A note on Kostka numbers

Matthew Fayers
Queen Mary University of London, Mile End Road, London E1 4NS, U.K.
m.fayers@qmul.ac.uk

Abstract

We prove a minor result on Kostka numbers, following a question from Mark Wildon on MathOverflow [MO]. We show that given partitions λ, μ, ν of n with $\mu \trianglerighteq \nu$, we have $\mathrm{K}_{\lambda\nu} \geqslant \mathrm{K}_{\lambda\mu}$. No attempt has been made to check for originality, and none is claimed.

1 Introduction

Recall that a *composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers which sum to n. Given compositions λ and μ of n, we say that λ *dominates* μ (written $\lambda \trianglerighteq \mu$) if $\lambda_1 + \dots + \lambda_r \geqslant \mu_1 + \dots + \mu_r$ for every r.

A composition is a *partition* if it is weakly decreasing. The *Young diagram* of a partition λ is the set

$$[\lambda] = \left\{ (r, c) \in \mathbb{N}^2 \;\middle|\; c \leqslant \lambda_r \right\},$$

which we draw as an array of boxes with the English convention (so that r increases down the page, and c from left to right). A λ-tableau is a function from $[\lambda]$ to $\mathbb{N}$, and we depict a tableau T by drawing $[\lambda]$ and filling each box with its image under T. The *type* of T is the composition μ, where μ_i is the number of is appearing in the diagram.

A λ-tableau is *semistandard* if the entries weakly increase from left to right along rows, and strictly increase down the columns. Given a partition λ of n and a composition μ of n, the *Kostka number* $\mathrm{K}_{\lambda\mu}$ is the number of different λ-tableaux of type μ.

This note concerns the following well-known result.

Theorem 1.1. *Suppose λ and μ are partitions of n. Then $\mathrm{K}_{\lambda\mu} > 0$ if and only if $\lambda \trianglerighteq \mu$.*

The 'only if' part of Theorem 1.1 is easy to see: if T is a semistandard λ-tableau of type μ, then all the numbers less than or equal to r in T must occur in the first r rows, so $\lambda_1 + \dots + \lambda_r \geqslant \mu_1 + \dots + \mu_r$. The converse is trickier to prove combinatorially, though a construction is given by the author in [MO]. The objective here is to prove the following result.

Proposition 1.2. *Suppose λ, μ, ν are partitions of n with $\mu \trianglerighteq \nu$. Then $\mathrm{K}_{\lambda\mu} \leqslant \mathrm{K}_{\lambda\nu}$.*

Since obviously $\mathrm{K}_{\lambda\lambda} = 1$, this proves the 'if' part of Theorem 1.1. We remark in passing that our Proposition 1.2 works when λ is a skew Young diagram.

2 The proof of Proposition 1.2

First we require an elementary lemma. Given non-negative integers $x_1, \dots, x_r, a$, let $S(x_1, \dots, x_r; a)$ be the number of ways choosing integers $y_1, \dots, y_r$ such that $0 \leqslant y_i \leqslant x_i$ for each i and $y_1 + \dots + y_r = a$. Now we have the following.

Lemma 2.1. *Suppose $x_1, \dots, x_r, a, b$ are non-negative integers, and let $m = x_1 + \dots + x_r$. If $|a - \frac{m}{2}| \geqslant |b - \frac{m}{2}|$, then $S(x_1, \dots, x_r; a) \leqslant S(x_1, \dots, x_r; b)$.*

Proof. Note first that $S(x_1, \dots, x_r; a) = S(x_1, \dots, x_r; m - a)$, since we have a bijection defined by $y_i \mapsto x_i - y_i$. So (replacing a with $m - a$ if necessary, and similarly for b) we can assume $a \leqslant b \leqslant \frac{m}{2}$. Assuming $r \geqslant 1$ and $x_1 \geqslant 1$, we write

$$S(x_1, \dots, x_r; a) = T(x_1, \dots, x_r; a) + U(x_1, \dots, x_r; a),$$

where $T(x_1, \dots, x_r; a)$ is the number of ways of choosing the y_i with $y_1 = x_1$, and $U(x_1, \dots, x_r; a)$ is the number of ways of choosing the y_i with $y_1 < x_1$. Obviously we have

$$T(x_1, \dots, x_r; a) = S(x_2, \dots, x_r; a - x_1), \qquad U(x_1, \dots, x_r; a) = S(x_1 - 1, x_2, \dots, x_r; a)$$

so it suffices to show that

$$S(x_2, \dots, x_r; a - x_1) \leqslant S(x_2, \dots, x_r; b - x_1), \qquad S(x_1 - 1, x_2, \dots, x_r; a) \leqslant S(x_1 - 1, x_2, \dots, x_r; b).$$

The first of these follows by induction, since $b - x_1$ is at least as close to $(m - x_1)/2$ as $a - x_1$ is. And the second also follows, since b is at least as close to $(m-1)/2$ as a is. So we can use induction on m. □

Using this, we can prove the following result which is the main ingredient in the proof of Proposition 1.2.

