text{ has a } 2h \text{-bar with leg length } 2l + 1 \\
\beta & \text{ has two } h \text{-bars each with leg length } l.
\end{align*}

So \( \hat{\alpha}, \beta, \hat{\gamma} \) are all black if \( l + \Gamma(\sigma) \) is odd, and all white otherwise.

Now we prove part (3). By Proposition 6.2, \( \alpha, \gamma \) and \( \lambda \) are comparable in the dominance order. Since \( \partial \psi_1(\lambda) = \partial \lambda, \psi_1(\lambda) \) and \( \hat{\beta} \) are also comparable in the dominance order. Now we claim that \( \lambda \prec \alpha \) if and only if \( \psi_1(\lambda) \prec \hat{\beta} \): if \( \lambda \prec \alpha \), then \( \lambda \prec_{\text{colex}} \alpha \), so by Lemma 4.10 \( \psi_1(\lambda) \prec_{\text{colex}} \hat{\beta} \), and so \( \psi_1(\lambda) \prec \hat{\beta} \). On the other hand, if \( \lambda \succ \alpha \), then \( \lambda \succ_{\text{lex}} \alpha \), so by Lemma 4.10 \( \psi_1(\lambda) \succ_{\text{lex}} \hat{\beta} \), so \( \psi_1(\lambda) \succ \hat{\beta} \). So \( \lambda \prec \alpha \) if and only if \( \psi_1(\lambda) \prec \hat{\beta} \), as claimed. In exactly the same way we prove that \( \lambda \prec \gamma \) if and only if \( \psi_1(\lambda) \prec \hat{\beta} \), which gives (3). Part (4) is proved in the same way.

We now note another result which we will need in view of the factor \( q_{\lambda, \mu} \) appearing in Theorem 6.4(1).

**Lemma 8.7.** Suppose \( \lambda \in P_{\sigma, 2} \) is unexceptional. Then \( \lambda \) contains \( h \) or \( 2h \) if and only if \( \psi_1(\lambda) \) does.

**Proof.** If \( i \geq 1 \) then this is obvious, since adding \( i \)-nodes cannot affect parts divisible by \( h \). So assume \( i = 0 \). If \( \lambda \) contains \( ah \) (where \( a = 1 \) or \( 2 \)), then \( \lambda \) must also contain \( ah - 1 \) (otherwise \( \lambda \) would have a removable 0-node, so would be exceptional). Hence \( \psi_0(\lambda) \) contains \( ah \). The converse is similar.

Now we come to the special partitions in \( P_{\sigma, 2} \) and their counterparts in \( P_{\tau, 2} \). Our first task is to show how these partitions compare in \( P_{\sigma, 2} \) and \( P_{\tau, 2} \).

**Lemma 8.8.**

1. Suppose \( \star \) is one of the symbols \( \own, \oown, \own. \) Then \( \star_{\sigma} \) is unexceptional for the pair \( (B_{\sigma, 2}, B_{\tau, 2}) \), and \( \psi_1(\star_{\sigma}) = \star_{\tau} \).

2. Suppose \( \star \) is one of the symbols \( \oslash, \oslash^\circ, \oslash. \) Then one of the following happens:

   (a) \( \star_{\sigma} \) is unexceptional for the pair \( (B_{\sigma, 2}, B_{\tau, 2}) \), and \( \psi_1(\star_{\sigma}) = \star_{\tau} \).

   (b) Either \( k = 1 \) or \( i = 0 \) and \( k = 3 \), and \( \star_{\sigma} = \beta, \star_{\tau} = \hat{\alpha} \).
**Proof.** We consider the six special partitions separately. Recall from Proposition 8.1(1) and Proposition 8.2(1) that if there are exceptional partitions (i.e., if $i = 0$ and $k = 3$, or if $k = 1$) then the exceptional partitions in $\mathcal{P}_{\tau, 2}$ have abacus notation $\langle -i \rangle, \langle i, i+1 \rangle, \langle i+1 \rangle$.

$\mathfrak{W}_\sigma$. As in the definition, take $1 \leq a < b$ minimal such that $a, b - h - a, h - b \notin \sigma$; then $\mathfrak{W}_\sigma = \sigma \cup (h - a, h - b, b, a)$, which has abacus notation $\langle a, b \rangle$. Given this abacus notation, $\mathfrak{W}_\sigma$ cannot be exceptional; note that we cannot have $i = a = b - 1$, because if $a = b - 1$ then $\sigma$ has no addable $a$-nodes.

Now $\psi_i(\mathfrak{W}_\sigma)$ has abacus notation $\langle \hat{a}, \hat{b} \rangle$, where

$$\hat{a} = \begin{cases} a - 1 & \text{if } i = a - 1 \\ a + 1 & \text{if } i = a + 1 \\ a & \text{otherwise,} \end{cases} \quad \hat{b} = \begin{cases} b - 1 & \text{if } i = b - 1 \\ b + 1 & \text{if } i = b + 1 \\ b & \text{otherwise,} \end{cases}$$

and a case-by-case check then shows that $\langle \hat{a}, \hat{b} \rangle = \mathfrak{W}_\tau$.

