

M. Sc. Examination by course unit 2011

MTHM023 Rings & modules

Duration: 3 hours

Date and time: 16th May 2011, 10 a.m.

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Fayers

In this paper, $\mathcal{M}_n(R)$ denotes the ring of $n \times n$ matrices over a ring R .

In any question, you may freely use the Coset Lemma: if L is a subgroup of an abelian group M and $a, b \in M$, then $a + L = b + L$ if and only if $a - b \in L$.

Question 1

- (a) Give the definition of a *ring*, and an *ideal*. [You may assume that the definition of an abelian group is understood.] [5]

Now suppose R is a ring and I is an ideal in R .

- (b) Define the *quotient ring* R/I . [You do not have to prove that R/I is a ring, but you should show that any operations you introduce are well-defined.] [4]
- (c) Suppose M is a simple R/I -module. Prove that M is a simple R -module with the R -action $rm = (r + I)m$ for $r \in R, m \in M$. [5]
- (d) Now suppose M is an R -module. What additional condition is needed in order to be able to make M into an R/I -module? [2]
- (e) If M is an R -module, prove that the set

$$\text{Ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$$

is an ideal in R . [4]

- (f) Now let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}/4\mathbb{Z} \right\}.$$

Write down six different ideals in R . [You do not need to prove anything.] [5]

Question 2

- (a) Give the definitions of a *simple module*, a *semisimple module* and a *semisimple ring*. [You may assume that the terms *ring* and *module* are understood.] [5]
- (b) State Wedderburn's Theorem on semisimple rings. [4]
- (c) Suppose M is an abelian group and N, P, K are subgroups such that $N \cap P = 0$ and $N \cap (P + K) \neq 0$. Prove that $(N + P) \cap K \neq 0$. [2]
- (d) Prove that if M is a semisimple R -module and $N \leq M$, then there is a submodule $P \leq M$ such that $M = N \oplus P$. [If you use Zorn's Lemma, you need not go into full detail about chain-completeness.] [8]
- (e) Let I be the ideal in $\mathbb{R}[X]$ consisting of all polynomials divisible by X^2 , and let $R = \mathbb{R}[X]/I$. Is R a semisimple ring? Justify your answer. [6]

Question 3 Let R be a ring. In this question, you may assume that a left ideal I of R is maximal if and only if ${}_R R/I$ is a simple R -module.

- (a) Give the definition of the *Jacobson radical* $J(R)$. [2]
- (b) Suppose r is an element of R such that $1 - ar$ has a left inverse for every $a \in R$. Prove that r lies in $J(R)$. [5]
- (c) Prove that if R is artinian, then $J(R)$ can be written as the intersection of a finite collection of maximal left ideals of R . [You may assume that artinian modules satisfy the minimal condition on submodules.] [6]
- (d) Suppose n is a positive integer. Describe (with proof) all the left ideals of $\mathbb{Z}/n\mathbb{Z}$. [6]
- (e) Draw a diagram of all the left ideals of $\mathbb{Z}/48\mathbb{Z}$, showing which left ideals contain which others. [4]
- (f) Hence find $J(\mathbb{Z}/48\mathbb{Z})$. [2]

Question 4

- (a) Suppose R is a ring and $r \in R$. Define what is meant by a *left inverse* for r in R . [2]
- (b) Show that if every non-zero element of R has a left inverse, then every non-zero element has a unique right inverse. [5]
- (c) Give an example of a ring R and an element $r \in R$ such that r has a left inverse but not a right inverse. Briefly justify your answer. [4]

Now suppose R is a division ring and $r \in R$.

- (d) Define what is meant by the *centraliser* $Z(r)$. [2]
- (e) Prove that $Z(r)$ is a subring of R , and is a division ring. [5]

Now let $R = \mathbb{R}^2$, with the usual addition of vectors, and with multiplication defined by

$$\begin{pmatrix} a \\ b \end{pmatrix} \times \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad + bc \end{pmatrix}.$$

[You may assume that R is a ring.]

- (f) What is the identity element of R ? [1]
- (g) Which elements of R have inverses? Justify your answer [3]
- (h) Prove that the set of non-invertible elements is an ideal in R . [3]

Question 5 Suppose R is a ring.

- (a) Define what is meant by a *central idempotent* in R . Define what it means for two central idempotents to be *orthogonal*. Define what it means for a central idempotent to be *primitive*. [5]
- (b) Suppose $e \in R$ is a central idempotent other than 0 or 1. Show that there are two non-zero ideals I, J of R such that $I + J = R$ and $I \cap J = 0$. [7]
- (c) Find (with proof) all the central idempotents in $\mathcal{M}_2(\mathbb{Z}/6\mathbb{Z})$. [5]

Now suppose $(M_i)_{i \in I}$ is a family of R -modules.

- (d) Define what is meant by a *coproduct* of this family with canonical injections ι_i . [3]
- (e) Using only the definition of a coproduct, prove that the canonical injections are injective. [5]

Question 6

- (a) State and prove one of the three Isomorphism Theorems for modules. [9]
- (b) What does it mean to say that a module is *noetherian*? What is meant by a *composition series of length r* for a module? What does it mean to say that two composition series are *equivalent*? [6]
- (c) For each of the following cases, write down either
- an infinite ascending chain of submodules of M ,
 - an infinite descending chain of submodules of M , or
 - a composition series for M .

[You do not need to prove anything.]

- (i) $R = \mathbb{C}[X], M = {}_R R$. [2]
- (ii) $R = \mathbb{Z}, M = \mathbb{Z}/8\mathbb{Z}$. [2]
- (iii) $R = \mathbb{Z}, M = \left\{ \frac{a}{2^n} \mid a, n \in \mathbb{Z} \right\} / \mathbb{Z}$. [2]
- (iv) $R = \mathcal{M}_2(\mathbb{Z}), M = {}_R R$. [2]
- (v) $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \in \mathbb{R}, b, c \in \mathbb{C} \right\}, M = \mathbb{C}^2$ (with the usual action of matrices on vectors). [2]

End of Paper