An interesting family of posets

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2010 Mathematics subject classification: 06A07

Abstract

We introduce a new family of posets which we call bichains. In the finite case these first arose in the study of 0-Hecke algebras, but they admit a variety of different characterisations. We give these characterisations, prove that they are equivalent and derive some numerical results concerning finite bichains.

1 Introduction

In this paper we introduce a family of partially ordered sets which we call bichains. These are remarkable in that (particularly in the finite case) they can be described in several quite different ways. Some of these descriptions are by direct construction, and some by restrictive properties. Bichains originally arose in the author’s paper [F] on 0-Hecke algebras, where they appear as structure posets for certain induced modules for Hecke algebras of type A. Apart from this, however, bichains appear not to have been considered before. The fact that they admit so many different descriptions suggests that they should be studied further, and various generalisations suggest themselves. We hope that this paper is the start of an interesting new avenue in poset theory.

The author must express his gratitude to Jeremy Rickard. Early in this work, the author posted a question about these posets on MathOverflow, and Jeremy’s answer [R] provided some key results which underpin a lot of this paper, and without which most of this work would not have been done. We indicate later exactly which parts are due to Jeremy.

We give a brief outline of the structure of the paper. We begin in Section 2 with a definition of bichains. In Section 3 we consider maximal elements, giving a reduction theorem which allows all finite bichains to be constructed recursively. In Section 4 we give a direct construction of posets from binary sequences, and show that these give all finite bichains. In Section 5 we introduce a method of joining posets together which we call splicing, and show that this can be used to join smaller bichains to make larger ones. In this way, we give another construction of all finite bichains as splices of certain simple bichains. In Section 6 we consider a connection with graphs: writing down a definition for graphs analogous to the definition of bichains leads to another characterisation of finite bichains, in terms of their incomparability graphs. In Section 7 we return to the connection with Coxeter groups of type A, showing how to characterise finite bichains in terms of Coxeter elements of symmetric groups.
## 2 Bichains

In this section we will introduce bichains. We begin by recalling some basic definitions for posets.

As usual, a **partial order** is a reflexive asymmetric transitive binary relation. A **partially ordered set** or **poset** is a set $P$ with a given partial order $\leq$. Given $p, q \in P$ with $p \leq q$, we say that $q$ is above $p$, and that $p$ is below $q$. $p, q \in P$ are incomparable if $p \not\leq q \not\geq p$. $\leq$ is a total order if for all $p, q \in P$ either $p \leq q$ or $q \leq p$. If $Q \subseteq P$, then $\leq$ induces a partial order on $Q$, which will also be denoted $\leq$. $Q$ is called:

- a **chain** if this induced partial order is a total order;
- an **antichain** if $p \not\leq q$ for all $p, q \in Q$;
- an **ideal** in $P$ if $p \not\leq q$ whenever $q \in Q$ and $p \in P \setminus Q$.

The dual poset $P^\circ$ has the same underlying set, with $p \leq q$ in $P^\circ$ if and only if $q \leq p$ in $P$.

A **refinement** of $\leq$ is a partial order $\leq^+$ such that $p \leq q$ whenever $p \leq q$; if there is at least one pair $p, q$ with $p \leq q$ but $p \not\leq q$, then $\leq^+$ is a proper refinement of $\leq$.

An element $p \in P$ is maximal if there is no $q \in P$ with $p \preceq q$. $p$ is the greatest element of $P$ if $q \preceq p$ for all $q \in Q$. **Minimal** and **least** elements are defined similarly.

In this paper, a partial order will almost always be written as $\leq$, and if we say that $P$ is a poset without specifying the partial order, we mean $(P, \leq)$ is a poset. We will use terms such as **chain**, **maximal** and **above** without explicit reference to a particular partial order, and it should be understood that we are referring to $\leq$. If we use such terms with respect to any other partial order, we will be explicit about which order we are referring to. An **$n$-element poset** means a poset in which the underlying set $P$ has exactly $n$ elements.

An **isomorphism** between two posets $P$ and $R$ is a bijection $\phi : P \to R$ such that for $p, q \in P$ we have $p \leq q$ if and only if $\phi(p) \leq \phi(q)$. $P$ and $R$ are **isomorphic** (written $P \cong R$) if there is at least one isomorphism from $P$ to $R$. An **automorphism** of $P$ is an isomorphism from $P$ to $P$.

The **Hasse diagram** of a poset $P$ is a graph drawn on the page with vertices corresponding to the elements of $P$, with $p$ further down the page than $q$ whenever $p \preceq q$, and an edge from $p$ to $q$ whenever $p \prec q$ and there is no $r$ with $p \prec r \prec q$.

Now we can give our main definition. Suppose $P$ is a poset. We say that $P$ is a **bichain** if the following two conditions are satisfied.

1. There is a unique way to write $P$ as the union of two chains.

2. $\leq$ is maximal subject to (1), i.e. if $\leq^+$ is a proper refinement of $\leq$, then there is more than one way to write $P$ as the union of two $\leq^+$-chains.

Note that when we say “union” in (1), we simply mean set union; so it is permissible to have an element of one chain lying above an element of the other.

There are four isomorphism classes of 5-element bichains, given by the following Hasse diagrams.
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The first example can be generalised to give a bichain of arbitrary size: just take a poset consisting of an arbitrary chain \( C \), together with an element \( p \) not lying above or below any element of \( C \).

For another class of bichains of arbitrary finite size, take \( n \) to be a positive integer, set \( Q_n = \{1, \ldots, n\} \), and define a partial order \( \preceq \) on \( Q_n \) by putting \( i \preceq j \) if and only if \( i \leq j - 2 \) in the usual order on \( \{1, \ldots, n\} \).

**Proposition 2.1.** If \( n \in \mathbb{N} \), then \( Q_n \) is a bichain.

**Proof.** \( Q_n \) can be expressed as the union of two chains, namely the set of even integers in \( Q_n \) and the set of odd integers in \( Q_n \). This is the unique such expression, since if \( Q_n \) is expressed in any other way as the union of two subsets, then one of these subsets contains two consecutive integers and so is not a chain.

Now suppose \( \preceq^+ \) is a proper refinement of \( \preceq \). Then there must be \( i \in \{1, \ldots, n-1\} \) such that either \( i \preceq^+ i+1 \) or \( i+1 \preceq^+ i \). Either way, we can find a new way to express \( Q_n \) as the union of two \( \preceq^+ \)-chains, namely

\[ \{ \ldots, i-4, i-2, i, i+1, i+3, i+5, \ldots \} \]

and

\[ \{ \ldots, i-3, i-1, i+2, i+4, \ldots \}. \]

This applies for any proper refinement \( \preceq^+ \), so \( Q_n \) is a bichain. \( \square \)

We now proceed to give several other characterisations of bichains. For the rest of the paper, we only consider finite posets.

## 3 Maximal elements

In this section we consider maximal elements in finite bichains; this leads to some numerical results and a recursive construction for all finite bichains. Several of the results in this section are due to Jeremy Rickard.

First we introduce some terminology. Say that a maximal element in a poset is **supermaximal** if it lies above all the non-maximal elements.