$\mathfrak{U}_\sigma$. In this case take $a \geq 1$ minimal such that $a, h - a \notin \sigma$; then $\mathfrak{U}_\sigma = \sigma \cup (h, h - a, a)$, with abacus notation $\langle a, 0 \rangle$. Hence $\mathfrak{U}_\sigma$ is not exceptional (in the case $i = 0$, observe that $a$ cannot equal 1 because we would then have $k = 1$). Now $\psi_i(\mathfrak{U}_\sigma) = \langle \hat{a}, 0 \rangle$, where $\hat{a}$ is defined as in the previous case, and we get $\psi_i(\mathfrak{U}_\sigma) = \mathfrak{U}_\tau$.

$\mathfrak{A}_\sigma$. The abacus notation for $\mathfrak{A}_\sigma$ is $\langle 0, 0 \rangle$, so $\mathfrak{A}_\sigma$ is unexceptional. And it is easily seen that $\psi_i(\mathfrak{A}_\sigma)$ also has abacus notation $\langle 0, 0 \rangle$, so equals $\mathfrak{A}_\tau$.

$\mathfrak{D}_\sigma$. Take $a > h$ minimal such that $a, 2h - a \notin \sigma$. Then

$$\mathfrak{D}_\sigma = \begin{cases} \langle a - h \rangle & \text{if } a - h \notin \sigma \\ \langle h - a \rangle & \text{if } a - h \in \sigma. \end{cases}$$

Now if $i = 0$, then $\mathfrak{D}_\sigma$ cannot be exceptional (note that in this case $1 \notin \sigma$, so $\mathfrak{D}_\sigma$ cannot equal $\langle 1 \rangle$). If $i \geq 1$ and $k = 1$, then $\mathfrak{D}_\sigma$ is exceptional provided $i = a$ and $a - h \in \sigma$: then the $[2 : 1]$-pair is of type $B$, and so $\mathfrak{D}_\sigma$ is the exceptional partition $\beta$. For every $h < c < a$ either $c$ or $2h - c$ lies in $\sigma$, so the same is true in $\tau$; in addition $a, a - h, 2h - a \notin \tau$, so that $\mathfrak{D}_\tau = \langle a \rangle$, which is the exceptional partition $\hat{a}$ for the pair $(B_{\tau, 2}, B_{\tau, 2})$.

When $\mathfrak{D}_\sigma$ is unexceptional, it is easily checked (by a similar argument to that used in the cases above) that $\psi_i(\mathfrak{D}_\sigma) = \mathfrak{D}_\tau$.

$\mathfrak{B}_\sigma$. Take a minimal such that $a \in \sigma \cup (h)$ but $a + h \notin \sigma$. Then

$$\mathfrak{B}_\sigma = \begin{cases} \langle a, 0 \rangle & \text{if } a \leq n \\ \langle h - a, 0 \rangle & \text{if } n < a < h \\ \langle 0 \rangle & \text{if } a = h. \end{cases}$$

Clearly then $\mathfrak{B}_\sigma$ is unexceptional if $i \geq 1$, so suppose $i = 0$ and $k \geq 3$. If $a \leq n$ then we would need $a - 1$ in order for $\mathfrak{B}_\sigma$ to be exceptional; but $i = 0$ implies that $1 \notin \sigma$, a contradiction. If $n < a < h$ and $a - h - 1$ then we get $\mathfrak{B}_\sigma = \langle 0, 1 \rangle$, which is the exceptional partition $\beta$. Then it is easily checked that $\mathfrak{B}_\tau = \langle 0 \rangle$, which is the exceptional partition $\hat{a}$. Finally, if $a = h$ then the definition of $a$ means that $2h - 1 \in \sigma$, so that $k \geq 5$, so $\mathfrak{B}_\sigma$ is unexceptional.

In the cases where $\mathfrak{B}_\sigma$ is unexceptional it is easily checked that $\psi_i(\mathfrak{B}_\sigma) = \mathfrak{B}_\tau$. 
\[\Delta_{_{\sigma}}\] Take the minimal integers \(a, b\) such that \(a, b \neq \sigma, a - h \in \sigma\) and \(b - h \in \sigma \cup \{a\}\), and let \(\hat{a}, \hat{b}\) be the integers in \(-n, \ldots, -1, 1, \ldots, n\) congruent to \(a\) and \(b\) modulo \(h\). Then

\[
\Delta_{_{\sigma}} = \begin{cases} 
\langle \hat{a}, \hat{b} \rangle & \text{if } b < a + h \\
\langle \hat{a} \rangle & \text{if } b = a + h.
\end{cases}
\]

Clearly then \(\Delta_{_{\sigma}}\) is exceptional for \(i = 0\). Assuming \(i \geq 1\) and \(b < a + h\), \(\Delta_{_{\sigma}}\) is exceptional provided \(b = a + h - 1\) and \(i\) equals either \(b\) or \(-1 - b\) (whichever is positive).

Then \((B_{_{c,2}}, B_{_{r,2}})\) is a \([2 : 1]\)-pair of type \(A\) (if \(b > 0\)) or \(C\) (if \(-1 - b > 0\)), and \(\Delta_{_{\sigma}}\) is the exceptional partition \(\beta\). \(\Delta_{_{\tau}}\) is obtained from \(\Delta_{_{\sigma}}\) by replacing \(a - h\) with \(a - h + 1\), and so coincides with the exceptional partition \(\hat{a}\).

If \(b = a + h\), then \(\Delta_{_{\sigma}}\) is not exceptional: for this we would need to have \(a - 1 \in \sigma\) but \(a - 1 + h \notin \sigma\), but this contradicts the definition of \(b\).