Lemma 2.2. *Suppose $i \in \mathbb{N}$, λ is a partition of n, and μ is a composition of n with $\mu_i > \mu_{i+1}$. Define a composition ν by*

$$\nu_i = \mu_i - 1, \quad \nu_{i+1} = \mu_{i+1} + 1, \quad \nu_j = \mu_j \text{ for all other } j.$$

Then $\mathrm{K}_{\lambda\mu} \leqslant \mathrm{K}_{\lambda\nu}$.

Proof. We define an equivalence relation $\sim$ on semistandard λ-tableaux by setting $S \sim T$ if all the entries different from i and $i+1$ are the same in S as they are in T. We show that within any one equivalence class there are at least as many semistandard tableaux of type ν as of type μ.

So fix an equivalence class C, and consider how to construct semistandard tableaux in C. The positions of the entries different from i and $i+1$ are determined, and we may as well assume there are μ_j entries equal to j for each $j \neq i, i+1$ (otherwise C contains no tableaux of type μ or ν). We are left with some positions in which to put is and $(i+1)$s – call these *available* positions. There are at most two available positions in each column, and if there are two, then these must be filled with i and $i+1$. So we need only consider columns having exactly one available position. Given $j \geqslant 1$, let x_j be the number of columns having an available position in row j only; these columns are consecutive, and can be filled in any way with is and $(i+1)$s as long as the i are to the left of the $(i+1)$s, to produce a semistandard tableau.

So choosing a semistandard tableau in C amounts to choosing integers $y_1, y_2, \dots$ such that $0 \leqslant x_j \leqslant y_j$ for each j: y_j is just the number of is placed in available positions in row j. In order for this semistandard tableau to have type μ, we must have $y_1 + y_2 + \cdots = a$, where $a = \frac{1}{2}(\mu_i - \mu_{i+1} + x_1 + x_2 + \dots)$. Similarly, to obtain a semistandard tableau of type ν we must have $y_1 + y_2 + \cdots = b$, where $b = \frac{1}{2}(\mu_i - \mu_{i+1} - 2 + x_1 + x_2 + \dots)$. Since $\mu_i > \mu_{i+1}$, b is at least as close to $\frac{1}{2}(x_1 + x_2 + \dots)$ as a is, so by Lemma 2.1 there are at least as many tableaux of type ν in C as there are of type μ. □

In order to use Lemma 2.2 we need to describe the covers in the dominance order on partitions. We leave the proof of the following results as an easy exercise.

Proposition 2.3. *Suppose μ and ν are partitions of n with $\mu \rhd \nu$. Then μ covers μ in the dominance order on partitions (i.e. there is no* partition *ξ with $\mu \rhd \xi \rhd \nu$) if and only if one of the following occurs:*

- *for some $i \in \mathbb{N}$ we have*

$$\nu_i = \mu_i - 1, \quad \nu_{i+1} = \mu_{i+1} + 1, \quad \nu_j = \mu_j \text{ for all other } j;$$

- *for some $i, j \in \mathbb{N}$ with $i < j$ we have*

$$\mu_{i+1} = \cdots = \mu_j = \mu_i - 1, \quad \nu_i = \mu_i - 1, \quad \nu_j = \mu_j + 1, \quad \nu_k = \mu_k \text{ for all other } k.$$

Informally, μ covers ν if and only if ν is obtained by moving one box down and to the right, either to an adjacent row or to an adjacent column.

Proof of Proposition 1.2. We may assume μ covers ν in the dominance order, and consider the two cases in Proposition 2.3. In the first case it is immediate from Lemma 2.2 that $\mathrm{K}_{\lambda\mu} \leqslant \mathrm{K}_{\lambda\nu}$. In the second case, define compositions $\xi^{i+1}, \dots, \xi^{j-1}$ by

$$\xi^k_i = \mu_i - 1, \quad \xi^k_k = \mu_k + 1, \quad , \xi^k_l = \mu_l \text{ for all other } l.$$

Then by Lemma 2.2 we have

$$\mathrm{K}_{\lambda\mu} \leqslant \mathrm{K}_{\lambda\xi^{i+1}} \leqslant \cdots \leqslant \mathrm{K}_{\lambda\xi^{j-1}} \leqslant \mathrm{K}_{\lambda\nu}\,.$$

□

References

[MO] M. Wildon, *Is there a short proof that the Kostka number $\mathrm{K}_{\lambda\mu}$ is non-zero whenever λ dominates μ?*, mathoverflow.net/questions/226537. 1