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Multiple-elimination knockout tournaments with the fixed-win property

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Abstract

We classify those triples (n, l, w) for which there exists a 'knockout' tournament for n players in which the winner always wins exactly w games and each loser loses exactly l games.

1 Introduction

In sporting competitions where a winner needs to be chosen in a relatively short time, a knockout tournament is frequently used. If more time is available, a double-elimination knockout tournament, in which a player or team is knocked out if it loses twice, may be employed. More generally, we may define an l-tuple-elimination knockout tournament to be one in which a player or team is eliminated if it loses l times. Specifically, an l-tuple elimination knockout tournament is a schedule in which:

- each game involves two players, one of whom wins and one of whom loses;
- the schedule for later games may depend upon the results of earlier games;
- if a player loses l games, he is eliminated from the tournament, and plays no further games;
- the tournament ends when exactly one player remains.

A great deal of literature is concerned with fairness in single- and (in [2]) double-elimination tournaments, assessing the probabilities of each player's winning the tournament given his probabilities of beating each other player in a single game. Here, we approach the issue of fairness from a different angle, and consider the number of games a player needs to win in order to win the tournament. As far as we can tell, this issue has not been considered before. In the most frequentlly used double-elimination tournaments (such as in [1]), a player losing a game at an early stage will tend to need to win many more games in order to win the tournament than a player winning his early games. We say that an l-tuple elimination tournament for $n \geqslant 2$ players has the *fixed-win* property if, for some w, the winner of the tournament is guaranteed to have won exactly w games. We refer to such a tournament as a $\mathrm{FW}(n, l, w)$.

The aim of this paper is to classify those triples (n, l, w) for which a $\mathrm{FW}(n, l, w)$ exists. Two obvious examples are as follows.

- The standard single-elimination knockout tournament for 2^w players is a $\mathrm{FW}(2^w, 1, w)$.
- Two players playing a 'first to w' match is a $\mathrm{FW}(2, w, w)$.

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In fact, these are almost the only examples. Our main theorem is as follows.

Theorem 1.1. *Suppose that $n \geqslant 2$, and that l, w are positive integers. Then a* $\mathrm{FW}(n, l, w)$ *exists if and only if*

- $l = 1$ *and* $n = 2^w$,
- $n = 2$ *and* $l = w$ *or*
- $(n, l, w) = (16, 2, 6)$.

2 The proof of Theorem 1.1

In order to prove Theorem 1.1, we need to prove a more general result. Given $n > 1$ and positive integers $w_1, \dots, w_n, l_1, \dots, l_n$, we ask whether we can arrange a tournament for n players (numbered $1, \dots, n$) in which $n - 1$ players are eliminated and the remaining player wins the tournament, and in which:

- player i is eliminated if and only if he loses l_i games;
- player i wins the tournament if and only if he wins w_i games.

We call such a tournament a $T(w_1, \dots, w_n, l_1, \dots, l_n)$, and say that the matrix

$$\begin{pmatrix} w_1 & \dots & w_n \\ l_1 & \dots & l_n \end{pmatrix}$$

is *good* if a $T(w_1, \dots, w_n, l_1, \dots, l_n)$ exists. In particular, a $\mathrm{FW}(n, l, w)$ exists if and only if the $2 \times n$ matrix

$$\begin{pmatrix} w & \dots & w \\ l & \dots & l \end{pmatrix}$$

is good. We shall classify good matrices; clearly

$$\begin{pmatrix} w_1 & \dots & w_n \\ l_1 & \dots & l_n \end{pmatrix}$$

is good if and only if

$$\begin{pmatrix} w_{\sigma(1)} & \dots & w_{\sigma(n)} \\ l_{\sigma(1)} & \dots & l_{\sigma(n)} \end{pmatrix}$$

is good for some permutation σ.

Given a matrix

$$\mathbf{M} = \begin{pmatrix} w_1 & \dots & w_n \\ l_1 & \dots & l_n \end{pmatrix}$$

and given $1 \leqslant i < j \leqslant n$, we define the (i, j)-*descendants* of $\mathbf{M}$ to be the matrices

$$\begin{pmatrix} w_1 & \dots & w_i - 1 & \dots & w_j & \dots & w_n \\ l_1 & \dots & l_i & \dots & l_j - 1 & \dots & l_n \end{pmatrix}$$

and

$$\begin{pmatrix} w_1 & \dots & w_i & \dots & w_j - 1 & \dots & w_n \\ l_1 & \dots & l_i - 1 & \dots & l_j & \dots & l_n \end{pmatrix}$$

where we delete any column in which the lower entry is 0.