In the cases where \(\Delta_{_{\sigma}}\) is unexceptional it is easily checked that \(\psi_{_{1}}(\Delta_{_{\sigma}}) = \Delta_{_{\tau}}\).

\[9\] The inductive step

In this section we use the results of Section 8 to carry out the inductive step in the proof of Theorem 6.4.

**Assumptions in force for Section 9**

\(\sigma\) and \(\tau\) are h-bar-cores, and \(B_{_{c,2}}\) and \(B_{_{r,2}}\) form a \([2 : k]\)-pair of residue \(i\), where either \(i \geq 1\) or \(k \geq 3\).

**Proposition 9.1.** Suppose \(B_{_{c,2}}\) and \(B_{_{r,2}}\) are Scopes–Kessar equivalent. If Theorem 6.4 holds for \(B_{_{c,2}}\) then it holds for \(B_{_{r,2}}\).

**Proof.** By Proposition 4.5(2) the canonical basis for \(B_{_{r,2}}\) is obtained from the canonical basis for \(B_{_{c,2}}\) by applying \(\psi_{_{i}}\) and extending linearly. By Propositions 8.3 and 8.4, \(\partial \psi_{_{i}}(\lambda) = \partial \lambda\) and \(\text{cl } \psi_{_{i}}(\lambda) = \text{cl } \lambda\) for every \(\lambda \in \mathcal{P}_{_{c,2}}\); furthermore, by Lemma 8.7, \(\lambda\) contains \(h\) or \(2h\) if and only if \(\psi_{_{i}}(\lambda)\) does. Next, by Proposition 8.5, for every pair \(\lambda, \mu \in \mathcal{P}_{_{c,2}}\) with \(|\partial \lambda - \partial \mu| \leq 1\), we have \(\lambda \leq \mu\) if and only if \(\psi_{_{i}}(\lambda) \leq \psi_{_{i}}(\mu)\). Finally, \(\psi_{_{i}}\) sends each special partition in \(\mathcal{P}_{_{c,2}}\) to the corresponding special partition in \(\mathcal{P}_{_{r,2}}\), by Lemma 8.8. So Theorem 6.4 holds in \(B_{_{r,2}}\) if and only if it holds in \(B_{_{c,2}}\).

Now we consider the more difficult case where \(B_{_{c,2}}\) and \(B_{_{r,2}}\) are not Scopes–Kessar equivalent. So we assume that for the rest of this section that either \(k = 1\) and \(1 \leq i < n\), or \(k = 3\) and \(i = 0\). We let \(\alpha \prec \beta \prec \gamma\) be the exceptional partitions in \(\mathcal{P}_{_{c,2}}\), and \(\hat{\alpha} \prec \hat{\beta} \prec \hat{\gamma}\) the corresponding partitions in \(\mathcal{P}_{_{r,2}}\). Let \(d = \partial \alpha = \partial \beta + 1 = \partial \gamma\).

First we look at the partition \(\mu^{+}\) defined in Theorem 6.4(1) for a restricted h-strict partition \(\mu \neq \emptyset, \phi_{_{c,2}}, \phi_{_{r,2}}\); this is the least dominant partition such that \(\mu \prec \mu^{+}, \partial \mu = \partial \mu^{+}\) and \(\text{cl } \mu = \text{cl } \mu^{+}\). Part of the statement of Theorem 6.4(1) is that \(\mu^{+}\) is defined, and we address this first. For this we need to define another bijection from \(\mathcal{P}_{_{c,2}}\) to \(\mathcal{P}_{_{r,2}}\); if \(\lambda \in \mathcal{P}_{_{c,2}}\), define

\[
\omega_{i}(\lambda) = \begin{cases} 
\psi_{_{i}}(\lambda) & \text{if } \lambda \text{ is unexceptional} \\
\hat{\beta} & \text{if } \lambda = \alpha \\
\hat{\alpha} & \text{if } \lambda = \beta \\
\hat{\gamma} & \text{if } \lambda = \gamma.
\end{cases}
\]
Lemma 9.2. Suppose $\mu \in B_{\sigma,2}$ is restricted, with $\mu \neq \emptyset_{\sigma}, \emptyset_{\sigma}, \emptyset_{\sigma}$, and that $\mu^+$ is defined. Then $\psi_i(\mu)^+$ is defined and equals $\omega_i(\mu^+)$.

Proof. The case $\mu = \alpha$ is easy: here $\psi_i(\mu) = \hat{\alpha}$, and from Proposition 8.6 $\hat{\alpha}^+$ is defined and equals $\hat{\gamma}$.