**Lemma 3.1 (Rickard).** Suppose \( n \geq 3 \), and \( P \) is a finite bichain with \( |P| \geq 3 \). Then \( P \) contains exactly two maximal elements, exactly one of which is supermaximal.

For example, looking at the four 5-element bichains depicted in Section 2, we can see that each has exactly two maximal elements; in each diagram, the left-hand maximal element is supermaximal, but the right-hand one is not.
Proof. By definition $P$ is the union of two chains. No two maximal elements can lie in the same chain, so there can be at most two maximal elements in $P$. If there is only one maximal element $p$, then $p$ is the greatest element of $P$. But then there cannot be a unique way to express $P$ as the union of two chains, since if $B$ and $C$ are chains with $B \cup C = P$ and $p \in B$, then $B \setminus \{p\}$ and $C \cup \{p\}$ are also chains with union $P$.

So $P$ has exactly two maximal elements $p$ and $q$. We show by contradiction that exactly one of $p$ and $q$ is supermaximal.

If both $p$ and $q$ are supermaximal, then $P$ cannot be uniquely expressed as the union of two chains: indeed, if $P = B \cup C$ with $B, C$ chains, then without loss of generality we may assume that $p \in B$ and $q \in C$. But $p$ and $q$ lie above all other elements of $P$, so $B \setminus \{p\} \cup \{q\}$ and $C \setminus \{q\} \cup \{p\}$ are also chains whose union is $P$, so there is more than one way to write $P$ as the union of two chains, a contradiction.

So suppose instead that neither $p$ nor $q$ is supermaximal. This means that there are $r, s \in P \setminus \{p, q\}$ such that $r \not\less q$ and $s \not\less p$. Now let $P = B \cup C$ be the unique decomposition of $P$ as the union of two chains. Then $p$ and $r$ lie in one of these chains, say $B$, and $q$ and $s$ lie in the other. Let $\less^+$ be the refinement of $\less$ obtained by setting $r \less^+ q$ and extending transitively; that is, we define $a \less^+ b$ if and only if $a \less b$ or $(a \less r$ and $b = q)$. Since $\less^+$ is a proper refinement of $\less$, the definition of a bichain means there are at least two ways to write $P$ as the union of two $\less^+$-chains. $P = B \cup C$ is one of these; let $P = D \cup E$ be another. Note that $p$ and $q$ are $\less^+$-incomparable, and so are $p$ and $s$ and also $r$ and $s$. So $p$ and $r$ lie in one of the two chains, say $D$, while $q$ and $s$ lie in the other. But now $D$ and $E$ are also $\less$-chains, contradicting the fact that $P$ can be uniquely written as the union of two $\less$-chains. \qed

Now we give a reduction result which leads to a recursive construction of all finite bichains.

Proposition 3.2. (Rickard). Suppose $P$ is a poset with $|P| \geq 3$. Suppose $P$ has exactly two maximal elements $p$ and $q$, and that $p$ is supermaximal. Then $P$ is a bichain if and only if $P \setminus \{p\}$ is a bichain.

Proof. For this proof we write $P^- = P \setminus \{p\}$. First consider the case where $q$ is also supermaximal. Then neither $P$ nor $P^-$ is a bichain: $P^-$ fails to be a bichain because it has a greatest element $q$, so (as in the proof of Lemma 3.1) is not uniquely expressible as the union of two chains; $P$ fails to be a bichain by Lemma 3.1 because it has two supermaximal elements.

So we can assume that $q$ is not supermaximal, and we consider decompositions of $P$ and $P^-$ as unions of two chains. Suppose we can decompose $P$ as the union of two chains $B, C$, with $p \in B$ and $q \in C$. Then we can decompose $P^-$ as the union of two chains $B \setminus \{p\}$ and $C$. Conversely, if we can decompose $P^-$ as the union of two chains $D, C$ with $q \in C$, then $q \not\less D$ (because $q$ is not supermaximal in $P$, and is therefore incomparable with some element of $P \setminus \{p\}$) so $P$ is the union of the two chains $D \cup \{p\}$ and $C$. Hence $P$ has a unique expression as the union of two chains if and only if $P^-$ does. So we can assume for the rest of the proof that $P$ and $P^-$ each have a unique decomposition as the union of two chains.

Suppose $P^-$ is not a bichain. Then there is a proper refinement $\less^+$ of $\less$ on $P^-$ such that $P^-$ has a unique expression as the union of two $\less^+$-chains. If we extend $\less^+$ to all of $P$ by setting $r \less^+ p$ for all $r \not\less q$ and keeping $p, q$ incomparable, then $\less^+$ is a proper refinement of $\less$ on $P$. Moreover, $P$ is uniquely expressible as the union of two $\less^+$-chains (since if there were more than one such expression, there would be more than one such expression for $P^-$). So $P$ is not a bichain.

Conversely, suppose $P$ is not a bichain. Then there is a proper refinement $\less^+$ of $\less$ such that $P$ is uniquely expressible as the union of two $\less^+$-chains. This means in particular that $P$ must
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have two \( \preceq^+ \)-maximal elements (namely \( p \) and \( q \)), so the restriction of \( \preceq^+ \) to \( P^- \) is a proper refinement of the restriction of \( \preceq \) to \( P^- \). Now \( P^- \) is uniquely expressible as the union of two \( \preceq^+ \)-chains because \( P \) is, so \( P^- \) is not a bichain.

An immediate consequence is that we can determine all automorphisms of finite bichains.

**Proposition 3.3.** Suppose \( P \) is a finite bichain with \( |P| \geq 3 \). Then \( P \) has no automorphisms except the identity.

**Proof.** We use induction on \( |P| \). The case where \( |P| = 3 \) is trivial to check, so assume \( |P| \geq 4 \).

By Lemma 3.1 \( P \) has a unique supermaximal element \( p \). Clearly \( p \) must be preserved under any automorphism of \( P \). Hence any automorphism of \( P \) restricts to an automorphism of \( P \setminus \{p\} \).

By Proposition 3.2 \( P \setminus \{p\} \) is a bichain, so by induction \( P \setminus \{p\} \) has no non-trivial automorphisms. Hence neither does \( P \).

Rickard has pointed out that Proposition 3.2 gives a way to construct all finite bichains recursively: starting from the unique 2-element bichain, we repeatedly add new supermaximal elements; at each step, we simply have to choose which of the two existing maximal elements should remain maximal. For example, we may construct a 6-element bichain by adding elements to form a sequence of bichains as follows.

![Diagram](https://via.placeholder.com/150)

We can make this precise in the following proposition, in which we determine the number of \( n \)-element bichains up to isomorphism.

**Proposition 3.4.** Suppose \( n \geq 3 \). Then there are exactly \( 2^{n-3} \) bichains of size \( n \) up to isomorphism.

**Proof.** We use induction on \( n \), with the case \( n = 3 \) being easy. Assuming \( n \geq 4 \), Proposition 3.2 shows that we have a function \( P \mapsto P^- \) from the set of \( n \)-element bichains to the set of \((n-1)\)-element bichains, defined by removing the unique supermaximal element from a bichain. By the induction hypothesis it suffices to show that given an \((n-1)\)-element bichain \( Q \), there are exactly two bichains \( P \) up to isomorphism with \( P^- = Q \). To reconstruct \( P \) from \( Q \), we just have to add a new element \( p \) which is supermaximal in \( P \), with \( P \) having exactly one maximal element other than \( p \). Since \( Q \) has two maximal elements, we just choose which of these two elements will not lie below \( p \) in \( P \). So there are two possibilities for \( P \); these are non-isomorphic, since any isomorphism from one to the other would restrict to a non-trivial automorphism of \( Q \); but by Proposition 3.3 \( Q \) has no non-trivial automorphisms.