Proposition 2.1. *The matrix* $\begin{pmatrix} w_1 & w_2 \\ l_1 & l_2 \end{pmatrix}$ *is good if and only if* $w_1 = l_2$ *and* $l_1 = w_2$. *If* $n > 2$, *then the matrix* $\begin{pmatrix} w_1 & \dots & w_n \\ l_1 & \dots & l_n \end{pmatrix}$ *is good if and only if for some* $1 \leqslant i < j \leqslant n$ *both of the* (i, j)*-descendants are good.*

Proof. For $n = 2$, the result is obvious, since each game must be between the only two players. If $n \geqslant 3$, suppose that we wish to construct a $T(w_1, \dots, w_n, l_1, \dots, l_n)$ in which the first game is between players i and j. If i beats j, then the remainder of the tournament may be viewed as a $T(w_1, \dots, w_{i-1}, w_i - 1, w_{i+1}, \dots, w_n, l_1, \dots, l_{j-1}, l_j - 1, l_{j+1}, \dots, l_n)$, while if j beats i, then the remainder of the tournament is a $T(w_1, \dots, w_{j-1}, w_j - 1, w_{j+1}, \dots, w_n, l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots, l_n)$. So a $T(w_1, \dots, w_n, l_1, \dots, l_n)$ exists if and only if both the latter two tournaments exist, for some i and j. □

It will turn out that there are remarkably few good matrices. We begin with the case where each l_i equals 1.

Proposition 2.2. *The matrix*

$$\mathbf{M} = \begin{pmatrix} w_1 & \dots & w_n \\ 1 & \dots & 1 \end{pmatrix}$$

is good if and only if $\sum_{k=1}^{n} 2^{-w_k} = 1$.

To prove this, we need the following simple lemma.

Lemma 2.3. *If* $n > 1$ *and* $a_1, \dots, a_n$ *are integers such that* $2^{a_1} + \dots + 2^{a_n} = 2^a$ *for some integer a, then* $a_i = a_j$ *for some* $i \neq j$.

Proof. If the a_i are all distinct, let a_m be the smallest. Then $2^{a_1} + \dots + 2^{a_n}$ is an odd integer multiple of 2^{a_m}, and so cannot be a power of 2 (since it does not equal 2^{a_m}). □

Proof of Proposition 2.2. The case $n = 2$ follows from Proposition 2.1. For $n > 2$, suppose that $\mathbf{M}$ is good, and let i and j be as in Proposition 2.1. Then by induction we must have

$$2^{-(w_i - 1)} + \sum_{k \neq i,j} 2^{-w_k} = 1 \tag{1}$$

and

$$2^{-(w_j - 1)} + \sum_{k \neq i,j} 2^{-w_k} = 1. \tag{2}$$

In particular, we have $w_i = w_j$, and so

$$\sum_{k=1}^{n} 2^{-w_k} = 1. \tag{3}$$

Conversely, suppose that $n \geqslant 3$ and that (3) holds. By Lemma 2.3 we can find i and j such that $w_i = w_j$; then equations (1–2) above hold, and so by induction both of the (i, j)-descendants of $\mathbf{M}$ are good. □

Now we consider the case where some of the l_i equal 2.

Proposition 2.4. *Suppose $n \geqslant 3$. Then the matrix*

$$\mathbf{M} = \begin{pmatrix} u_1 & \dots & u_{n-1} & v \\ 1 & \dots & 1 & 2 \end{pmatrix}$$

is good if and only if

$$v = 1 \text{ or } 2 \quad \text{and} \quad \sum_{k=1}^{n-1} 2^{-u_i} = \frac{v}{4}.$$

Proof. We begin with the case $n = 3$. The $(1,2)$-descendants of $\mathbf{M}$ are

$$\begin{pmatrix} u_1 - 1 & v \\ 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} u_2 - 1 & v \\ 1 & 2 \end{pmatrix},$$

and these are both good if and only if $v = 1$ and $u_1 = u_2 = 3$. The $(1,3)$-descendants are

$$\begin{pmatrix} u_1 - 1 & u_2 & v \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} u_2 & v - 1 \\ 1 & 2 \end{pmatrix},$$

and these are both good if and only if $u_2 = 2$, $v - 1 = 1$ and $2^{-(u_1-1)} + 2^{-u_2} + 2^{-v} = 1$, i.e. if and only if $(u_1, u_2, v) = (2, 2, 2)$; similarly for the $(2,3)$-descendants.