So assume that $\mu \neq \alpha$, and for the rest of the proof write $\bar{\mu} = \psi_i(\mu)$ and $\bar{\gamma} = \omega_i(\mu^+)$. The assumption $\mu \neq \alpha$ means that $\mu^+ \neq \gamma$. Using Propositions 8.3 to 8.6 we find that $\bar{\mu} < \bar{\gamma}$, $\bar{\mu} = \bar{\gamma}$ and $\text{cl} \bar{\mu} = \text{cl} \bar{\gamma}$. So $\bar{\mu}^+$ is certainly well-defined. If $\bar{\mu}^+ \neq \bar{\mu}$, then there is a partition $\rho \in P_{\tau,2}$ such that $\bar{\mu} < \rho < \bar{\mu}$, $\bar{\mu} = \bar{\rho}$ and $\text{cl} \bar{\mu} = \text{cl} \rho$. We can assume $\rho \neq \bar{\gamma}$ (because if $\rho = \bar{\gamma}$, we can replace it with $\hat{\alpha}$ and it will still have the same properties), so $\rho = \omega_i(\epsilon)$ for some $\epsilon \neq \gamma$. But then we find that $\mu < \epsilon < \mu^+$, $\bar{\mu} = \bar{\epsilon}$ and $\text{cl} \mu = \text{cl} \epsilon$, contradicting the definition of $\mu^+$. \qed

Now for a restricted partition $\mu \in P_{\sigma,2}$ for which $\mu^+$ is defined, set

$$A_{\mu} = \{ \lambda \in P_{\sigma,2} \mid \mu < \lambda < \mu^+ \land |\partial \lambda - \partial \mu| = 1 \}.$$ 

Lemma 9.3. Suppose $\mu \in B_{\sigma,2}$ is restricted, with $\mu \neq \emptyset_{\sigma}, \emptyset_{\sigma}, \emptyset_{\sigma}$, and that $\mu^+$ is defined.

1. If $\lambda \in P_{\sigma,2}$ is unexceptional, then $\lambda \in A_{\mu}$ if and only if $\psi_i(\lambda) \in A_{\psi_i(\mu)}$.

2. $\alpha \in A_{\mu} \Rightarrow \gamma \in A_{\mu} \Rightarrow \hat{\beta} \in A_{\psi_i(\mu)}$.

3. $\beta \in A_{\mu} \Rightarrow \hat{\alpha} \in A_{\psi_i(\mu)} \Rightarrow \hat{\gamma} \in A_{\psi_i(\mu)}$.

Proof. This follows from Propositions 8.3, 8.5 and 8.6 and Lemma 9.2. \qed

We need one more technical lemma before we can complete our inductive step.

Lemma 9.4. Suppose $i = 0$, and $\mu \in B_{\sigma,2}$ is restricted, with $\mu \neq \emptyset_{\sigma}, \emptyset_{\sigma}, \emptyset_{\sigma}$ and $h \in \mu$ or $2h \in \mu$. If the partition $\mu^+$ is defined, then $\mu^+ \neq \alpha$.

Proof. In order to have $\mu^+ = \alpha$, we would need to have $\mu < \alpha$. Since by assumption $\mu \neq \sigma \uplus (h, h)$, the only way this can happen is if $\mu = \sigma \uplus (h, a, h - a)$ for some $a$ with $a, h - a \notin \sigma$. In particular, we may assume $\Gamma(\sigma) \leq n - 1$. Under this assumption, we will construct a partition $\xi \in B_{\sigma,2}$ such that $\xi < \alpha$, $\partial \xi = \partial \alpha$, $\text{cl} \xi = \text{cl} \alpha$ and $\xi \notin \mu$. This is sufficient to rule out the possibility that $\mu^+ = \alpha$. (In fact the $\xi$ we construct will satisfy $\xi^+ = \alpha$, but we will not need this.)

Take $b \leq h - 2$ maximal such that $b, h - b \notin \sigma$, and let $l \leq h - 2$ be maximal such that either $l \in \sigma$ or $l = b$. Let $\xi = \sigma \uplus (b, h - b, l + h) \setminus (l)$. Then $\xi$ has fewer non-zero parts than $\mu$, so $\xi \notin \mu$. So we just need to check that $\partial \alpha = \partial \xi$ and $\text{cl} \alpha = \text{cl} \xi$.

$\alpha = \sigma \uplus (2h - 1, 1)$, so

$$\partial \alpha = 1 + \{1, h - 1\}_{\sigma} - \{h - 1, 2h - 1\}_{\sigma}$$

$$= \{0, h\}_{\sigma} - \{h, 2h\}_{\sigma}$$

since $h - 1 \in \sigma \neq 1, h + 1, 2h - 1$. On the other hand,

$$\partial \xi = h - b + \{h - b, b\}_{\sigma} - \{l, l + h\}_{\sigma}$$

$$= \{0, h\}_{\sigma} - \{l, l + h\}_{\sigma}$$
Hence the coefficients in $G_{\beta}$.

\[ \text{Example.} \] We give an example to illustrate the proof of Proposition 9.5. Suppose $\mu = \gamma$, with $(d_{\alpha \gamma}, d_{\beta \gamma}, d_{\gamma \gamma}) = (0, 0, 1)$. Then $\tilde{\mu} = \hat{\beta}$, and by Proposition 8.1(6) or Proposition 8.2(6), $(d_{\alpha \hat{\beta}}, d_{\beta \hat{\beta}}, d_{\hat{\beta} \hat{\beta}}) = (0, 0, q^2)$ while $d_{\psi(\lambda) \hat{\beta}} = d_{\lambda \gamma}$ for unexceptional $\lambda$. $\gamma^+$ is unexceptional, so $\tilde{\beta} = \psi(\gamma^+)$ by Lemma 9.2. Now by Propositions 8.3, 8.5 and 8.6

\[ A_{\tilde{\beta}} = \{ \psi(\lambda) \mid \lambda \in A_{\gamma} \} \cup \{ \gamma \}. \]

Hence the coefficients in $G(\tilde{\beta})$ satisfy Theorem 6.4. (Note that $\tilde{\beta}$ contains $h$ or $2h$ if and only if $\gamma$ does; also, Theorem 6.4 gives $d_{\gamma \gamma}$ as $q^2$, not $q$, because if $h$ or $2h = \hat{\gamma}$, then $i = 0$, so that $h \in \hat{\beta}$.)
Proposition 9.7. \( \hat{\mathfrak{r}} \) special partitions that are defined in Defect we find seven possibilities for the intersection of \( \psi \).