Next we show that the number of pairs of comparable elements in an \( n \)-element bichain depends only on \( n \); this will be very useful in later sections.

**Proposition 3.5.** Suppose \( P \) is a bichain with \( |P| = n \). Then there are exactly \( \binom{n}{2} \) + 1 pairs \( \{p,q\} \) in \( P \) with \( p \preceq q \), and hence exactly \( n - 1 \) pairs \( \{p,q\} \) with \( p \not\preceq q \not\preceq p \).

**Proof.** Suppose \( n \geq 3 \). By Lemma 3.1 \( P \) has a supermaximal element \( p \) and one other maximal element \( q \), and by Proposition 3.2 \( P \setminus \{p\} \) is a bichain. By induction there are exactly \( \binom{n-1}{2} \) + 1 comparable pairs in \( P \setminus \{p\} \); adding the supermaximal element \( p \) adds \( n - 1 \) comparable pairs (i.e. \( r \preceq p \) for every \( r \neq q \)), giving \( \binom{n}{2} \) + 1 comparable pairs in \( P \).
We can also count ideals in bichains.

**Proposition 3.6.** Suppose \( P \) is a bichain with \( n \) elements, and \( 1 \leq m < n \). Then \( P \) has exactly two \( m \)-element ideals, and if \( m \geq 2 \) then exactly one of these two ideals is a bichain.

**Proof.** We use induction on \( n \), with the case \( n = 2 \) being trivial. Assuming \( n \geq 3 \), let \( p \) be the unique supermaximal element of \( P \). Then by Proposition 3.2 \( P^- = P \setminus \{ p \} \) is a bichain. Since \( p \) is supermaximal and there is only one other maximal element \( q \), there are \( n - 2 \) elements \( r \in P \) with \( r \prec p \). Hence \( p \) cannot be contained in any ideal with fewer than \( n - 1 \) elements. So if \( m \leq n - 2 \), then any \( m \)-element ideal of \( P \) is contained in \( P^- \), and the result follows by the inductive hypothesis. So it remains to consider the case \( m = n - 1 \). There are clearly two ideals with \( n - 1 \) elements, namely \( P^- \) and \( P \setminus \{ q \} \). \( P^- \) is a bichain but \( P \setminus \{ q \} \) is not, because it contains only one maximal element, namely \( p \). \( \square \)

**Example.** Consider the bichain \( Q_n \) introduced in Section 2. For \( m \in \{ 1, \ldots, n - 1 \} \), the two \( m \)-element ideals of \( Q_n \) are \( \{ 1, \ldots, m \} \) and \( \{ 1, \ldots, m - 1 \} \cup \{ m + 1 \} \). Of these, only the first is a bichain.

In fact, there is a converse to Proposition 3.6, which provides another characterisation of finite bichains.

**Proposition 3.7.** Suppose \( P \) is an \( n \)-element poset, and that \( P \) has exactly two \( m \)-element ideals, for each \( m \in \{ 1, \ldots, n - 1 \} \). Then \( P \) is a bichain.

**Proof.** We use induction on \( n \), and for the inductive step we assume \( n \geq 3 \). Since \( P \) has exactly two \( (n - 1) \)-element ideals, it has exactly two maximal elements \( p, q \). We claim that at least one of them must be supermaximal. If not, then there are elements \( r, s \in P \setminus \{ p, q \} \) with \( r \not\prec q \) and \( s \not\prec p \). If we take \( r, s \) to be maximal with these properties, then \( P \) has three \( (n - 2) \)-element ideals

\[
P \setminus \{ p, q \}, \quad P \setminus \{ p, r \}, \quad P \setminus \{ q, s \},
\]

a contradiction. So at least one of \( p, q \), say \( p \), is supermaximal. As observed in the proof of Proposition 3.6, this means that every ideal with fewer than \( n - 1 \) elements is contained in \( P^- = P \setminus \{ p \} \). So \( P^- \) has exactly two \( m \)-element ideals, for each \( m \in \{ 1, \ldots, n - 2 \} \), so by induction \( P^- \) is a bichain. Hence by Proposition 3.2 \( P \) is a bichain too. \( \square \)

## 4 Two concrete constructions of finite bichains

Here we give a concrete construction of finite posets, which will turn out to be bichains and in fact to give all finite bichains up to isomorphism. This construction comes originally from the author’s paper [F].

First we set out some notation for binary sequences. A binary sequence of length \( n \) means a word \( a = a_1 \ldots a_n \) in the alphabet \( \{ 0, 1 \} \). Given such a sequence \( a \), we define \( \bar{a} = \bar{a}_1 \ldots \bar{a}_n \), where \( \bar{0} = 1 \) and \( \bar{1} = 0 \). We say that two binary sequences \( b_1 \ldots b_r \) and \( c_1 \ldots c_r \) are opposed if for some \( i, j \) we have \( b_i > c_i \) and \( b_j < c_j \).
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Now we can give our constructions. We fix a binary sequence \( a = a_1 \ldots a_n \) of length \( n \geq 0 \), and define \( n+2 \) sequences \( a(0), \ldots, a(n+1) \) of length \( n+1 \) as follows:

\[
\begin{align*}
a(n+1) &= a_1 a_2 a_3 \ldots a_{n-1} a_n 0 \\
a(n) &= a_1 a_2 a_3 \ldots a_{n-1} 0 1 \\
a(n-1) &= a_1 a_2 a_3 \ldots 0 1 a_n \\
\vdots \\
a(2) &= a_1 0 1 \ldots a_{n-2} a_{n-1} a_n \\
a(1) &= 0 1 a_2 \ldots a_{n-2} a_{n-1} a_n \\
a(0) &= 1 a_2 \ldots a_{n-2} a_{n-1} a_n.
\end{align*}
\]

Lemma 4.1 [F, Proposition 6.1]. The sequences \( a(0), \ldots, a(n+1) \) are distinct.

**Proof.** If the sequences \( a(i) \) and \( a(j) \) are equal, for \( i < j \), then we get

\[
1 = a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0,
\]

a contradiction. \( \square \)

In fact, there is another description of the set \( \{ a(0), \ldots, a(n+1) \} \).

**Lemma 4.2.** \( a(0), \ldots, a(n+1) \) are precisely the sequences that can be obtained by inserting 0 or 1 at some point in \( a \).

**Proof.** First we observe that each \( a(i) \) is obtained by inserting a 0 or a 1 in \( a \). For \( i = 0, n+1 \) this is trivial, so suppose \( 1 \leq i \leq n \). If \( a_i = 0 \), then \( a(i) \) is obtained from \( a \) by inserting a 1 immediately after \( a_i \), while if \( a_i = 1 \), then \( a(i) \) is obtained from \( a \) by inserting a 0 immediately before \( a_i \).

