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# Another runner removal theorem for $v$-decomposition numbers of Iwahori–Hecke algebras and $q$-Schur algebras

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**Abstract**

Let $\mathcal{F}$ denote the Fock space representation of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$. The '$v$-decomposition numbers' are the coefficients when the canonical basis for this representation is expanded in terms of the basis of partitions, and the evaluations at $v = 1$ of these polynomials give the decomposition numbers for Iwahori–Hecke algebras and $q$-Schur algebras over $\mathbb{C}$. James and Mathas have proved a theorem which relates $v$-decomposition numbers for different values of $e$, by adding empty runners to the abacus displays for the labelling partitions. Here we prove a similar theorem, which involves adding 'full' runners to these abacus displays.

## 1 Introduction

Let $\mathbb{F}$ be a field, $q$ a non-zero element of $\mathbb{F}$ and $n$ a non-negative integer. Let $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ denote the Iwahori–Hecke algebra of the symmetric group $\mathfrak{S}_n$, and let $\mathcal{S}_{\mathbb{F},q}(n,n)$ denote the $q$-Schur algebra. The representation theory of these algebras (which generalise the group algebra $\mathbb{F}\mathfrak{S}_n$ and the classical Schur algebra $\mathcal{S}_{\mathbb{F}}(n,n)$) is the subject of a great deal of research, with the most important outstanding problem being the determination of the *decomposition numbers*, i.e. the composition multiplicities of the simple modules $D^\mu$ in the Specht modules $S^\lambda$ (for the Iwahori–Hecke algebra) or of the simple modules $L(\mu)$ in the Weyl modules $\Delta(\lambda)$ (for the $q$-Schur algebra). The *decomposition matrix* records these multiplicities.

It is known that if we change the characteristic of the field $\mathbb{F}$ while not changing the multiplicative order $e$ of $q$, then the decomposition matrix 'does not change much'. Specifically, changing $\mathbb{F}$ from a field of infinite characteristic to a field of prime characteristic causes the decomposition matrix to be post-multiplied by a square unitriangular matrix (the *adjustment matrix*), which is conjectured by James to equal the identity matrix in certain cases. Thus, knowing the decomposition numbers in the case $\mathbb{F} = \mathbb{C}$ is an important first step in working out the decomposition numbers over any field. Fortunately,

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the decomposition numbers can be computed when $\mathbb{F} = \mathbb{C}$; they are the values at $v = 1$ of certain polynomials $d_{\lambda\mu}(v)$, which have accordingly become known as '$v$-decomposition numbers'; this was proved for the Iwahori–Hecke algebras by Ariki [1] and extended to the $q$-Schur algebras by Varagnolo and Vasserot [9]. The $v$-decomposition numbers arise from the Fock space representation of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$. This has a natural basis indexed by the set of partitions, and a 'canonical basis' which is invariant under the bar involution. The $v$-decomposition numbers are are the entries of the transition matrix between these two bases.

In studying decomposition matrices for Iwahori–Hecke algebras and $q$-Schur algebras, James noticed that certain parts of decomposition matrices corresponding to different values of $e$ were equal. James and Mathas proved a theorem relating $v$-decomposition numbers for different values of $e$, which partially explains this phenomenon. The author used this theorem extensively in his proof [2] that the decomposition numbers for blocks of Iwahori–Hecke algebras of weight three are at most 1. Here, we prove a closely related theorem; the two theorems are vital to the author's forthcoming paper [3] on blocks of weight four.

## 2 Background and notation

An excellent introduction to the representation theory of Iwahori–Hecke algebras and $q$-Schur algebras can be found in the book by Mathas [8], which also contains an account of the Fock space. More detail on the latter is given in the paper by Lascoux, Leclerc and Thibon [6].

Throughout this paper, $e$ denotes a fixed integer greater than or equal to 2. We identify $\mathbb{Z}/e\mathbb{Z}$ with the set $\{0, \ldots, e-1\}$.