Proof. We know from Lemma 8.8 that \( Q \) is restricted if and only if Proposition 9.6.

Proposition 9.6. Suppose \( \Gamma(\sigma) \leq n - 2 \) and Theorem 6.4 holds for \( \mu = \hat{\mathfrak{r}} \). Then it holds for \( \hat{\mathfrak{r}} \).

Proof. We know from Lemma 8.8 that \( \hat{\mathfrak{r}} \) and \( \hat{\mathfrak{r}} \) are unexceptional, with \( \psi(\hat{\mathfrak{r}}) = \hat{\mathfrak{r}} \), \( \psi(\hat{\mathfrak{r}}) = \hat{\mathfrak{r}} \), and \( \psi(\hat{\mathfrak{r}}) = \hat{\mathfrak{r}} \). In addition, either \( \hat{\mathfrak{r}} \) is unexceptional or \( \hat{\mathfrak{r}} = \beta \), and if \( \hat{\mathfrak{r}} \) is unexceptional then \( \psi(\hat{\mathfrak{r}}) = \hat{\mathfrak{r}} \).

Now let
\[
D(\sigma) = \{ \lambda | \hat{\mathfrak{r}} < \lambda \leq \hat{\mathfrak{r}}, \hat{\lambda} = 2 \}
\]
\[
E(\sigma) = \{ \lambda | \hat{\mathfrak{r}} < \lambda \leq \hat{\mathfrak{r}}, \hat{\lambda} = 2 \}
\]
\[
F(\sigma) = D(\sigma) \cup E(\sigma) \cup \{ \hat{\mathfrak{r}}, \hat{\mathfrak{r}} \},
\]
and define \( D(\tau), E(\tau), F(\tau) \) similarly. By considering \( \hat{\lambda} \)-values and the dominance order, we find six possibilities for the intersection of \( \{ \alpha, \beta, \gamma \} \) with \( F(\sigma) \).

1. \( \alpha, \beta, \gamma \notin F(\sigma) \). In this case \( D(\tau) = \psi(\hat{\mathfrak{r}}) \) and \( E(\tau) = \psi(\hat{\mathfrak{r}}) \).
2. \( \beta \in D(\sigma), \alpha, \gamma \notin F(\sigma) \). In this case \( D(\tau) = \psi(\hat{\mathfrak{r}}) \cup \{ \hat{\lambda} \} \) and \( E(\tau) = \psi(\hat{\mathfrak{r}}) \).
3. \( \alpha, \gamma \in D(\sigma), \beta \notin F(\sigma) \). In this case \( D(\tau) = \psi(\hat{\mathfrak{r}}) \backslash \{ \hat{\lambda} \} \), and \( E(\tau) = \psi(\hat{\mathfrak{r}}) \).
4. \( \beta \in E(\sigma), \alpha, \gamma \notin F(\sigma) \). In this case \( D(\tau) = \psi(\hat{\mathfrak{r}}) \) and \( E(\tau) = \psi(\hat{\mathfrak{r}}) \cup \{ \hat{\lambda} \} \).
5. \( \alpha, \gamma \in E(\sigma), \beta \notin F(\sigma) \). In this case \( D(\tau) = \psi(\hat{\mathfrak{r}}) \) and \( E(\tau) = \psi(\hat{\mathfrak{r}}) \backslash \{ \hat{\lambda} \} \).
6. \( \alpha \in E(\sigma), \beta = \hat{\mathfrak{r}}, \gamma \notin F(\sigma) \). In this case \( D(\tau) = \psi(\hat{\mathfrak{r}}) \), \( E(\tau) = \psi(\hat{\mathfrak{r}}) \backslash \{ \hat{\lambda} \} \) and \( \hat{\mathfrak{r}} = \hat{\lambda} \).

In fact the second and third cases cannot occur, by the inductive assumption, Proposition 8.1(7) and Proposition 8.2(8). In the other four cases, the values \( d_{1,\hat{\mathfrak{r}}} \) are given for all \( \lambda \) by Proposition 8.1(6) and Proposition 8.2(6), and we see that Theorem 6.4 holds for \( \hat{\mathfrak{r}} \). (Note that by Proposition 8.2(7) cases 5 and 6 can only happen when \( i \geq 1 \).)

Proposition 9.7. Suppose \( \Gamma(\sigma) \leq n - 1 \) and Theorem 6.4 holds for \( \mu = \hat{\mathfrak{r}} \). Then it holds for \( \hat{\mathfrak{r}} \).

