Semigroups generated by projections

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1 Introduction

Let V be a finite-dimensional vector space over $\mathbb{C}$. A *reflection* of V can be defined as an invertible linear map $V \to V$ of finite order which fixes pointwise a subspace of codimension 1. A *complex reflection group* is a subgroup of $\mathrm{GL}(V)$ generated by reflections. The finite complex reflection groups were classified by Shephard and Todd [ST], and are an important research topic from many points of view.

Here (out of curiosity rather than any potential application) we consider what happens if we drop the requirement for our linear maps to be invertible. So we consider the semigroup of all linear endomorphisms of V, and as well as reflections, we can consider *projections*. In this note we use the word "projection" in a very restricted sense to mean an idempotent linear endomorphism of V of nullity 1. Now we can consider finite semigroups generated by reflections and projections. Given such a semigroup, the set of invertible elements (if there are any) forms a reflection group, and the subsemigroup generated by the projections is normalised by the reflection subgroup. So to classify finite semigroups generated by reflections and projections, it suffices to find the finite semigroups generated by projections, and then for each such semigroup M find the reflection groups that normalise M. In this note we will mostly concentrate on finding finite semigroups generated by projections; let's call these *projection semigroups*. We'll say that two projection semigroups on V are *equivalent* if they are conjugate under $\mathrm{GL}(V)$.

Example. Take V of dimension $n \geqslant 2$, and take $n+1$ lines $l_1, \dots, l_{n+1}$ with the property that any n of them span V. Now take the set of all projections whose eigenspaces between them contain n of the lines $l_1, \dots, l_{n+1}$. Then it's not a very hard exercise to show that these projections generate a finite semigroup. In dimensions $2, 3, 4$ this semigroup has order 19, 229, 3001 respectively. In dimension n, we'll call (the isomorphism type of) this semigroup P_n.

As with reflection groups, we have a notion of reducibility: if we take a projection semigroup which fixes a proper subspace $W < V$, then in a natural way we obtain projection semigroups on W and on V/W. Note, however, that our starting semigroup is not in general isomorphic to the direct product of these semigroups; so there is a non-trivial extension theory.

We also have a notion of duality: if $\alpha : V \to V$ is a projection, then so is the dual map $\alpha^* : V^* \to V^*$. So for every projection semigroup there is a dual projection semigroup. A projection semigroup is anti-isomorphic to its dual, but (unlike in the reflection group case) need not be isomorphic to its dual.

2 Classification in dimension 2

The aim of the rest of this note is to classify finite projection semigroups in dimension 2 up to equivalence. So from now on all projections will be idempotent linear maps $\mathbb{C}^2 \to \mathbb{C}^2$ of rank 1. We'll identify linear endomorphisms of $\mathbb{C}^2$ with 2×2 complex matrices (acting on $\mathbb{C}^2$ on the left). Given a non-zero vector $\binom{x}{y}$, we'll write $\langle\binom{x}{y}\rangle$ for the line it spans.

We first deal with the reducible case, which is easy. By definition, a projection semigroup M on $\mathbb{C}^2$ is reducible if there is a line which is an eigenspace for every projection in M. But in fact if we choose a line l and take *any* finite set of projections which all have l as an eigenspace, they will generate a finite semigroup.

So from now on we concentrate on irreducible projection semigroups. We need a combinatorial lemma.

Lemma 2.1. *Suppose G is a finite directed graph, where we allow edges in both directions between a pair of vertices. Suppose G has at least four vertices, and that every vertex has at least one incident arrow. Say that a vertex a is a* start *(resp.* end*) if there is at least one arrow $a \to b$ (resp. $b \to a$). Suppose also that whenever $a \to c$ and $b \to d$ are independent arrows in G (i.e. a, b, c, d are distinct), then $a \to d$ and $b \to c$ are also arrows in G. Then at least one of the following occurs:*

- *there is a vertex incident to every arrow;*
- *for every start s and every end $t \neq s$, the arrow $s \to t$ lies in G.*