Conversely, suppose \( b \) is the sequence obtained by inserting \( x \in \{0,1\} \) immediately after position \( i \). We assume \( x = 0 \), as the case \( x = 1 \) is similar. Let \( j > i \) be maximal such that \( a_{i+1} = \cdots = a_{j-1} = 0 \). Then \( b = a(j) \); indeed, both sequences equal

\[
a_1 \ldots a_i 00 \ldots 0 a_j 1 \ldots a_n
\]

where there are \( j-i \) 0s, and the 1 should be omitted if \( j = n+1 \). \( \square \)

The description of the set \( \{ a(0), \ldots, a(n+1) \} \) given by Lemma 4.2 is arguably simpler than the original definition. However, the ordering \( a(0), \ldots, a(n+1) \) is significant for defining our partial order, which we do next.

Set \( \mathcal{P}_a = \{ a(0), \ldots, a(n+1) \} \) and define a binary relation \( \preceq \) on \( \mathcal{P}_a \) by setting \( a(i) \prec a(j) \) whenever \( i < j \) and \( a(i) \) and \( a(j) \) are opposed.

**Example.** Take \( a = 1101 \). Then we have

\[
\begin{align*}
a(5) &= 11010 \\
a(4) &= 11001 \\
a(3) &= 11011 \\
a(2) &= 10101 \\
a(1) &= 01101 \\
a(0) &= 11101
\end{align*}
\]
and the relation \( \preceq \) is a partial order, with the following Hasse diagram.

\[
\begin{array}{c}
11011 \\
| \\
11010 \\
| \\
11001 \\
| \\
11011 \\
| \\
11101 \\
| \\
10101 \\
| \\
01100 \\
\end{array}
\]

It is not obvious from the definition that \( \preceq \) is a partial order in general – in particular, that it is transitive. In fact this was shown in [F], where \( \mathcal{P}_a \) is shown to be the “structure poset” of a multiplicity-free module for a 0-Hecke algebra. We will show it in a different (and purely combinatorial) way by giving a different definition of \( \mathcal{P}_a \).

Keeping the binary sequence \( a \) fixed, define a second list of binary sequences \( a[0], \ldots, a[n+1] \) by

\[
\begin{align*}
a[n+1] &= a_1 a_2 a_3 \ldots a_{n-1} a_n 1 \\
a[n] &= a_1 a_2 a_3 \ldots a_{n-1} 1 0 \\
a[n-1] &= a_1 a_2 a_3 \ldots 1 0 a_n \\
\vdots \\
a[2] &= a_1 1 0 \ldots a_{n-2} a_{n-1} a_n \\
a[1] &= 1 0 a_2 \ldots a_{n-2} a_{n-1} a_n \\
a[0] &= 0 a_1 a_2 \ldots a_{n-2} a_{n-1} a_n.
\end{align*}
\]

**Lemma 4.3.** \( \mathcal{P}_a = \{a[0], \ldots, a[n+1]\} \).

**Proof.** Lemma 4.2 says that \( \mathcal{P}_a = \{a(0), \ldots, a(n+1)\} \) is the set of all sequences that can be obtained by inserting a symbol in \( a \). But the latter description is symmetric in 0 and 1, and therefore the same applies to the set \( \{a[0], \ldots, a[n+1]\} \). \( \square \)

As a consequence of Lemma 4.3, we have a permutation \( w_a \) of \( \{0, \ldots, n+1\} \) defined by \( a(i) = a[w_a(i)] \). This allows us to give a second description of the relation \( \preceq \). First we give a simple description the permutation \( w_a \); we leave the proof as an exercise.

**Lemma 4.4.** Suppose \( i \in \{0, \ldots, n+1\} \).

- If \( i = 0 \) or \( a_i = 0 \), then \( w_a(i) \) is the smallest \( k > i \) such that \( k = n+1 \) or \( a_k = 0 \).
- If \( i = n+1 \) or \( a_i = 1 \), then \( w_a(i) \) is the largest \( k < i \) such that \( k = 0 \) or \( a_k = 1 \).

**Proposition 4.5.** Given \( i, j \in \{0, \ldots, n+2\} \) we have \( a(i) \preceq a(j) \) if and only if \( i \preceq j \) and \( w_a(i) \preceq w_a(j) \). Hence \( \preceq \) is a partial order on \( \mathcal{P}_a \).

**Proof.** We need to show that for \( 0 \leq i < j \leq n+1 \) the sequences \( a(i) \) and \( a(j) \) are opposed if and only if \( w_a(i) < w_a(j) \). We consider several cases; in the remainder of this proof we read \( a_0 \) as 0 and \( a_{n+1} \) as 1.

1. Suppose \( a_i = a_j \). Then \( a(i) \) and \( a(j) \) are certainly opposed because they are different but have the same sum. Moreover, \( w_a(i) < w_a(j) \), by Lemma 4.4.
2. Suppose \( a_i = 1 \) and \( a_j = 0 \). Then \( a(i) \) and \( a(j) \) are opposed, since
\[
a(i)_k = a(j)_k = 0, \quad a(i)_l = a(j)_l = 1,
\]
where \( k > i \) is minimal such that \( a_k = 0 \). Furthermore, \( w_a(i) < i < j < w_a(j) \) by Lemma 4.4.

3. Finally suppose \( a_i = 0 \) and \( a_j = 1 \).
   - If \( w_a(j) < i \), then certainly \( w_a(i) > w_a(j) \). And in this case \( a_{i+1} = \cdots = a_{j-1} = 0 \) by Lemma 4.4, so that \( a(i) \) and \( a(j) \) differ only in the \((i+1)\)th position and are not opposed.
   - Similarly if \( w_a(i) > j \), then \( w_a(i) > w_a(j) \) and \( a(i), a(j) \) are not opposed.
   - If \( w_a(i) = w_a(j) + 1 \), then the sequence \( a_i a_{i+1} \ldots a_{j} \) has the form 011 \ldots 1100 \ldots 001. So \( a(i) \) and \( a(j) \) are not opposed: they differ only in the \( w_a(i) \)th position.
   - The remaining possibility is that \( i < w_a(i) < w_a(j) < j \). In this case let \( k = w_a(i) \), and let \( l > k \) be minimal such that \( a_l = 1 \). Then by assumption \( l < j \), so
\[
a(i)_k = a(j)_l = 1, \quad a(i)_l = a(j)_k = 0,
\]
and \( a(i) \) and \( a(j) \) are opposed. \( \square \)

Proposition 4.5 shows that \( \leq \) is the intersection of the two total orders
\[
a(0) < \cdots < a(n+1) \quad \text{and} \quad a[0] < \cdots < a[n+1],
\]
so the poset \( P_a \) has dimension 2 (in general, the dimension of a poset \( P \) is the smallest \( d \) such that the partial order on \( P \) can be expressed as the intersection of \( d \) total orders). In fact, this follows from a more general result of Pretzel [P, Theorem 1], which says that the dimension of a poset is no more than its width, i.e. the size of its largest antichain; clearly in a bichain the largest antichain has size 2.

In Section 7 we will examine the permutations \( w_a \) in more detail.