Now we assume $n > 3$, and suppose that the (i, j)-descendants of $\mathbf{M}$ are both good. There are two cases to consider.

[$i, j < n$] By induction we have

$$2^{-(u_i-1)} + \sum_{k \neq i,j} 2^{-u_k} = \frac{v}{4} = 2^{-(u_j-1)} + \sum_{k \neq i,j} 2^{-u_k},$$

and so $u_i = u_j$. Thus $\sum_{k=1}^{n-1} 2^{-u_k}$ equals $\frac{v}{4}$ as well, and so $\mathbf{M}$ satisfies one of the above criteria.

[$i < j = n$] Now the (i, j)-descendants of $\mathbf{M}$ are

$$\begin{pmatrix} u_1 & \dots & u_{i-1} & u_i - 1 & u_{i+1} & \dots & u_{n-1} & v \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} u_1 & \dots & u_{i-1} & u_{i+1} & \dots & u_{n-1} & v - 1 \\ 1 & \dots & 1 & 1 & \dots & 1 & 2 \end{pmatrix},$$

so by induction we must have $v - 1 = 1$ or 2 and

$$\sum_{k \neq i} 2^{-u_k} = \frac{v-1}{4}, \qquad \sum_{k \neq i} 2^{-u_k} + 2^{-(u_i-1)} + 2^{-v} = 1.$$

This gives

$$2^{-(u_i-1)} = 1 - 2^{-v} - \frac{v-1}{4};$$

if $v - 1 = 2$, then we get $2^{-(u_i-1)} = \frac{3}{8}$, which is impossible, so we have $v - 1 = 1$, which gives $2^{-(u_i-1)} = \frac{1}{2}$, whence $u_i = 2$ and we have $v = 2$, $\sum_{k=1}^{n-1} 2^{-u_k} = \frac{1}{2}$.

Conversely, suppose that $\mathbf{M}$ satisfies the conditions of the proposition. By Lemma 2.3 we must have $u_i = u_j$ for some $1 \leqslant i < j < n$; then the (i, j)-descendants of $\mathbf{M}$ both satisfy this criterion as well. □

In order to deal with the case where two or more of the l_i equal 2, we need to introduce a certain function. Suppose $\{a_1, \dots, a_n\}$ is a multiset of integers such that $\sum_{k=1}^{n} 2^{-a_k} = \frac{1}{4}$. Then we define $h(\{a_1, \dots, a_n\})$ recursively by:

- $h(\{2\}) = \frac{1}{2}$;
- $h(\{a_1 + 1, a_1 + 1, a_2, \dots, a_n\}) = h(\{a_1, a_2, \dots, a_n\}) - 2^{-a_1-1}$.

For example, we have

$$h(\{2\}) = \frac{1}{2}, \qquad h(\{3,3\}) = \frac{3}{8}, \qquad h(\{3,4,4\}) = \frac{5}{16},$$
$$h(\{3,4,5,5\}) = \frac{9}{32}, \qquad h(\{4,4,4,4\}) = \frac{1}{4}.$$

It is an easy exercise to show that h is well-defined; in fact, we can give a closed form for h: we have

$$h(\{a_1, \dots, a_n\}) = \frac{3}{4} - \sum_{i=1}^{n} a_i 2^{-a_i-1}.$$

However, the recursive characterisation of h will be more helpful.

Lemma 2.5. *For $a \geqslant 2$, let S be the multiset with 2^{a-2} elements all equal to a. Then*

$$h(S) = \frac{6-a}{8}.$$

Proof. This follows from the closed form for h, or a simple induction on a. □

Now we can deal with the general case in which each l_i equals 1 or 2.

Proposition 2.6. *Suppose that $n \geqslant 3$ and $0 \leqslant r \leqslant n-2$. Then the matrix*

$$\mathbf{M} = \begin{pmatrix} u_1 & \dots & u_r & v_{r+1} & \dots & v_n \\ 1 & \dots & 1 & 2 & \dots & 2 \end{pmatrix}$$

is good if and only if

$$\sum_{k=r+1}^{n} 2^{-v_k} = \frac{1}{4} \quad \textit{and} \quad \sum_{k=1}^{r} 2^{-u_k} = h(\{v_{r+1}, \dots, v_n\}).$$

Proof. For the 'if' part, we have by Lemma 2.3 that $v_i = v_j$ for some $i \neq j$. Then, by the definition of h and by induction (or by Proposition 2.4 in the case $r = n-2$), the (i, j)-descendants of $\mathbf{M}$ are both good.