Proof. Suppose there is no vertex which is incident to every arrow, and that s is a start and t an end with $s \neq t$. We need to show that there is an arrow $s \to t$. There are certainly arrows $s \to u$ and $v \to t$ for some u, v. If $u = t$ or $v = s$ then we're done, while if $u \neq v$ then the independent arrow condition gives the result. So we can assume $u = v$, and that $s \to u$ is the only arrow from s in G, and $u \to t$ the only arrow to t in G. This means that any other arrow in G must be incident with both of the arrows $s \to u$ and $u \to t$ (otherwise the independent arrow condition would yield another arrow from s or to t). In other words, every arrow either meets u, or meets s and t. By assumption u is not incident to every arrow, so if there is not an arrow $s \to t$ then there must be an arrow $t \to s$. But now take a vertex w different from s, t, u. By assumption there is an arrow incident with w. This arrow is incident with both of the arrow $s \to u$ and $u \to t$, so is one of the arrows $u \to w$ or $w \to u$. Now repeated application of the independent arrow condition yields an arrow $s \to t$. $\square$

Now suppose M is a finite projection semigroup on $\mathbb{C}^2$. Let's say that a line l in $\mathbb{C}^2$ is a *kernel* of M if it is the kernel of some non-zero element of M. Since M is generated by projections, this means in particular that l is the kernel of some projection in M. We'll write $\mathcal{K}_M$ for the set of kernels of M. We define *images* of M similarly, and write $\mathcal{I}_M$ for the set of images of M.

Now take two projections $\alpha, \beta \in M$, and suppose α and β have no common eigenspace. Then $\mathrm{Tr}(\alpha\beta) \neq 0$, and $\alpha\beta$ equals $\mathrm{Tr}(\alpha\beta)$ times the projection with kernel $\ker\beta$ and image $\mathrm{im}\,\alpha$. Since M is finite, this means that $\mathrm{Tr}(\alpha\beta)$ is a root of unity (we will call this the "trace condition"), and that the projection with kernel $\ker\beta$ and image $\mathrm{im}\,\alpha$ lies in M. If we define the *trace group* of M to be the group G generated by all the non-zero traces of elements of M, then G is a finite subgroup of $\mathbb{C}^\times$.

If M is irreducible, then M has at least two kernels and at least two images. If $|\mathcal{I}_M \cup \mathcal{K}_M| \leqslant 3$, then clearly M is isomorphic to a subsemigroup of of the semigroup P_2 defined above. So we assume that $|\mathcal{I}_M \cup \mathcal{K}_M| \geqslant 4$. Now consider a directed graph whose vertex set is $\mathcal{I}_M \cup \mathcal{K}_M$, with an arrow $s \to t$ if the projection with kernel s and image t lies in M. The previous paragraph shows that the independent arrow condition from Lemma 2.1 holds, and so using Lemma 2.1 and the assumption that M is irreducible we find that for every $k \in \mathcal{K}_M$ and every $i \in \mathcal{I}_M$, the projection with kernel k and image i lies in M. Hence M is determined by its images and kernels.

Now we can give the classification. First we describe the semigroups in the classification; in view of the last paragraph, it suffices to list the images and kernels of each M. Henceforth we fix a primitive cube root of unity $\omega \in \mathbb{C}$.

1. Let S be a finite set of roots of unity in $\mathbb{C}$ such that $1 \in S$. Then there are projection semigroups $A_S^{(0)}$, $A_S^{(1)}$, $A_S^{(2)}$ defined as follows.

	images	kernels
$A_S^{(0)}$	$\langle\begin{smallmatrix}1\\s\end{smallmatrix}\rangle$ for $s \in S$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle$
$A_S^{(1)}$	$\langle\begin{smallmatrix}1\\s\end{smallmatrix}\rangle$ for $s \in S$ and $\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle$
$A_S^{(2)}$	$\langle\begin{smallmatrix}1\\s\end{smallmatrix}\rangle$ for $s \in S$ and $\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle$

The trace group of $A_S^{(i)}$ is the group G generated by S, and

$$|A_S^{(i)}| = \begin{cases} 2|S||G| & (i=0) \\ 2(|S|+1)|G|+1 & (i=1) \\ 2(|S|+2)|G|+1 & (i=2). \end{cases}$$

2. Take any root of unity $w \neq \pm 1$, and let B_w be the semigroup with

$$\text{images } \left\langle\begin{matrix}-1\\1\end{matrix}\right\rangle, \left\langle\begin{matrix}1\\w\end{matrix}\right\rangle, \left\langle\begin{matrix}w\\1\end{matrix}\right\rangle, \qquad \text{kernels } \left\langle\begin{matrix}-1\\1\end{matrix}\right\rangle, \left\langle\begin{matrix}1\\0\end{matrix}\right\rangle, \left\langle\begin{matrix}0\\1\end{matrix}\right\rangle.$$

The trace group of B_w is the subgroup G_w of $\mathbb{C}^\times$ generated by w and -1. The order of B_w is $9|G_w|+1$.