**Corollary 4.6.** There is an isomorphism
\[
\phi : P_a \rightarrow P_{\bar{a}}
\]
\[
a(i) \mapsto \bar{a}(i).
\]

**Proof.** Notice that \( \bar{a}(i) = \bar{a}[i] \) and \( \bar{a}[i] = \bar{a}(i) \) for each \( i \), so \( \phi \) is certainly a bijection from \( P_a \) to \( P_{\bar{a}} \), and Proposition 4.5 guarantees that \( \phi(a(i)) \leq \phi(a(j)) \) if and only if \( a(i) \leq a(j) \). \( \square \)

We note in passing that this corollary yields a third way to describe the partial order on \( P_a \): we have \( a[i] \preceq a[j] \) if and only if \( i < j \) and \( a[i] \) and \( a[j] \) are opposed.

Now we prove the main result of this section.

**Theorem 4.7.** Suppose \( n \geq 0 \).

1. If \( a \) is a binary sequence of length \( n \), then \( P_a \) is a bichain.
2. If \( P \) is an \((n+2)\)-element bichain, then \( P \cong P_a \) for some binary sequence \( a \).
3. If \( a, b \) are binary sequences with \( P_a \cong P_b \), then \( b = a \) or \( b = \bar{a} \).
Proof.
1. We use induction on \( n \), and Proposition 3.2. The case \( n = 0 \) is trivial, so take \( n \geq 1 \). Using Corollary 4.6 we can assume that \( a_n = 1 \). First we claim that \( P_a \) has exactly two maximal elements \( a(n+1) \) and \( a[n+1] \), with \( a(n+1) \) being supermaximal. Proposition 4.5 certainly shows that \( a(n+1) \) and \( a[n+1] \) are maximal. Furthermore, the assumption \( a_n = 1 \) means that \( a(n+1) = a[n] \). So (again using Proposition 4.5) \( a(n+1) \) lies above every element of \( P_a \) except \( a[n+1] \). So \( a(n+1) \) is supermaximal, and there are no maximal elements except \( a(n+1) \) and \( a[n+1] \).

Now define \( a^- \) to be the binary sequence \( a_1 \ldots a_{n-1} \). Let \( P_a^- = \{a(0), \ldots, a(n)\} \), with the partial order \( \preceq \) induced from \( P_a \). Then we observe that \( P_a^- \) is isomorphic to \( P_a^- : \) each of the sequences \( a(0), \ldots, a(n) \) ends in a 1, and deleting this 1 from each sequence yields the sequences \( a^-(0), \ldots, a^-(n) \), preserving opposedness. So by induction \( P_a^- \) is a bichain, and hence by Proposition 3.2 so is \( P_a \).

2. Again we use induction on \( n \), and the case \( n = 0 \) is trivial. By Lemma 3.1 \( P \) has one supermaximal element \( p \), and one other maximal element \( q \). Let \( P^- = P \setminus \{p\} \). Then by Proposition 3.2 \( P^- \) is a bichain, so by induction there is an isomorphism \( P^+ \cong P_a^- \) for some binary sequence \( a^- \) of length \( n - 1 \). Under this isomorphism, the two maximal elements of \( P^- \) (one of which is \( q \)) map to \( a^-(n) \) and \( a^-[n] \). We suppose \( q \) maps to \( a^-[n] \) (the other case is similar). Define a sequence \( a \) by adding a 1 at the end of \( a^- \); then we claim that \( P \cong P_a \). As explained in the proof of (1), \( a(n+1) \) is a supermaximal element of \( P_a \), and \( P \setminus \{a(n+1)\} \cong P_a^- \), and so there is an isomorphism \( \theta : P \setminus \{a(n+1)\} \rightarrow P^- \), with \( \theta(a[n+1]) = q \). Since \( a(n+1) \) is incomparable with \( a[n+1] \) in \( P_a \) and above every other element, and \( p \) is incomparable with \( q \) in \( P \) but above every other element, we can extend \( \theta \) by setting \( \theta(a(n+1)) = p \) to give an isomorphism from \( P_a \) to \( P \).

3. Again, we use induction on \( n \), and assume here that \( n \geq 2 \). Let \( \phi : P_a \rightarrow P_b \) is the unique isomorphism from \( P_a \) to \( P_b \). Then \( \phi \) must map the supermaximal element of \( P_a \) to the supermaximal element of \( P_b \), and the other maximal element of \( P_a \) to the other maximal element of \( P_b \). Arguing as in the proof of (1) above, we see that the supermaximal element of \( P_a \) is \( a_1 \ldots a_n a_n \), and the other maximal element is \( a_1 \ldots a_n a_n \). A similar statement holds for \( P_b \), so \( \phi \) maps \( a_1 \ldots a_n a_n \) to \( b_1 \ldots b_n b_n \).

Let \( P_a^- \) be the bichain obtained by deleting the supermaximal element of \( P_a \), and define \( P_b^- \) similarly. Then (as in the proof of (1), interchanging 0 and 1 if necessary) \( P_a^- \cong P_a^- \), with the unique isomorphism \( \theta : P_a^- \rightarrow P_a^- \) defined by deleting the last digit \( a_n \) from the end of each element of \( P_a^- \). Similarly, there is a unique isomorphism \( \kappa : P_b^- \rightarrow P_b^- \), given by deleting the digit \( b_n \) at the end of each element of \( P_b^- \).

Now we have an isomorphism \( \kappa \circ \phi \circ \theta^{-1} : P_a^- \rightarrow P_b^- \) which maps \( a \) to \( b \). By induction we see that \( b^- \) equals either \( a^- \) or \( a^- \). But the unique isomorphism from \( P_a^- \) to \( P_a^- \) maps \( a \) to \( a \), while the unique isomorphism from \( P_a^- \) to \( P_a^- \) maps \( a \) to \( \bar{a} \) by Corollary 4.6. So \( b^- \) must equal either \( a \) or \( \bar{a} \).

Theorem 4.7 allows to label finite bichains in a canonical way: each finite bichain can be written as \( P_a \) for some sequence \( a \) which is unique up to replacing \( a \) with \( \bar{a} \). Note that given Proposition 3.4 and Corollary 4.6, any two parts of Theorem 4.7 imply the other part. But proving all three parts directly yields a new proof of Proposition 3.4.
An interesting family of posets

Now we consider duality. Let us write $P^\circ$ for the poset dual to $P$. It is obvious from the definition in Section 2 that the dual of a bichain is again a bichain. Using the labelling for bichains provided by Theorem 4.7, we can be more specific.

**Proposition 4.8.** Suppose $a$ is a binary sequence of length $n$, and let $\text{rev} \ a$ be the binary sequence $a_n \ldots a_1$. Then $P^\circ_a \cong P_{\text{rev} \ a}$.

**Proof.** This follows from Proposition 4.5, which says that $P_a$ is the intersection of the total orders

$$a(0) < \cdots < a(n+1) \quad \text{and} \quad a[0] < \cdots < a[n+1]$$

on $P_a$. Hence the partial order on $P^\circ_a$ is the intersection of the total orders

$$a(n+1) < \cdots < a(0) \quad \text{and} \quad a[n+1] < \cdots < a[0].$$

But now observe that $(\text{rev} \ a)(i) = \text{rev}(a[n+1-i])$ and $(\text{rev} \ a)[i] = \text{rev}(a(n+1-i))$ for each $i$. So we have an isomorphism from $P^\circ_a$ to $P_{\text{rev} \ a}$ given by mapping $a(i) \leftrightarrow \text{rev}(a(i))$ for each $i$. \qed

## 5 Splicing bichains

In this section we give a simple way of joining two finite bichains together to create a larger one. This yields a further construction of all finite bichains, starting from a family of “indecomposable” bichains.