Proof. We know from Lemma 8.8 that \( \hat{\mathfrak{r}} \) and \( \hat{\mathfrak{r}} \) are unexceptional, with \( \psi(\hat{\mathfrak{r}}) = \hat{\mathfrak{r}} \), \( \psi(\hat{\mathfrak{r}}) = \hat{\mathfrak{r}} \). In addition, \( \hat{\mathfrak{r}} \) either coincides with \( \beta \) or is unexceptional, and if \( \hat{\mathfrak{r}} \) is unexceptional then \( \psi(\hat{\mathfrak{r}}) = \hat{\mathfrak{r}} \), and a similar statement holds for \( \hat{\mathfrak{r}} \).

Now let
\[
D(\sigma) = \{ \lambda | \hat{\mathfrak{r}} < \lambda \leq \hat{\mathfrak{r}}, \hat{\lambda} = 2 \}
\]
\[
E(\sigma) = \{ \lambda | \hat{\mathfrak{r}} < \lambda \leq \hat{\mathfrak{r}}, \hat{\lambda} = 1 \}
\]
\[
F(\sigma) = D(\sigma) \cup E(\sigma) \cup \{ \hat{\mathfrak{r}}, \hat{\mathfrak{r}} \},
\]
and define \( D(\tau), E(\tau), F(\tau) \) similarly. Now by considering \( \hat{\lambda} \)-values and the dominance order, we find seven possibilities for the intersection of \( \{ \alpha, \beta, \gamma \} \) with \( F(\sigma) \).

1. \( \alpha, \beta, \gamma \notin F(\sigma) \). In this case \( D(\tau) = \psi(\hat{\mathfrak{r}}) \) and \( E(\tau) = \psi(\hat{\mathfrak{r}}) \).
2. $\beta \in D(\sigma), \alpha, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma)) \cup \{\lambda\}$ and $E(\tau) = \psi_i(E(\sigma))$.

3. $\alpha, \gamma \in D(\sigma), \beta \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma)) \setminus \{\lambda\}$, and $E(\tau) = \psi_i(E(\sigma))$.

4. $\alpha \in D(\sigma), \beta = \Delta_\sigma, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma)) \setminus \{\lambda\}$, $\Delta_\tau = \lambda$ and $E(\tau) = \psi_i(E(\sigma)) \cup \{\gamma\}$.

5. $\beta \in E(\sigma), \alpha, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma))$ and $E(\tau) = \psi_i(E(\sigma)) \cup \{\lambda\}$.

6. $\alpha, \gamma \in E(\sigma), \beta \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma))$ and $E(\tau) = \psi_i(E(\sigma)) \setminus \{\lambda\}$.

7. $\alpha \in E(\sigma), \beta = \Xi_\sigma, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma))$, $E(\tau) = \psi_i(E(\sigma)) \setminus \{\lambda\}$ and $\Xi_\tau = \lambda$.

In the seven cases, the values $d_{\lambda, \beta, \gamma}$ are given for all $\lambda$ by Proposition 8.1(6) and Proposition 8.2(6), and we see that Theorem 6.4 holds for $\Xi_\tau$. (Note that by Proposition 8.1(7) and Proposition 8.2(8,9) cases 3, 6 and 7 can only happen when $i \geq 1$, while case 7 can only happen if $i = 0$.)

**Proposition 9.8.** Suppose $\Gamma(\sigma) \geq 1$ and Theorem 6.4 holds for $\mu = \Xi_\tau$. Then it holds for $\Delta_\tau$.

**Proof.** We know from Lemma 8.8 that $\Xi_\tau$ is unexceptional, with $\psi_i(\Xi_\tau) = \Xi_\tau$. In addition, $\Xi_\sigma$ either coincides with $\beta$ or is unexceptional, and if $\Xi_\sigma$ is unexceptional then $\psi_i(\Xi_\sigma) = \Xi_\tau$, and a similar statement holds for $\Delta_\sigma$.

Now let

$$D(\sigma) = \{\lambda \mid \Delta_\sigma \prec \lambda \prec \Xi_\sigma, \partial \lambda = 1\}$$

$$E(\sigma) = \{\lambda \mid \Xi_\sigma \prec \lambda \prec \Delta_\sigma, \partial \lambda = 1\}$$

$$F(\sigma) = D(\sigma) \cup E(\sigma) \cup \{\Xi_\sigma, \Delta_\sigma\},$$

and define $D(\tau), E(\tau), F(\tau)$ similarly. Now by considering $\partial$-values and the dominance order, we find seven possibilities for the intersection of $[\alpha, \beta, \gamma]$ with $F(\sigma)$.

1. $\alpha, \beta, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma))$ and $E(\tau) = \psi_i(E(\sigma))$.

2. $\beta \in D(\sigma), \alpha, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma)) \cup \{\lambda\}$ and $E(\tau) = \psi_i(E(\sigma))$.

3. $\alpha, \gamma \in D(\sigma), \beta \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma)) \setminus \{\lambda\}$, and $E(\tau) = \psi_i(E(\sigma))$.

4. $\alpha \in D(\sigma), \beta = \Xi_\sigma, \gamma \in E(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma)) \setminus \{\lambda\}$, $\Xi_\tau = \lambda$ and $E(\tau) = \psi_i(E(\sigma))$.

5. $\beta \in E(\sigma), \alpha, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma))$ and $E(\tau) = \psi_i(E(\sigma)) \cup \{\lambda\}$.

6. $\alpha, \gamma \in E(\sigma), \beta \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma))$ and $E(\tau) = \psi_i(E(\sigma)) \setminus \{\lambda\}$.