3. Now we define sporadic examples $C^{(i)}$ for $i = 0,1,2,3,4$. In each case the trace group is the group of 6th roots of unity in $\mathbb{C}$. The order of $C^{(i)}$ is then $6(k+i)+1$, where k is the number of kernels and i the number of images.

	images	kernels
$C^{(0)}$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega\\1\end{smallmatrix}\rangle$
$C^{(1)}$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega^2\\1\end{smallmatrix}\rangle$
$C^{(2)}$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega\\1\end{smallmatrix}\rangle$
$C^{(3)}$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega^2\\1\end{smallmatrix}\rangle$
$C^{(4)}$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega\\1\end{smallmatrix}\rangle$	$\langle\begin{smallmatrix}1\\0\end{smallmatrix}\rangle, \langle\begin{smallmatrix}0\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}-1\\1\end{smallmatrix}\rangle, \langle\begin{smallmatrix}\omega\\1\end{smallmatrix}\rangle$

Now we can give our main theorem.

Theorem 2.2. *Suppose M is an irreducible finite projection semigroup on $\mathbb{C}^2$. Then M or its dual is isomorphic to one of:*

- ◇ *a subsemigroup of P_2;*
- ◇ *$A_S^{(i)}$, for $i \in \{0,1,2\}$ and a finite set S of roots of unity with $1 \in S$;*
- ◇ *B_w for some root of unity $w \neq \pm 1$; or*
- ◇ *$C^{(i)}$ for $i \in \{0,1,2,3,4\}$.*

The proof splits up into several propositions.

Proposition 2.3. *Suppose M is a finite irreducible projection semigroup on $\mathbb{C}^2$ with $|\mathcal{I}_M \cup \mathcal{K}_M| \geqslant 4$ and $|\mathcal{K}_M| = 2$. Then M is isomorphic to $A_S^{(i)}$ for some $i \in \{0,1,2\}$ and some set S of roots of unity with $1 \in S$.*

Proof. By replacing M with an equivalent semigroup, we can assume that the two kernels of M are $\langle\binom{1}{0}\rangle$ and $\langle\binom{0}{1}\rangle$, and that one of the images is $\langle\binom{1}{1}\rangle$. Hence M contains the projections

$$\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}.$$

If $\langle\binom{1}{w}\rangle$ is also an image with $w \neq 0$, then M also contains the projection

$$\begin{pmatrix}1 & 0\\ w & 0\end{pmatrix},$$

and the trace condition shows that w is a root of unity.

So the images of M have the form $\langle\binom{1}{w}\rangle$ for w in some set of roots of unity including 1, possibly together with $\langle\binom{1}{0}\rangle$ and/or $\langle\binom{0}{1}\rangle$. Now we see that M equals $A_S^{(i)}$ for some i, except in the case where $\langle\binom{0}{1}\rangle \in \mathcal{I}_M \not\ni \langle\binom{1}{0}\rangle$, in which case M is equivalent to $A_{S'}^{(1)}$, where $S' = \{ s^{-1} \mid s \in S\}$. $\square$

So we can assume from now on that $|\mathcal{K}_M| \geqslant 3$, and dually that $|\mathcal{I}_M| \geqslant 3$. We distinguish the possibilities further according to $|\mathcal{I}_M \cap \mathcal{K}_M|$.