### 2.1 Partitions and the abacus

We assume that the reader is familiar with the combinatorics of partitions, $e$-regular partitions, $e$-cores and the dominance order $\trianglerighteq$. We often identify a partition $\lambda$ with its Young diagram $\{(i, j) \in \mathbb{N}^2 \mid j \leqslant \lambda_i\}$, whose elements we call nodes. If $(i, j)$ and $(i', j')$ are nodes, we say that $(i, j)$ is *above* $(i', j')$ if $i < i'$. The *residue* of a node $(i, j)$ is defined as the residue of $j - i$ modulo $e$, and we refer to a node of residue $i$ as an $i$-node. An node $\mathfrak{n}$ of $\lambda$ is removable if $\lambda \setminus \{\mathfrak{n}\}$ is (the Young diagram of) a partition, and pair $\mathfrak{n} = (i, j)$ not in $\lambda$ is an addable node of $\lambda$ if $\lambda \cup \{\mathfrak{n}\}$ is a partition. We write $\varnothing$ for the unique partition of zero.

The notion of *ladders* will be very useful in this paper. Given $l \geqslant 1$, the $l$th ladder in $\mathbb{N}^2$ is defined as the set

$$\{(i, j) \in \mathbb{N}^2 \mid (e-1)j + i = l + e - 1\};$$

the intersection of this set with a Young diagram $\lambda$ is called the $l$th ladder of $\lambda$. If $\lambda \neq \varnothing$, the *outermost ladder* of $\lambda$ is the last (i.e. indexed by the largest value of $l$) non-empty ladder of $\lambda$. Note that all the nodes in any ladder have the same residue, and that if $\lambda$ is $e$-regular then the highest removable node of $\lambda$ is contained in the outermost ladder.

We shall make extensive use of the abacus, which is a useful device for representing partitions. Given a partition $\lambda$, take an integer $r \geqslant \lambda'_1$, and define $\beta_i = \lambda_i + r - i$ for $i = 1, \ldots, r$; $\beta_1, \ldots, \beta_r$ are called the *beta-numbers* for $\lambda$. Now draw an abacus with $e$ vertical runners numbered $0, 1, \ldots, e-1$ from left to right, and mark positions $0, 1, \ldots$ on the runners of the abacus, reading from left to right and from top to bottom (so that the integers congruent to $i$ modulo $e$ are marked on runner $i$, in increasing order from the top down). Place a bead on the abacus at position $\beta_i$ for each $i$. The resulting configuration is called an abacus display for $\lambda$.

The abacus is most useful in determining the $e$-core of a partition. If we have abacus displays for two partitions $\lambda$ and $\mu$ with the same numbers of beads, then (as observed by James) $\lambda$ and $\mu$ have the same $e$-core if and only if for each $i$ the number of beads on runner $i$ of one abacus equals the number of beads on runner $i$ of the other.

We may also use an abacus display to find addable and removable nodes. Take $i \in \mathbb{Z}/e\mathbb{Z}$, and let $j \in \{0, \dots, e-1\}$ be given by $j \equiv i + r \pmod{e}$. Then removable $i$-nodes of $\lambda$ correspond to beads on runner $j$ of the abacus with no bead immediately to the left, while addable $i$-nodes correspond to empty spaces on runner $j$ with beads immediately to the left; here, we regard position $x-1$ as being immediately to the left of position $x$ even if $x \equiv 0 \pmod{e}$, and we regard position 0 as having a bead immediately to its left. So the sequence of addable and removable $i$-nodes may be found by examining only runners $j$ and $j-1$; reading from top to bottom of the Young diagram (with the non-French convention) corresponds to reading from bottom to top on these runners of the abacus. This is very useful when calculating the action of $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ on $\lambda$, which is described in the next section.