For the 'only if' part, we proceed by induction on $n - r$ and r, beginning with the case $n = 3, r = 1$; notice that the conditions of the proposition cannot hold here. The $(1, 2)$-descendants of

$$\begin{pmatrix} u_1 & v_2 & v_3 \\ 1 & 2 & 2 \end{pmatrix}$$

are

$$\begin{pmatrix} u_1 - 1 & v_2 & v_3 \\ 1 & 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} v_2 - 1 & v_3 \\ 2 & 2 \end{pmatrix}.$$

For both of these to be good we should need $v_2 = 3, v_3 = 2$ and $2^{-(u_1-1)} + 2^{-v_2} = \frac{v_3}{4}$, which is impossible. Similarly for the $(1, 3)$-descendants. The $(2, 3)$-descendants are

$$\begin{pmatrix} u_1 & v_2 - 1 & v_3 \\ 1 & 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} u_1 & v_2 & v_3 - 1 \\ 1 & 1 & 2 \end{pmatrix};$$

similarly, these cannot both be good.

Now we examine the case $r > 1, n - r = 2$, and we suppose that the (i, j)-descendants of $\mathbf{M}$ are both good. We look at the various possibilities for i, j.

$[i, j \leqslant r]$ By induction we have $2^{-v_{n-1}} + 2^{-v_n} = \frac{1}{4}$ and

$$\sum_{k\neq i,j} 2^{-u_k} + 2^{-(u_i-1)} = h(\{v_{n-1}, v_n\}) = \sum_{k\neq i,j} 2^{-u_k} + 2^{-(u_j-1)},$$

so that $u_i = u_j$ and thus

$$\sum_{k=1}^{r} 2^{-u_k} = h(\{v_{n-1}, v_n\})$$

and **M** satisfies the criteria of the proposition.

$[(i, j) = (n-1, n)]$ By Proposition 2.4 we have

$$v_m - 1 = 1 \text{ or } 2, \qquad \sum_{k=1}^{r} 2^{-u_k} + 2^{-v_{2n-1-m}} = \frac{v_m - 1}{4}$$

for $m = n-1, n$. This gives a contradiction unless we have $v_{n-1} = v_n = 3$, whence $2^{-v_{n-1}} + 2^{-v_n} = \frac{1}{4}$ and $\sum_{k=1}^{r} 2^{-u_k} = \frac{3}{8}$, as required.

$[i \leqslant r < j]$ By induction we must have

$$v_j - 1 = v_{2n-1-j} = 3, \qquad \sum_{k\neq i} 2^{-u_k} = \frac{3}{8}$$

while by Proposition 2.4 we need

$$v_{2n-1-j} = 1 \text{ or } 2,$$

a contradiction.

Finally, we examine the case where $n - r > 2$, and suppose that the (i, j)-descendants of **M** are good.

$[i, j \leqslant r]$ Exactly as above we find that $u_i = u_j$ and that **M** satisfies the criteria of the proposition.

$[i, j > r]$ By induction we have

$$\sum_{k\neq i,j} 2^{-v_k} + 2^{-(v_i-1)} = \frac{1}{4} = \sum_{k\neq i,j} 2^{-v_k} + 2^{-(v_j-1)}$$

so that $v_i = v_j$, and

$$\begin{aligned}\sum_{k=1}^{r} 2^{-u_k} + 2^{-v_i} &= h(v_{r+1}, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n, v_j - 1)\\ &= h(v_{r+1}, \dots, v_n) + 2^{-v_j},\end{aligned}$$

as required.

$[i \leqslant r < j]$ In this case we have

$$\sum_{k\neq j} 2^{-v_k} = \frac{1}{4} = \sum_{k\neq j} 2^{-v_k} + 2^{-(v_j-1)},$$

a contradiction.

□

Now we consider those cases where one of the l_i is at least 3.