7. $\alpha \in E(\sigma), \beta = \Delta_\sigma, \gamma \notin F(\sigma)$. In this case $D(\tau) = \psi_i(D(\sigma))$, $E(\tau) = \psi_i(E(\sigma)) \setminus \{\lambda\}$ and $\Delta_\tau = \lambda$.

In the seven cases, the values $d_{\lambda, \beta, \gamma}$ are given for all $\lambda$ by Proposition 8.1(6) and Proposition 8.2(6), and we see that Theorem 6.4 holds for $\Delta_\tau$. (Note that by Proposition 8.1(7) and Proposition 8.2(8,9) cases 3, 6 and 7 can only happen when $i \geq 1$, while case 4 can only happen if $i = 0$.)
10 Proof of Theorem 6.4

Finally we can complete the proof of the main theorem. We proceed by induction on $|\tau|$. Let $l = \text{len}(\tau)$, and consider the three possibilities in Lemma 4.11. If $\tau = (l, l-1, \ldots, 1)$ with $l \leq n$, then the results of Section 7 show that Theorem 6.4 holds for $\tau$. Alternatively, there is a residue $i$ such that either $i \neq 0$ and $\tau$ has a removable $i$-node, or $i = 0$ and $\tau$ has at least three removable $i$-nodes. So define the $h$-bar-core $\sigma$ by removing all the removable $i$-nodes from $\tau$. Now the result follows from Propositions 9.1 and 9.5 to 9.8, and the inductive hypothesis.

11 Application to spin representations of symmetric groups

This paper is motivated by the decomposition number problem for spin representations of symmetric groups. Here we briefly summarise the background, and discuss the implications of our results.

Take $m \geq 4$, and let $\tilde{S}_m$ denote one of the two Schur covers of $S_m$. Any representation of $\tilde{S}_m$ lifts to a representation of $\tilde{S}_m$; the irreducible representations which do not come from $S_m$ in this way are called spin representations of $\tilde{S}_m$. Given a representation (or character), the associate representation is obtained by tensoring with the (lift of the) sign representation of $S_m$.

The classification of irreducible representations of $\tilde{S}_m$ over $\mathbb{C}$ goes back to Schur [Sch] (though construction of the actual representations was achieved much later, by Nazarov [N]), and can be stated as follows. Say that a strict partition $\lambda$ is even or odd as the number of positive even parts of $\lambda$ is even or odd. For each even strict $\lambda$ of $m$, there is an irreducible self-associate character $\chi(\lambda)$ of $\tilde{S}_m$ if $\lambda$ is even, and a pair of associate irreducible characters $\chi(\lambda)_+ , \chi(\lambda)_-$ if $\lambda$ is odd. These characters are pairwise distinct, and yield all the ordinary irreducible spin characters of $\tilde{S}_m$.

The classification of irreducible modular representations is due to Brundan and Kleshchev [BK1, BK2]. Suppose $h$ is an odd prime, and let $\mathbb{F}$ be a field of characteristic $h$ which is a splitting field for $\tilde{S}_m$. Say that a partition is $h$-even or $h$-odd as the number of nodes of non-zero residue is even or odd. Then for each restricted $h$-strict partition $\mu$ of $m$, there is a self-associate irreducible Brauer character $\phi(\mu)$ if $\mu$ is $h$-even, and a pair of associate irreducible Brauer characters $\phi(\mu)_+, \phi(\mu)_-$ if $\mu$ is $h$-odd. These Brauer characters are distinct, and give all the irreducible $h$-modular spin Brauer characters of $\tilde{S}_m$.

Two characters (ordinary or modular) lie in the same $h$-block of $\tilde{S}_m$ if and only if the labelling partitions have the same $h$-bar-core (except in the case of an odd partition $\lambda$ of $h$-bar-weight $0$, where $\chi(\lambda)_+$ and $\chi(\lambda)_-$ lie in separate simple blocks). So (apart from this slight caveat for $h$-bar-weight $0$) $h$-blocks correspond precisely to the combinatorial blocks studied in this paper. The defect group of a block with bar-weight $\omega$ is isomorphic to a Sylow $h$-subgroup of $S_{\omega h}$; in particular, for blocks with bar-weight less than $h$, the defect group is abelian and the defect coincides with the bar-weight.

The spin decomposition number problem then asks for the decomposition of $\chi(\lambda)$ or $\chi(\lambda)_\pm$ as a sum of irreducible Brauer characters $\phi(\mu)$ or $\phi(\mu)_\pm$. A close approximation to this problem is to consider the reduced decomposition number obtained by adding the indecomposable projective characters corresponding to associate Brauer characters. So for a strict partition $\lambda$ of $m$ and a
restricted $h$-strict partition $\mu$ of $m$ we define the reduced decomposition number

$$D_{\lambda \mu} =
\begin{cases}
[\chi(\lambda) : \phi(\mu)] & \text{if } \lambda \text{ is even and } \mu \text{ is } h\text{-even} \\
[\chi(\lambda) : \phi(\mu) ] + [\chi(\lambda)_- : \phi(\mu)_-] & \text{if } \lambda \text{ is even and } \mu \text{ is } h\text{-odd} \\
[\chi(\lambda)_+ : \phi(\mu) ] & \text{if } \lambda \text{ is odd and } \mu \text{ is } h\text{-even} \\
[\chi(\lambda)_+ : \phi(\mu)_+ ] + [\chi(\lambda)_+ : \phi(\mu)_-] & \text{if } \lambda \text{ is odd and } \mu \text{ is } h\text{-odd}.
\end{cases}$$