## 2.2 The Fock space

Let $v$ be an indeterminate over $\mathbb{C}$, and define the *Fock space* to be the free $\mathbb{C}[v^{\pm 1}]$-module $\mathcal{F}$ with basis the set of all partitions. This has the structure of a module for the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$; we describe the action of the Chevalley generators $\mathfrak{f}_0, \dots, \mathfrak{f}_{e-1}$ of the negative part $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$, and refer the reader to [6] for a full description of the module action. If $\lambda$ and $\xi$ are partitions, then we write $\lambda \xrightarrow{a:i} \xi$ to indicate that $\xi$ is obtained from $\lambda$ by adding $a$ addable $i$-nodes. If this is the case, then we define the integer

$$N_i(\lambda, \xi) = \sum_{\mathfrak{n} \in \xi \setminus \lambda} \left( (\text{number of addable } i\text{-nodes of } \xi \text{ above } \mathfrak{n}) - (\text{number of removable } i\text{-nodes of } \lambda \text{ above } \mathfrak{n}) \right).$$

Then for any partition $\lambda$ we have

$$\mathfrak{f}_i^{(a)} \lambda = \sum_{\xi \xleftarrow{a:i} \lambda} v^{N_i(\lambda, \xi)} \xi.$$

The submodule $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)\varnothing$ of $\mathcal{F}$ is frequently identified with the irreducible integrable highest weight module $L(\Lambda_0)$. This module inherits an involutory $\mathbb{C}[v + v^{-1}]$-linear map – the *bar involution* – from $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$. This is also described in [6]; for us it suffices to note that for any $a_1, \dots, a_s$ and $i_1, \dots, i_s$, the element $\mathfrak{f}_{i_s}^{(a_s)} \dots \mathfrak{f}_{i_1}^{(a_1)} \varnothing$ of $L(\Lambda_0)$ is bar-invariant. $L(\Lambda_0)$ possesses a basis $\{G(\mu) \mid \mu \text{ an } e\text{-regular partition}\}$ which is uniquely determined by the properties that each $G(\mu)$ is bar-invariant and $G(\mu) \equiv \mu \pmod{vL(\Lambda_0)}$. This is the Lusztig–Kashiwara *canonical basis* of $L(\Lambda_0)$.

Lascoux, Leclerc and Thibon gave an algorithm for computing the $G(\mu)$. First, one defines an intermediate basis $\{A(\mu) \mid \mu \text{ an } e\text{-regular partition}\}$: define $\overline{\mu}$ to be the partition obtained by removing the outermost ladder of the Young diagram for $\mu$, and suppose that this ladder consists of $a$ nodes of residue $i$. Now define (recursively)

$$A(\mu) = \mathfrak{f}_i^{(a)}(A(\overline{\mu})).$$

$A(\mu)$ has the form $\mu + \sum_\lambda c_{\lambda\mu}(v)\lambda$, where $c_{\lambda\mu}(v)$ is non-zero only if $|\lambda| = |\mu|$, $\lambda$ and $\mu$ have the same $e$-core and $\mu \rhd \lambda$. So there exist unique Laurent polynomials $\alpha_{\nu\mu}(v) \in \mathbb{C}[v + v^{-1}]$ for $\mu, \nu$ $e$-regular such that

$$G(\mu) = A(\mu) + \sum_{\nu \lhd \mu} \alpha_{\nu\mu} A(\nu).$$

Using the action of the Heisenberg algebra, Leclerc and Thibon [7] found a way to extend the bar involution to the whole of the Fock space; this enabled them to extend the basis $\{G(\mu) \mid \mu \text{ an } e\text{-regular partition}\}$ to a canonical basis $\{G(\mu) \mid \mu \text{ a partition}\}$ for the whole Fock space. The elements of this basis have the form

$$G(\mu) = \sum_{\lambda} d_{\lambda\mu}(v)\lambda,$$

where the $d_{\lambda\mu}(v)$ are polynomials such that:

- $d_{\mu\mu}(v) = 1$;
- $d_{\lambda\mu}(v)$ is divisible by $v$ for $\lambda \neq \mu$;
- $d_{\lambda\mu}(v) = 0$ unless $\lambda$ and $\mu$ are partitions of the same integer with the same $e$-core and with $\mu \unrhd \lambda$.

These polynomials $d_{\lambda\mu}(v)$ are the '$v$-decomposition numbers' mentioned in the introduction.