We start with a more general definition. Suppose $P$ and $Q$ are posets, and that $P$ has exactly two maximal elements $p_0, p_1$, with only $p_0$ being supermaximal, and suppose $Q$ has exactly two minimal elements $q_0, q_1$ with only $q_0$ being superminimal (i.e. lying below every non-minimal element). We define a new poset $P \sqcup Q$ called the *splice* of $P$ and $Q$. Informally, this is defined by placing $Q$ above $P$, and identifying $p_0$ with $q_0$ and $p_1$ with $q_1$. Formally, we start by defining a partial order on the disjoint union $P \sqcup Q$ via the following rules:

- if $a, b \in P$, then $a \preceq b$ in $P \sqcup Q$ if and only if $a \preceq b$ in $P$;
- if $a, b \in Q$, then $a \preceq b$ in $P \sqcup Q$ if and only if $a \preceq b$ in $Q$;
- if $a \in P$ and $b \in Q$, then $a \preceq b$ if and only if for some $i \in \{0, 1\}$ we have $a \preceq p_i$ in $P$ and $q_i \preceq b$ in $Q$;
- if $a \in Q$ and $b \in P$, then $a \not\preceq b$.

Now define $P \sqcap Q$ to be the quotient poset obtained by identifying $p_0$ with $q_0$ and $p_1$ with $q_1$. Note that $P$ and $Q$ are then naturally subposets of $P \sqcup Q$.

Now we consider the special case of bichains. Recall that a finite bichain of size at least 3 has exactly two maximal elements, with exactly one being supermaximal. Dually, a finite bichain of size at least 3 has exactly two minimal elements, with exactly one being superminimal. Hence the splice of two finite bichains each with at least three elements is well-defined. In fact we can easily extend this definition to include the case where either or both of the posets is the unique 2-element bichain $P_{\varnothing}$, even though both elements of $P_{\varnothing}$ are both supermaximal and superminimal. We then get $P \sqcup P_{\varnothing} = P$ and $P_{\varnothing} \sqcup Q = Q$.

**Proposition 5.1.** If $P$ and $Q$ are finite bichains, then so is $P \sqcup Q$. 

**Proof.** Since $P$ and $Q$ can each be expressed as the union of two chains, so can $P \boxtimes Q$: we take the union of the chain in $P$ containing $p_0$ and the chain in $Q$ containing $q_0$ to give one chain, and do the same for $p_1, q_1$ to give the other. Moreover, this is the unique way to express $P \boxtimes Q$ as the union of two chains, since any other such expression would restrict to give a new expression for either $P$ or $Q$ as the union of two chains.

Now suppose $\leq^+$ is a proper refinement of $\leq$ on $P \boxtimes Q$. Note that for every $p \in (P \boxtimes Q) \setminus Q$ and $q \in (P \boxtimes Q) \setminus P$ we already have $p \leq q$, since $p \leq p_0$ in $P$ and $q_0 \leq q$ in $Q$. So in order for $\leq^+$ to be a proper refinement of $\leq$ on $P \boxtimes Q$, we must have $a \leq^+ b$ but $a \not\leq b$ either for some $a, b \in P$ or for some $a, b \in Q$. We assume the former case; then the restriction of $\leq^+$ to $P$ is a proper refinement of $\leq$ on $P$. Since $P$ is a bichain, this means that there are at least two different ways to express $P$ as the union of two $\leq^+$-chains. But now (via the construction in the first paragraph of the proof) there are at least two ways to express $P \boxtimes Q$ as the union of two $\leq^+$-chains. \( \square \)

**Example.** Take $P = Q = \mathcal{P}_{001}$. Then the Hasse diagram of $P$ with the elements $p_0, p_1, q_0, q_1$ marked is as follows.

Hence the splice $P \boxtimes Q$ is given by the following diagram.

We can make Proposition 5.1 more explicit using the labelling for finite bichains introduced in Section 4. Given binary sequences $a, b$, write $a|b$ for their concatenation.

**Proposition 5.2.** Suppose $a = a_1 \ldots a_r$ and $b = b_1 \ldots b_s$ are binary sequences, and define

$$c = \begin{cases} a|b & \text{if } a_r = b_1 \\ a|\overline{b} & \text{if } a_r \neq b_1. \end{cases}$$

Then $\mathcal{P}_a \boxtimes \mathcal{P}_b \cong \mathcal{P}_c$.

(Note that we include the case where $r$ or $s$ equals 0; in this case the condition $a_r = b_1$ doesn’t make sense, but the conclusion is trivially true regardless.)

**Proof.** By replacing $b$ with $\overline{b}$ if necessary and using the fact that $\mathcal{P}_b \cong \mathcal{P}_{\overline{b}}$, we can assume that $a_r = b_1$. In fact, by Corollary 4.6 we can assume that $a_r = 0 = b_1$. Then the supermaximal
element of \( P_a \) is \( a|1 = a[r+1] \), with the other maximal element being \( a|0 = a(r+1) \). Similarly, the superminimal element of \( P_b \) is \( 1|b = b(0) \), and the other minimal element is \( 0|b = b[0] \).

We define an injective function

\[
\phi_b : P_a \rightarrow P_a|b \\
\phi_b[a[i]] \rightarrow (a|b)[i].
\]

This function can be more simply described as mapping \( p \mapsto p|b \) for each \( p \in P_a \), which shows that it is order-preserving, using the observation at the end of Section 4: \( a[i] \) and \( a[j] \) are opposed if and only if \( a[i]|b \) and \( a[j]|b \) are opposed.

We also define an injective function

\[
\phi_a : P_b \rightarrow P_a|b \\
\phi_a[b(i)] \rightarrow (a|b)(r+i).
\]

This function can be more simply described as mapping \( q \mapsto a|q \), which shows that it too is order-preserving.

The images of \( \phi_b \) and \( \phi_a \) intersect in the two points

\[
\phi_b(a|0) = a|0|b = \phi_a(0|b)
\]

and

\[
\phi_b(a|1) = a|1|b = \phi_a(1|b).
\]

Since \( a|0 \) is supermaximal in \( P_a \) and \( 0|b \) is superminimal in \( P_b \), we get

\[
p \preceq a|0|b \preceq q
\]

for all \( p \in \text{im } \phi_b \setminus \{a|1|b\} \) and \( q \in \text{im } \phi_a \setminus \{a|1|b\} \). Putting this together with the partial orders on \( \text{im } \phi_b \) and \( \text{im } \phi_a \), we see that \( P_a|b \cong P_a \sqcup P_b \). \( \square \)

As a consequence of Proposition 5.2, we can see that whenever a binary sequence \( a \) contains two consecutive equal entries, \( P_a \) can be expressed as a splice of two smaller bichains. This means that we can construct all finite bichains from the bichains of the form \( P_{1010\ldots} \). In fact, we have already seen these bichains constructed in a very simple way: they are the posets \( Q_n \) defined in Section 2, as we now show.