Proposition 2.4. *Suppose M is a finite projection semigroup on $\mathbb{C}^2$, and that a, b, c are distinct kernels of M and d, e, f are distinct images of M. Then $\{a, b, c\} \cap \{d, e, f\} \neq \emptyset$.*

Proof. Without loss of generality we can assume that a, b, c are the lines $\langle\binom{1}{0}\rangle$, $\langle\binom{0}{1}\rangle$ and $\langle\binom{1}{-1}\rangle$. Assuming for a contradiction that a, b, c, d, e, f are distinct, we write $d = \langle\binom{x}{1}\rangle$, $e = \langle\binom{y}{1}\rangle$, $f = \langle\binom{z}{1}\rangle$ for distinct $x, y, z \neq 0, -1$. Now M contains the projections

$$\alpha = \begin{pmatrix}1 & 0\\ x^{-1} & 0\end{pmatrix}, \beta = \begin{pmatrix}0 & y\\ 0 & 1\end{pmatrix}, \gamma = \begin{pmatrix}\frac{x}{x+1} & \frac{x}{x+1}\\ \frac{1}{x+1} & \frac{1}{x+1}\end{pmatrix}.$$

Now the trace condition implies that y/x and $(y+1)/(x+1)$ are both roots of unity. Similarly z/x and $(z+1)/(x+1)$ are roots of unity. But this means in particular that $|x| = |y| = |z|$ and $|x+1| = |y+1| = |z+1|$, which is impossible for three distinct complex numbers. $\square$

Proposition 2.5. *Suppose M is a finite projection semigroup on $\mathbb{C}^2$, with $|\mathcal{I}_M| = |\mathcal{K}_M| = 3$ and $|\mathcal{I}_M \cap \mathcal{K}_M| = 1$. Then M is isomorphic to B_w for some root of unity $w \neq \pm 1$.*

Proof. Without loss of generality we assume that the kernels of M are $\langle\binom{1}{0}\rangle$, $\langle\binom{0}{1}\rangle$ and $\langle\binom{1}{-1}\rangle$, and that $\langle\binom{1}{-1}\rangle$ is also an image of M. We write the other two images as $\langle\binom{w}{1}\rangle$ and $\langle\binom{x}{1}\rangle$ for some $w, x \neq 0, -1$. Now M includes the projections

$$\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & w\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & x\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\frac{x}{x+1} & \frac{x}{x+1}\\ \frac{1}{x+1} & \frac{1}{x+1}\end{pmatrix},$$

and the trace condition shows that $-w$, $-x$ and $\frac{w+1}{x+1}$ are roots of unity. The only way this can happen is if $x = w^{-1}$, which means in particular that $w \neq -1$, so $M = B_w$. $\square$

Now we consider the case where $|\mathcal{I}_M \cap \mathcal{K}_M| \geqslant 2$.

Lemma 2.6. *Suppose M is a finite projection semigroup on $\mathbb{C}^2$, and that $\langle\binom{1}{0}\rangle$ and $\langle\binom{0}{1}\rangle$ are both images and kernels of M.*

1. *Suppose $\langle\binom{-1}{1}\rangle$ is a kernel of M. Then the only possible other images of M are $\langle\binom{-1}{1}\rangle$, $\langle\binom{\omega}{1}\rangle$ and $\langle\binom{\omega^2}{1}\rangle$.*
2. *Suppose $\langle\binom{-1}{1}\rangle$ is an image of M. Then the only possible other kernels of M are $\langle\binom{-1}{1}\rangle$, $\langle\binom{\omega}{1}\rangle$ and $\langle\binom{\omega^2}{1}\rangle$.*

3. *Suppose $\langle{\omega \atop 1}\rangle$ is an image of M. Then the only possible other kernels of M are $\langle{-1 \atop 1}\rangle$, $\langle{\omega \atop 1}\rangle$ and $\langle{-\omega^2 \atop 1}\rangle$.*

4. *Suppose $\langle{\omega^2 \atop 1}\rangle$ is an image of M. Then the only possible other kernels of M are $\langle{-1 \atop 1}\rangle$, $\langle{\omega^2 \atop 1}\rangle$ and $\langle{-\omega \atop 1}\rangle$.*

Proof. For (1), suppose $\langle{x \atop 1}\rangle$ is an image, with $x \neq 0, -1$. By computing the traces of products of projections in M, we find that x and $(x+1)/x$ are both roots of unity. The only way this can happen is if $x = \omega^{\pm 1}$.