The point of this paper is to compare canonical bases (and hence $v$-decomposition numbers) for different values of $e$. Accordingly, we introduce notation to avoid ambiguity. We may write $G(\mu)$ and $d_{\lambda\mu}(v)$ as $G^e(\mu)$ and $d^e_{\lambda\mu}(v)$ if necessary. As above, we write the Chevalley generators for $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ as $\mathfrak{f}_0 \dots, \mathfrak{f}_{e-1}$, but we denote the corresponding generators of $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_{e+1})$ as $\mathfrak{F}_0, \dots, \mathfrak{F}_e$. If $k$ is an integer, then we write $\overline{k}$ to denote the residue of $k$ modulo $e$ or modulo $e+1$, as appropriate.

### 2.3 The James–Mathas runner removal theorem

It has long been observed that decomposition numbers for Iwahori–Hecke algebras are in some sense 'independent of $e$'. James and Mathas proved a precise version of this statement for $v$-decomposition numbers.

**Theorem 2.1. [5, Theorem 4.5]** *Suppose $e \geqslant 3$, $\lambda$ and $\mu$ are partitions of $n$, and that we have abacus displays for $\lambda$ and $\mu$ on an abacus with $r$ beads. Suppose there is some $i$ such that in both abacus displays runner $i$ contains no beads. Delete runner $i$ from each abacus, and let $\lambda^-$ and $\mu^-$ be the partitions defined by the resulting abacus displays. Then*

$$d^e_{\lambda\mu}(v) = d^{e-1}_{\lambda^-\mu^-}(v).$$

Our main theorem is very similar to this, but it concerns removing a 'full' runner from the abacus rather than an empty one. For the case where $\mu^-$ is $(e-1)$-regular, we follow essentially the same line of proof, analysing how the addition of a runner interacts with the action of the Chevalley generators. The case where $\mu$ is $(e-1)$-singular is then deduced easily using Theorem 2.1.

After this paper was submitted, Kai Meng Tan pointed out to the author that there is a fairly straightforward way to deduce our main result from Theorem 2.1, by translating between 'full runner addition' and 'empty runner addition' using conjugation of partitions and the Mullineux map, and employing [6, Theorem 7.2]. We feel that the proof herein is still valuable, since it avoids any discussion of the weight of a partition or of the Mullineux map, and since it should indicate to the reader how to tidy up the rather poorly-presented proof of the regular part of Theorem 2.1.

We also discuss another possible method of proof. It has been pointed out by Fred Goodman that the regular case of Theorem 2.1 can be deduced from the interpretation [4, Theorem 5.3] of $v$-decomposition numbers as parabolic Kazhdan–Lusztig polynomials. The weight space $\mathfrak{h}^*$ for $\mathfrak{sl}_r$ (where $r$ is as in Theorem 2.1) is partitioned into '$e$-alcoves', which are naturally in bijection with elements of the affine Weyl group. Under this bijection, a Kazhdan–Lusztig polynomial may be indexed by a pair of $e$-alcoves rather than a pair of Weyl group elements (or rather, cosets of the finite Weyl group), and this indexing is independent of $e$, in the sense that changing an $e$-alcove to the corresponding $f$-alcove is achieved simply by dilating by a factor $\frac{f}{e}$. To each partition is associated an $e$-alcove, and [4, Theorem 5.3] says that the $v$-decomposition number indexed by a given pair of partitions equals the Kazhdan–Lusztig polynomial indexed by the pair of corresponding alcoves. To deduce Theorem 2.1, one observes that the $(e-1)$-alcove corresponding to the partition $\lambda^-$ is the dilation of the $e$-alcove corresponding to $\lambda$.

It does not seem that our result can be deduced as easily. For one thing, we do not have a natural choice of an integer $r$. Also, it seems that the procedure of adding a full runner does not behave so well with respect to the alcove geometry. Perhaps readers more expert than the author will see how to make this approach work.

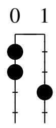
We now make our precise definitions, and prove our results.

## 3 Addition of a full runner and the canonical basis

Given a partition $\lambda$ and a non-negative integer $k$, we construct a new partition $\lambda^{+k}$ as follows. Take $r \geqslant \lambda'_1$ and construct the abacus display for $\lambda$ with $r$ beads. Write $r + k = ce + d$, with $0 \leqslant d \leqslant e - 1$, and add a runner to the abacus display immediately to the left of runner $d$; now put $c$ beads on this new runner, in the top $c$ positions, i.e. the positions labelled $d, d + e + 1, \dots, d + (c - 1)(e + 1)$ in the usual labelling for an abacus with $e + 1$ runners. The partition whose abacus display is obtained is $\lambda^{+k}$. It is a straightforward exercise to check that the definition of $\lambda^{+k}$ does not depend on the choice of $r$.