**Proposition 5.3.** Suppose \( a \) is the alternating binary sequence 1010\ldots of length \( n \). Then \( P_a \cong Q_{n+2} \).

**Proof.** Suppose \( 0 \leq i < j \leq n+1 \). We can check that \( a(i) \) and \( a(j) \) are opposed unless (and only unless) one of the following happens:

- \( i \) is even and \( j = i + 1 \);
- \( i \) is even and \( j = i + 3 \);
- \( n \) is odd, \( i = n - 1 \) and \( j = n + 1 \).
Hence we can define an isomorphism from $P_a$ to $Q_{n+2}$ by mapping

$$ a(i) \mapsto \begin{cases} 
  i & \text{if } i \text{ is odd} \\
  i + 2 & \text{if } i \text{ is even and } i \neq n + 1 \\
  i + 1 & \text{if } i \text{ is even and } i = n + 1.
\end{cases} \qed
$$

As a consequence, we deduce the following.

**Proposition 5.4.** Suppose $P$ is a bichain with $n$ elements, where $n \geq 3$. Then $P$ is isomorphic to the bichain $Q_{n_1} \boxtimes \cdots \boxtimes Q_{n_s}$ for a unique choice of $n_1, \ldots, n_s \geq 3$ with $n_1 + \cdots + n_s = n + 2s - 2$.

**Proof.** The existence of such a decomposition comes from Propositions 5.2 and 5.3: writing $P \simeq P_a$ for a binary sequence $a$, we break $a$ into subsequences $a^{(i)}, \ldots, a^{(s)}$ where each $a^{(i)}$ is an alternating sequence $\ldots 0101 \ldots$, and the last term of $a^{(i)}$ equals the first term of $a^{(i+1)}$ for each $i$. Then $n_1, \ldots, n_s$ are just the lengths of the sequences $a^{(1)}, \ldots, a^{(s)}$.

For the uniqueness, it suffices to use Proposition 3.4 and to count the number of possible expressions $n_1 + \cdots + n_s = n + 2s - 2$, which is a simple exercise. \qed

**Example.** The finite bichain $P_{011011}$ is isomorphic to $Q_4 \boxtimes Q_5 \boxtimes Q_3$, as we see from the following Hasse diagrams.

![Hasse diagrams](image)

### 6 Graphs

In this section we consider the analogue of bichains for graphs. Let’s say that a graph $G = (V, E)$ is a *biclique* if $V$ can be uniquely expressed as the union of two cliques (i.e. complete subgraphs) and $G$ is maximal with this property: adding any edge to $E$ breaks the uniqueness.

In fact, bicliques are easy to understand: a graph is a biclique if and only if its complement is bipartite with a unique bipartition, and is minimal with this property. But a minimal bipartite graph is a tree, so a biclique is simply the complement of a tree.
However, bicliques do have a direct relationship with bichains. The *comparability graph* of a poset $P$ has vertex set $P$, with an edge from $p$ to $q$ if and only if either $p \prec q$ or $q \prec p$. Our main result here is the following, which gives yet another characterisation of bichains.

**Proposition 6.1.** Suppose $P$ is a finite poset. Then $P$ is a bichain if and only if its comparability graph is a biclique.

**Proof.** Let $G$ be the incomparability graph of $P$, in which there is an edge from $p$ to $q$ if and only if $p \not\prec q$. From the above discussion, we need to show that $P$ is a bichain if and only if $G$ is a tree. Let $n = |P|$.

Suppose first that $P$ is a bichain; then by Proposition 3.5, the number of edges in the incomparability graph is $n - 1$. So it suffices to show that the incomparability graph is connected. Suppose for a contradiction that it is not; that is, $P$ can be partitioned into two non-empty sets $Q, R$ such that every $q \in Q$ and $r \in R$ are comparable under $\preceq$. Since $P$ is a bichain, we can also partition $P$ as the union of two chains $S$ and $T$. But now notice that $P$ is also partitioned by the two sets

$$(S \cap Q) \cup (T \cap R) \quad \text{and} \quad (S \cap R) \cup (T \cap Q)$$

and that these sets are chains. This contradicts the uniqueness in the definition of a bichain.

For the converse, suppose $P$ is not a bichain. We consider three possibilities.

- Suppose $P$ cannot be written as the union of two chains. Then by Dilworth’s Theorem [D, Theorem 1.1] $P$ contains an antichain of size 3; but then $G$ contains a triangle, so is not a tree.

- Suppose $P$ can be written in more than one way as the union of two chains; we may assume the two chains are disjoint in each case, and write $P = S \sqcup T = U \sqcup V$ with $S, T, U, V$ all chains. But then every element of $(S \cap U) \cup (T \cap V)$ is comparable under $\preceq$ with every element of $(S \cap V) \cup (T \cap U)$, so that $G$ is disconnected.

- Finally suppose $P$ can be uniquely written as the union of two chains, but is not maximal with this property. Then it is possible to refine $\preceq$ to a partial order $\preceq^+$ such that $(P, \preceq^+)$ is a bichain. But then the incomparability graph of $(P, \preceq^+)$ has exactly $n - 1$ edges by Proposition 3.5, so $G$ has more than $n - 1$ edges and cannot be a tree. $\square$

One can ask which trees can occur as the incomparability graphs of posets (and hence of bichains). The general question of which finite graphs are incomparability graphs of posets was answered completely by Gallai [G]. His theorem gives an explicit list of graphs, and says that a finite graph is the incomparability graph of a poset if and only if it has no induced subgraph in the given list. We are interested in the special case of trees; for this, we just check that the only forest in Gallai’s list is the triskelion

![Triskelion Graph](image)

which leads to the following result.
Theorem 6.2 (Gallai). Suppose $G$ is a finite tree. Then $G$ is the incomparability graph of a poset if and only if $G$ is a caterpillar, i.e. if the non-leaves in $G$ form a path.

Proposition 6.1 and Theorem 6.2 show that the incomparability graph of a finite bichain is a caterpillar, and that every finite caterpillar arises as the incomparability graph of a bichain. We now show that this bichain is unique up to duality.

Proposition 6.3. Suppose $G$ is a finite caterpillar. Then $G$ is the incomparability graph of a bichain $P$, which is unique up to duality.

Proof. We use a counting argument. Sending a poset to its incomparability graph defines a function from (isomorphism classes of) finite bichains to finite caterpillars, and Proposition 6.1 and Theorem 6.2 show that this function is surjective. Moreover, dual bichains obviously map to the same caterpillar. So it suffices to show that the number of dual pairs of $n$-element bichains equals the number of isomorphism types of caterpillars with $n$ vertices. Proposition 4.8 and the discussion preceding Proposition 3.4 show that the former number is the number of binary sequences of length $n-2$ modulo reversal and modulo interchanging 0s and 1s; an easy exercise shows that this number is $2^{n-4} + 2^{(n-4)/2}$. [HS, Theorem 2.1] shows that this is also the number of isomorphism types of caterpillars on $n$ vertices. \hfill \square

By way of example, we consider two special cases of caterpillars. The first example is the star, where one vertex is attached to all the others. This is the incomparability graph of the bichain consisting of a chain and an isolated point. The second example is the path with $n$ vertices: this is the incomparability graph of the bichain $Q_n$ introduced in Section 2.