(2) now follows by duality, and (3) and (4) are proved in a similar way. □

Now we consider in general a finite projection semigroup on $\mathbb{C}^2$. We continue to assume that $|\mathcal{I}_M| \geqslant 3$, $|\mathcal{K}_M| \geqslant 3$ and $|\mathcal{I}_M \cup \mathcal{K}_M| \geqslant 4$. Proposition 2.4 shows that $|\mathcal{I}_M \cap \mathcal{K}_M| \geqslant 1$. Proposition 2.5 shows that if $|\mathcal{I}_M \cap \mathcal{K}_M| = 1$ (forcing $|\mathcal{I}_M| = |\mathcal{K}_M| = 3$) then M is isomorphic to B_w for some w. So assume $|\mathcal{I}_M \cap \mathcal{K}_M| \geqslant 2$. Then without loss of generality we can assume that $\langle{1 \atop 0}\rangle$ and $\langle{0 \atop 1}\rangle$ are both images and kernels of M, and that $\langle{-1 \atop 1}\rangle$ is also a kernel.

If $|\mathcal{I}_M \cap \mathcal{K}_M| = 2$, then $\langle{-1 \atop 1}\rangle$ is not an image of M. So by Lemma 2.6(1), the only possible other images of M are $\langle{\omega \atop 1}\rangle$ and $\langle{\omega^2 \atop 1}\rangle$. Let's assume that $\langle{\omega \atop 1}\rangle$ is an image. If $\langle{\omega^2 \atop 1}\rangle$ is also an image, then by Lemma 2.6(3,4) there are no other kernels, so M is $C^{(1)}$. If $\langle{\omega \atop 1}\rangle$ is an image but $\langle{\omega^2 \atop 1}\rangle$ is not, then by Lemma 2.6(3) the only other possible kernel is $\langle{-\omega^2 \atop 1}\rangle$. If this is indeed a kernel, then M is equivalent to the dual of $C^{(1)}$; if not, then M is $C^{(0)}$.

If $|\mathcal{I}_M \cap \mathcal{K}_M| = 3$, then we can assume $\langle{-1 \atop 1}\rangle$ is both an image and a kernel of M. If M is not isomorphic to P_2, then (up to duality) it has another image, and by Lemma 2.6(1) we can assume this is $\langle{\omega \atop 1}\rangle$. Now we have the possible further image $\langle{\omega^2 \atop 1}\rangle$ but no other possible kernel. So M is $C^{(2)}$ or $C^{(3)}$.

If $|\mathcal{I}_M \cap \mathcal{K}_M| \geqslant 4$, then we assume that $\langle{-1 \atop 1}\rangle$ and $\langle{\omega \atop 1}\rangle$ are both images and kernels. Now Lemma 2.6 shows that there can be no other images or kernels, and M is $C^{(4)}$.

This completes the proof of Theorem 2.2.

We end with by returning to our original question, considering finite semigroups generated by reflections and projections. As remarked in the introduction, once we have a projection semigroup M, we just need to find the reflections that normalise M. It is a simple matter to do this for each of the semigroups appearing in our classification. We do not give full details, but list the cases where the maximal reflection group normalising M is irreducible.

- ◇ P_2 and $C^{(3)}$ are both normalised by the reflection group of type $G(6,6,2)$ (also known as the Coxeter group of type $I_2(6)$, or the Weyl group of type G_2).

- ◇ Take a finite set of roots of unity S with $1 \in S$, and define

$$G = \left\{ g \in \mathbb{C}^{\times} \;\middle|\; gs \in S \text{ for all } s \in S \right\}.$$

  Then G is the group of mth roots of unity for some $m \geqslant 1$. Assume $m \geqslant 2$ and also that there is $g \in \mathbb{C}$ such that $gs^{-1} \in S$ for every $s \in S$. Then Now $A_S^{(0)}$ and $A_S^{(2)}$ are normalised by a reflection group of type $G(2m,2,2)$.

- ◇ $C^{(4)}$ is normalised by the exceptional reflection group of type G_4.

References

[ST] G. C. Shephard & J. A. Todd, 'Finite unitary reflection groups', *Canadian J. Math.* **6** (1954), 274–304. [cited on p. 1]