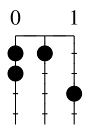
Note that we retain the original labelling of the runners, leaving the new runner unlabelled.

**Example.** Suppose $\lambda = (3, 1)$, and $e = 2$. Choosing $r = 3$, we get the following abacus display:

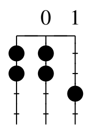



.

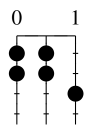
We obtain

$\lambda^{+0} =$ 

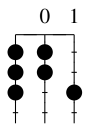
$= (5, 1)$,

$\lambda^{+1} =$ 

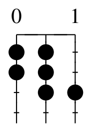
$= (4, 1^2)$,

$\lambda^{+2} =$ 

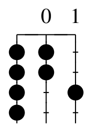
$= (4, 1^2)$,

$\lambda^{+3} =$ 

$= (3, 2, 1^2)$,

$\lambda^{+4} =$ 

$= (3^2, 1^2)$,

$\lambda^{+3} =$ 

$= (3^2, 2, 1^2)$.

We extend the operator ${}^{+k}$ linearly to the whole of the Fock space. We can now state our main theorem concerning canonical bases.

**Theorem 3.1.** *Suppose $\mu$ is an $e$-regular partition and $k \geqslant \mu_1$. Then $G^{e+1}(\mu^{+k}) = G^e(\mu)^{+k}$.*

Let $\mu$ be an $e$-regular partition, and define the element $A(\mu)$ of the Fock space as in Section 2.2. Theorem 3.1 will follow from the following proposition.

**Proposition 3.2.** *Suppose $\mu$ is an $e$-regular partition, and $k \geqslant \mu_1$. Then there exist $i_1, \dots, i_t$ and $a_1, \dots, a_t$ such that $A(\mu)^{+k} = \mathfrak{F}_{i_t}^{(a_t)} \dots \mathfrak{F}_{i_1}^{(a_1)} \varnothing$; in particular, $A(\mu)^{+k}$ is bar-invariant.*

We use two different inductive steps to prove this.

**Lemma 3.3.** *Suppose $\lambda$ is a partition and $k > \lambda_1$. Then there is a constant $a$, depending only on $k$ and the $e$-core of $\lambda$, such that $\lambda^{+k} = \mathfrak{F}_{\bar{k}}^{(a)}(\lambda^{+(k-1)})$.*

**Proof.** Choose some large $r$ and construct the abacus display for $\lambda$ with $r$ beads. Write $r+k-1 = ce+d$ with $0 \leqslant d \leqslant e-1$; then an abacus display for $\lambda^{+(k-1)}$ is obtained by inserting a runner containing $c$ beads to the immediate left of runner $d$, while an abacus display for $\lambda^{+k}$ may be obtained by adding this runner to the immediate right of runner $d$ instead; in the case where $e \mid r+k$, this is not the same abacus display for $\lambda^{+k}$ as that constructed above, but it is easy to see that adding a runner at the far right containing $c$ beads in the topmost positions gives the same partition as adding a runner at the far left containing $c+1$ beads in the topmost positions. Let $a$ equal $c$ minus the number of beads on runner $d$ in the abacus for $\lambda$.