7 Coxeter elements in the symmetric group

In this section we give yet another construction of finite bichains, relating to permutations. Recall from Section 4 the second construction of the poset $P_a$: we define a permutation $w_a$ of $\{0, \ldots, n + 1\}$ by $a(i) = a[w_a(i)]$; then we have $a(i) \leq a(j)$ in $P_a$ if $i \leq j$ and $w_a(i) \leq w_a(j)$. Our focus in this section is on the permutations $w_a$: we make an analogous definition for an arbitrary permutation, and determine (in the finite case) exactly which permutations yield bichains.

We recall some basic Coxeter theory of the symmetric group (for more details see the book by Humphreys [H]). Let $S_n$ denote the group of all permutations of $\{1, \ldots, n\}$. We write $s_1, \ldots, s_{n-1}$ for the Coxeter generators of $S_n$; here $s_i$ is the transposition $(i, i+1)$. Any permutation $w \in S_n$ can be written in the form $s_{i_1} \ldots s_{i_l}$ for some $i_1, \ldots, i_l$; the smallest $l$ for which we can do this is called the length $l(w)$. $l(w)$ is also the number of inversions of $w$, i.e. pairs $i < j$ such that $w(i) > w(j)$.

A Coxeter element of $S_n$ is an element which can be written as a product of the Coxeter generators, each appearing exactly once, in some order.

Example. The Coxeter elements in $S_4$ are the 4-cycles

$$(1,2,3,4), \quad (1,3,4,2), \quad (1,2,4,3), \quad (1,4,3,2).$$

Coxeter elements are defined for all finite Coxeter groups, and play an important role in Coxeter theory. For the symmetric group, we will need the following result, which is well-known.
**Proposition 7.1.** Suppose \( n \geq 2 \). Then there are exactly \( 2^{n-2} \) Coxeter elements in \( S_n \), and they all have length \( n - 1 \) and order \( n \). In particular, if \( n \geq 3 \), then no Coxeter element of \( S_n \) is self-inverse.

Now we use permutations to define posets. Given \( w \in S_n \), define \( R_w \) to be the set \( \{1, \ldots, n\} \), with \( i \preceq j \) if and only if \( i \leq j \) and \( w(i) \leq w(j) \). Note that if \( w \in S_n \) then \( R_w \) and \( R_{w^{-1}} \) are isomorphic; in fact \( w \) is itself an isomorphism \( R_w \to R_{w^{-1}} \).

Our main result in this section is the following.

**Theorem 7.2.**

1. Suppose \( w \in S_n \). Then \( R_w \) is a bichain if and only if \( w \) is a Coxeter element of \( S_n \).

2. Every \( n \)-element bichain is isomorphic to \( R_w \) for some Coxeter element \( w \in S_n \).

3. Suppose \( v \) and \( w \) are Coxeter elements of \( S_n \). Then \( R_v \cong R_w \) if and only if \( v = w \) or \( v = w^{-1} \).

**Proof.** We assume \( n \geq 3 \). We begin with the “only if” part of (1). The construction of \( R_w \) means that the number of pairs of incomparable elements in \( R_w \) is \( l(w) \). So by Proposition 3.5 \( R_w \) can only be a bichain if \( l(w) = n - 1 \). If \( l(w) = n - 1 \) but \( w \) is not a Coxeter element, then \( w \) can be written as a product of Coxeter generators with one Coxeter generator, say \( s_m \), not occurring. This means that \( w \) is contained in the Young subgroup \( S_{(m,n-m)} \); this is the subgroup of \( S_n \) consisting of all permutations \( v \in S_n \) such that \( v(\{1,\ldots,m\}) = \{1,\ldots,m\} \). So in \( R_w \) we have \( i \not\preceq j \) whenever \( i \leq m < j \). But now the incomparability graph of \( R_w \) is disconnected, so by Proposition 6.1 \( R_w \) is not a bichain.

So \( R_w \) is a bichain only if \( w \) is a Coxeter element. Proposition 4.5 and Theorem 4.7 show that every finite bichain is isomorphic to \( R_w \) for some permutation \( w \), which must therefore be a Coxeter element, so (2) is proved. Now consider the function \( \psi : w \mapsto R_w \) from the set of Coxeter elements of \( S_n \) to the set of isomorphism classes of \( n \)-element posets. We have just seen that the image of \( \psi \) contains all \( n \)-element bichains, and therefore this image has size at least \( 2^{n-3} \), by Proposition 3.4. On the other hand, since \( w \neq w^{-1} \) but \( \psi(w) = \psi(w^{-1}) \) for each Coxeter element \( w \), the image of \( \psi \) has size at most \( 2^{n-3} \), by Proposition 7.1. So equality holds everywhere; hence the image of \( \psi \) is precisely the set of \( n \)-element bichains (which is the “if” part of (1)), and different inverse pairs of Coxeter elements map to different bichains (which gives (3)). \( \square \)

### 8 Trichains and beyond

In this final section we briefly discuss a natural generalisation of bichains. Given \( r \in \mathbb{N} \), define an \( r \)-chain to be a poset \( P \) such that \( P \) is uniquely expressible as the union of \( r \) chains, and the partial order on \( P \) is maximal with this property.

For a first family of examples, we can generalise the posets \( Q_n \) defined in Section 2. Given \( n \in \mathbb{N} \), define \( Q'_n = \{1, \ldots, n\} \), with \( i \preceq j \) if and only if \( i \leq j - r \) in the usual order on \( \{1, \ldots, n\} \).

To construct all finite \( r \)-chains, it seems to be possible to generalise the recursive construction suggested after Proposition 3.3: starting from the unique \( r \)-element \( r \)-chain, repeatedly add supermaximal elements, at each stage choosing which \( r - 1 \) of the existing \( r \) maximal elements should remain maximal. From this it should follow that the number of isomorphism classes of labelled \( r \)-chains with \( P \) with \( n \geq r \) elements (i.e. with the \( r \) chains whose union is \( P \) labelled \( 1, \ldots, r \), and with isomorphisms required to preserve labelling) is \( r^{n-r} \).
However, the other results in this paper remain to be extended. The author cannot see an obvious way to extend the constructions in Section 4 to the cases \( r \geq 3 \). For the results in Section 5, it should be possible to splice \( r \)-chains to make larger ones, but a direct analogue of Proposition 5.4 fails: there are \( r \)-chains other than \( Q_1^r, Q_2^r, \ldots \) which cannot be written as splices of smaller \( r \)-chains. An example with \( r = 3 \) is as follows.

The results of Section 7 also have no obvious extension beyond \( r = 2 \); indeed, \( r \)-chains need not have dimension 2, so not all \( r \)-chains have the form \( R_w \) for \( w \) a permutation. For example, \( Q_3^3 \) has dimension 3.

We hope to say more about \( r \)-chains in future work.

References


[R] J. Rickard, answer to ‘Has anyone seen these posets before?’, mathoverflow.net/q/278244. [1]