Proposition 2.7. *If $n \geqslant 2$ and $x \geqslant 3$, then the matrix*

$$\mathbf{M} = \begin{pmatrix} u_1 & \dots & u_{n-1} & v \\ 1 & \dots & 1 & x \end{pmatrix}$$

is good if and only if

$$v = 1 \quad \textit{and} \quad \sum_{k=1}^{n-1} 2^{-u_k} = 2^{-x}.$$

Proof. We use induction on n and x, with the case $n = 2$ following from Proposition 2.1. If $n > 2$ and $\mathbf{M}$ satisfies the hypotheses, then $u_i = u_j$ for some $i \neq j$, and the (i, j)-descendants are good, by induction.

Now suppose that $n > 2$ and that the (i, j)-descendants of $\mathbf{M}$ are both good. If $i, j < n$, then by induction we have $v = 1$ and

$$2^{-(u_i-1)} + \sum_{k\neq i,j} 2^{-u_k} = 2^{-x} = 2^{-(u_j-1)} + \sum_{k\neq i,j} 2^{-u_k},$$

so that $u_i = u_j$ and $\mathbf{M}$ satisfies the hypotheses. If $i < j = n$, then by induction we have both $v - 1 = 1$ and $v = 1$ – contradiction – unless $x = 3$, when we may have

$$v = 2, \sum_{k\neq i} 2^{-u_k} = 2^{-x}, \sum_{k\neq i} 2^{-u_k} + 2^{-(u_i-1)} = \frac{1}{2};$$

but this gives $2^{-(u_i-1)} = \frac{3}{8}$, also a contradiction. □

Now we show that we have found all good matrices.

Proposition 2.8. *If $n \geqslant 3$ and $l_i \geqslant 3, l_j \geqslant 2$ for some $i \neq j$, then the matrix*

$$\mathbf{M} = \begin{pmatrix} w_1 & \dots & w_n \\ l_1 & \dots & l_n \end{pmatrix}$$

is not good.

Proof. Suppose that the (i, j)-descendants of $\mathbf{M}$ are both good. Then by induction neither of the (i, j)-descendants can satisfy the conditions of the proposition, and so we must have $l_i = 3$, $l_j = 2$ and $l_k = 1$ for all $k \neq i, j$ (if $n = 3$ then are two extra possible cases, but these are easily dealt with). But then we require $w_i = w_j - 1 = 3$ by Proposition 2.6, while $w_i - 1 = 1$ by Proposition 2.7. Contradiction. □

Proof of Theorem 1.1. As noted above, a FW(n, l, w) exists if and only if the matrix

$$\mathbf{M} = \begin{pmatrix} w & \dots & w \\ l & \dots & l \end{pmatrix}$$

is good. For $n = 2$, this is true if and only if $l = w$, by Proposition 2.1. For $l = 1$, we need $n.2^{-w} = 1$, so that $n = 2^w$. If $n > 2$ and $l = 2$, then by Proposition 2.6 we need $n.2^{-w} = \frac{1}{4}$, and $h(w, \dots, w) = 0$. By Lemma 2.5, this happens if and only if $n = 16, w = 6$.

If $n, l > 2$, then $\mathbf{M}$ is not good, by Proposition 2.8. □

3 Tournaments with multiple winners

An interesting extension to this problem is to ask about tournaments which produce multiple winners; this is applicable to the situation where a set of players is chosen to proceed to the next stage of a competition. We define a $\mathrm{FW}_r(n, l, w)$ to be a competition for n players which produces r winners, each of whom has won exactly w games, and $n - r$ losers, each of whom has lost exactly l games. Classifying such tournaments seems to be rather harder than the case $r = 1$; the following simple observations provide lots of 'trivial' examples of these tournaments:

1. a $\mathrm{FW}_r(n, l, w)$ exists if and only if a $\mathrm{FW}_{n-r}(n, w, l)$ exists;

2. if a $\mathrm{FW}_r(n, l, w)$ and a $\mathrm{FW}_s(m, l, w)$ exist, then a $\mathrm{FW}_{r+s}(n + m, l, w)$ exists.

But there are other examples. For instance, it is fairly easy to show that for any $x \geqslant 0$ a $\mathrm{FW}_{2^x}(2^{2^{x+3}-4}, 2, 2^{x+3} - 2)$ exists. Of course, the case $x = 0$ gives the 'sporadic' case $(16, 2, 6)$ of Theorem 1.1. We hope to be able to say more about tournaments with multiple winners in a future paper.

References

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