The fact that $\lambda_1 < k$ means that the positions of the beads on runner $d$ of the abacus display for $\lambda$ are among the top $c$ positions on that runner; for if any were lower, then one of the beta-numbers for $\lambda$ would be at least $ce+d = r+k-1$; hence the largest beta-number would be at least $r+k-1$, i.e. $\lambda_1$ would be at least $k$; contradiction. The abacus display for $\lambda^{+k}$ is obtained from the abacus display for $\lambda^{+(k-1)}$ by swapping runner $d$ and the inserted runner, i.e. simultaneously removing all removable nodes of residue $\bar{k}$ and adding all addable nodes of residue $\bar{k}$. But the fact that the inserted runner has beads in the top $c$ positions while runner $d$ can have beads only in the top $c$ positions means that there are no removable nodes of residue $\bar{k}$. Hence $\lambda^{+k}$ is obtained from $\lambda^{+(k-1)}$ simply by adding all the addable nodes of residue $\bar{k}$, and we have $\lambda^{+(k-1)} \xrightarrow{a:\bar{k}} \lambda^{+k}$ with $N_{\bar{k}}(\lambda^{+(k-1)}, \lambda^{+k}) = 1$ and hence

$$\mathfrak{F}_{\bar{k}}^{(a)}(\lambda^{+(k-1)}) = \lambda^{+k}.$$

It remains to show that $a$ depends only on $k$ and the $e$-core of $\lambda$. Any other partition with the same $e$-core as $\lambda$ will have the same number of beads on runner $d$ when displayed on an abacus with $r$ beads, so $a$ depends only on $c$, $d$ and the $e$-core of $\lambda$. $c$ and $d$ are determined by $k$ and $r$, so $a$ depends only on $r, k$ and the $e$-core of $\lambda$. But the statement of the lemma does not involve $r$, so in fact $a$ depends only on $k$ and the $e$-core of $\lambda$. □

**Corollary 3.4.** *If $\mu$ is an $e$-regular partition and $k > \mu_1$, then there is some $a$ such that*

$$A(\mu)^{+k} = \mathfrak{F}_{\bar{k}}^{(a)}(A(\mu)^{+(k-1)}).$$

**Proof.** We have $A(\mu) = \sum c_{\lambda\mu}(v)\lambda$, where the sum is over partitions $\lambda$ with the same $e$-core as $\mu$ with $\mu \trianglerighteq \lambda$. In particular, $k > \lambda_1$ for each such $\lambda$. By Lemma 3.3, we have

$$\lambda^{+k} = \mathfrak{F}_{\bar{k}}^{(a)}(\lambda^{+(k-1)}),$$

and the value $a$ is the same for each $\lambda$. The result follows. □

For our next lemma, we need to define a function $g : (\mathbb{Z}/e\mathbb{Z}) \setminus \{\overline{k}\} \to (\mathbb{Z}/(e+1)\mathbb{Z}) \setminus \{\overline{k}, \overline{k+1}\}$ by mapping

$$\overline{k-i} \longmapsto \overline{k-i}$$

for $i = 1, \dots, e-1$.

**Lemma 3.5.** *Suppose $i \in (\mathbb{Z}/e\mathbb{Z})\setminus\{\overline{k}\}$, and $\lambda$ and $\xi$ are partitions. Then $\lambda \xrightarrow{a:i} \xi$ if and only if $\lambda^{+k} \xrightarrow{a:g(i)} \xi^{+k}$, and if this happens then we have $N_i(\lambda, \xi) = N_{g(i)}(\lambda^{+k}, \xi^{+k})$.*

**Proof.** Take an abacus display with $r$ beads. We have $\lambda \xrightarrow{a:i} \xi$ if and only if the abacus display for $\xi$ may be obtained by moving $a$ beads from runner $i + r - 1$ to runner $i + r$, and in this case $N_i(\lambda, \xi)$ is determined by the configurations of these two runners in the two abacus displays. The fact that $i \neq \overline{k}$ means that in constructing the abacus displays for $\lambda^{+k}$ and $\xi^{+k}$ the new runner is *not* added in between these two runners, and so the condition $\lambda^{+k} \xrightarrow{a:g(i)} \xi^{+k}$ and the coefficient $N_{g(i)}(\lambda^{+k}, \xi^{+k})$ are determined from these two runners in exactly the same way. □

**Corollary 3.6.** *Suppose $i \in (\mathbb{Z}/e\mathbb{Z}) \setminus \{\overline{k}\}$, $a \geqslant 1$ and $\lambda$ is any partition. Then $\left(\mathfrak{f}_i^{(a)}(\lambda)\right)^{+k} = \mathfrak{F}_{g(i)}^{(a)}(\lambda^{+k})$.*

**Proof.** This is immediate from Lemma 3.5 and the description of the action of $\mathfrak{f}_i^{(a)}$ in Section 2.2. □

This immediately yields the following.