A conjecture due to Leclerc and Thibon [LT, Conjecture 6.2] says that if $h$ is large relative to $m$ then $D_{\lambda \mu}$ is determined by the integer $d_{\lambda \mu}(1)$ obtained by setting $q = 1$ in $d_{\lambda \mu}$, specifically, if we define $n_h(\lambda)$ to be the number of positive parts of $\lambda$ divisible by $h$, and

$$x_h(\lambda) =
\begin{cases}
n_h(\lambda) & \text{if } \lambda \text{ is even and } h\text{-even} \\
n_h(\lambda) + 1 & \text{if } \lambda \text{ is even and } h\text{-odd} \\
n_h(\lambda) - 1 & \text{if } \lambda \text{ is odd and } h\text{-even} \\
n_h(\lambda) & \text{if } \lambda \text{ is odd and } h\text{-odd},
\end{cases}$$

then $D_{\lambda \mu} = 2^{x_h(\lambda)/2}d_{\lambda \mu}(1)$.

In fact, the original Leclerc–Thibon conjecture asserts that this relationship should hold whenever $m < h^2$. A reasonable extension to a blockwise version would say that the formula should hold in all blocks of bar-weight less than $h$ (regardless of $m$), i.e. for all blocks with abelian defect group; this is analogous to the blockwise form of James’s conjecture for decomposition numbers of symmetric groups. Theorem 5.2 and Müller’s work [Mü] show that the Leclerc–Thibon conjecture is true for blocks of bar-weight 1. However, the recent demise of James’s conjecture (thanks to Williamson [W]) suggests that it is too optimistic to hope that the conjecture will be true for all blocks with abelian defect group. Nevertheless, it seems likely that it will hold for blocks of bar-weight 2; the decomposition numbers are known for all $m \leq 18$ thanks to Maas [Ma], and the conjecture can be checked in these cases. So it appears that setting $q = 1$ in Theorem 6.4 gives a formula for the reduced decomposition numbers for defect 2 spin blocks of symmetric groups. We hope to prove this in future work.

12 The Fock space of type $A_h^{(2)}$

The subject of this paper is the determination of canonical basis coefficients for the $q$-deformed Fock space in type $A_{h-1}^{(2)}$ where $h \geq 3$ is odd. In this section we briefly discuss the corresponding problem in type $A_h^{(2)}$. These two Kac–Moody types are related by folding of Dynkin diagrams, as illustrated below.
Defect 2 spin blocks and canonical basis coefficients

(n.b. These diagrams apply only for \( h \geq 5 \), but a similar folding takes place between types \( A_3^{(2)} \) and \( A_2^{(2)} \). In the Kac–Moody classification, type \( A_3^{(2)} \) is usually referred to as \( D_3^{(2)} \).

Following the general construction for all classical types in [KK] in terms of Young walls, the Fock space can be described combinatorially; we give a brief summary. Define a decorated \( h \)-strict partition to be an \( h \)-strict partition in which each non-zero part divisible by \( h \) is decorated with an accent ° or †, and two consecutive equal parts must have opposite decorations. The \( q \)-deformed Fock space has the set of all decorated \( h \)-strict partitions as a basis, and the actions of the generators \( e_i \) and \( f_i \) can be described in terms of adding and removing \( i \)-nodes (with a suitable definition of \( i \)-node for \( i = 0 \) or \( \tilde{0} \)). The weight spaces can be defined in term of “decorated \( h \)-bar-cores”, leading to the notion of bar-weight.

It appears that analogues of Theorems 5.2 and 6.4 hold in this setting, and can be proved by the same techniques; in fact, the situation for weight spaces of bar-weight 2 in type \( A_5^{(2)} \) is simpler in several ways:

- there is no exceptional behaviour for the special partitions, so a more direct analogue of Richards’s theorem is possible;
- the canonical basis coefficients are all equal to 0, 1, \( q \) or \( q^2 \), so there are fewer cases to consider in the analysis of \([2 : k]\)-pairs;
- \([2 : 1]\)-pairs of residue 0 or \( \tilde{0} \) are much more tractable, so that only the case \( \tau = \emptyset \) is needed as a base case.

The relationship between the quantum groups of types \( A_5^{(2)} \) and \( A_4^{(2)} \) means that the canonical basis coefficients in the two types are very closely related. The “folding” process involved in the transition introduces the exceptional behaviour for the special partitions. We illustrate this by giving an example for \( h = 5 \), showing the canonical basis coefficients for the weight spaces corresponding to the 5-bar-core (1), with the special partitions labelled. We can see that the first two columns of the right-hand matrix (labelled by the special partitions \( \emptyset \) and...
\(\hat{\mathfrak{h}}\), for which the exceptional behaviour occurs) are obtained by adding the first three columns of the left-hand matrix in pairs, with an (as yet mysterious) adjustment to the powers of \(q\) occurring.

It therefore appears that a promising approach to finding canonical basis coefficients in type \(A_{h-4}^{(2)}\) (and to the spin decomposition number problem for symmetric groups in characteristic \(h\)) is to work in type \(A_{h}^{(2)}\) first, and understand how folding affects decomposition numbers.

References