**Corollary 3.7.** *Suppose $\mu$ is an e-regular partition, and that the outermost ladder of $\mu$ consists of a nodes of residue $i \neq \overline{k}$. Let $\overline{\mu}$ be the partition obtained by removing this ladder. Then*

$$A(\mu)^{+k} = \mathfrak{F}_{g(i)}^{(a)}(A(\overline{\mu})^{+k}).$$

**Proof of Proposition 3.2.** We proceed by induction on $n$ and $k$. If $\mu_1 < k$, then by Corollary 3.4 we have $A(\mu)^{+k} = \mathfrak{F}_{\overline{k}}^{(a)}(A(\mu)^{+(k-1)})$ for some $a$, and the result follows by induction on $k$. If $\mu_1 = k$, then let $i$ be the residue of the highest removable node of $\mu$. $i$ cannot equal $\overline{k}$ (since otherwise $\mu$ would be $e$-singular), and so by Corollary 3.7 we have $A(\mu)^{+k} = \mathfrak{F}_{g(i)}^{(a)}(A(\overline{\mu})^{+k})$, and the result follows by induction on $n$. □

**Proof of Theorem 3.1.** We have $G^e(\mu) = \sum \alpha_{\nu\mu}(v)A(\nu)$, where the sum is over $e$-regular partitions $\nu$ such that $\mu \trianglerighteq \nu$. In particular, we have $k \geqslant \nu_1$ for each $\nu$, and hence, by Proposition 3.2, each $A(\nu)^{+k}$ is bar-invariant. Thus $G^e(\mu)^{+k}$ is bar-invariant; since we also have

$$G^e(\mu)^{+k} = \mu^{+k} + \sum_{\lambda} d^e_{\lambda\mu}(v)\lambda^{+k}$$

with each $d^e_{\lambda\mu}(v)$ divisible by $v$, $G^e(\mu)^{+k}$ must equal $G^{e+1}(\mu^{+k})$ by the uniqueness property of the canonical basis. □

# 4 $v$-decomposition numbers

Recall that the $v$-decomposition number $d^e_{\lambda\mu}(v)$ is defined even when $\mu$ is $e$-singular. We now express Theorem 3.1 in terms of $v$-decomposition numbers, extending it to include the $e$-singular case.

**Theorem 4.1.** *Suppose $e \geqslant 3$ and $\lambda$ and $\mu$ are partitions of $n$. Take abacus displays for $\lambda$ and $\mu$ on an abacus with $r$ beads, and suppose that there is some $i$ such that in both displays the first empty space on runner $i$ occurs after every bead on the abacus. Define partitions $\lambda^-$ and $\mu^-$ by deleting runner $i$ from each abacus display. Then $d^e_{\lambda\mu}(v) = d^{e-1}_{\lambda^-\mu^-}(v)$.*

**Proof.** The condition on runner $r$ is the same as saying that we can write $\lambda = (\lambda^-)^{+k}$ and $\mu = (\mu^-)^{+k}$ for some $k \geqslant \lambda^-_1, \mu^-_1$. The conclusion now follows immediately from Theorem 3.1 if $\mu$ is $e$-regular. If $\mu$ is $e$-singular, then we define new partitions $\hat{\lambda}, \hat{\mu}, \hat{\lambda}^-, \hat{\mu}^-$ by adding an empty runner at the left of each abacus display. $\hat{\mu}$ is $(e + 1)$-regular, and so we have

$$d^{e+1}_{\hat{\lambda}\hat{\mu}}(v) = d^e_{\hat{\lambda}^-\hat{\mu}^-};$$

on the other hand, Theorem 2.1 implies that

$$d^{e+1}_{\hat{\lambda}\hat{\mu}}(v) = d^e_{\lambda\mu}, \qquad d^e_{\hat{\lambda}^-\hat{\mu}^-}(v) = d^{e-1}_{\lambda^-\mu^-},$$

and the result follows. □

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