Queen Mary
University of London

M.Sci./M.Sc. EXAMINATION BY COURSE UNIT

MAS427/MTHM023 Rings and Modules (First Sit)

2008 tba

Duration: 3 hours

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

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THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Throughout this paper, R denotes a ring and all modules are left modules unless otherwise stated.

1. (a) Given a family $(M_i)_{i\in I}$ of R-modules, explain how the direct product $\prod_{i\in I} M_i$ and the direct sum $\bigoplus_{i\in I} M_i$ are defined. State and prove the universal property of the direct product.

(b) Let M_1, M_2 be submodules of an R-module M such that $M = M_1 + M_2$. Prove that

$$M/M_1 \cap M_2 \cong_R (M/M_1) \oplus (M/M_2).$$

(c) Given an R-homomorphism $\varphi : A \to B$, where A and B are R-modules, and an R-module X, explain how $\varphi_* : \mathrm{Hom}_R(X, A) \to \mathrm{Hom}_R(X, B)$ is defined.

Give an example to show that, if φ is surjective, φ_* need not be surjective.

2. Explain what is meant by a *projective* R-module. Prove the following.

(a) Free R-modules are projective.

(b) A direct summand of a projective module is projective.

(c) An R-module P is projective if and only if it is a direct summand of a free R-module.

(d) If $e \in R$ satisfies $e^2 = e$, then Re is a projective submodule of ${}_RR$.

(e) Show that, if M is a non-zero submodule of a free R-module, then there is a non-zero R-homomorphism $M \to {}_RR$.

3. (a) Define an *injective* R-module. Show that if $(Q_i)_{i\in I}$ is a family of injective R-modules, then $\prod_{i\in I} Q_i$ is injective.

(b) For any ring R and $\mathbb{Z}$-module J, explain how $\mathrm{Hom}_{\mathbb{Z}}(R, J)$ is made into an R-module. Prove that, if M is an R-module, M is isomorphic to an R-submodule of $\mathrm{Hom}_{\mathbb{Z}}(R, J)$ for some divisible $\mathbb{Z}$-module J. (You may assume any $\mathbb{Z}$-module is a submodule of a divisible $\mathbb{Z}$-module.)

(c) Is it true that a submodule of an injective module is injective? Give brief justification for your answer.

(d) Let n be a non-negative integer. If $n > 1$, find a $\mathbb{Z}$-homomorphism from $\mathbb{Z}$ to $\mathbb{Q}/\mathbb{Z}$ with kernel $n\mathbb{Z}$. Deduce that there is a non-zero $\mathbb{Z}$-homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Q}/\mathbb{Z}$ for all $n \neq 1$ (including $n = 0$).

[Next question overleaf]

4. Given a right R-module M and a left R-module N, define the tensor product $M \otimes_R N$ and give (without proof) a construction of it.

Give proofs of the following.

(a) If M is a right R-module, N is an (R, S)-bimodule and L is a left S module, where S is any ring, then $(M \otimes_R N) \otimes_S L$ is isomorphic to $M \otimes_R (N \otimes_S L)$.

(b) If R is commutative and M, N are R-modules, then $M \otimes_R N$ is isomorphic to $N \otimes_R M$.

Show that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = \{0\}$. Hence, or otherwise, show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. (You may assume $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, and that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$.)

5. (a) Prove the Correspondence Theorem for R-modules, that is, prove that if $\varphi : M \to K$ is an R-homomorphism, there is a 1-1 correspondence from the set of submodules of $M\varphi$ to the set of submodules of M containing $\ker(\varphi)$, given by $L \mapsto L\varphi^{-1}$. (You need not prove that if $L \leq_R M\varphi$ then $L\varphi^{-1} \leq_R M$, nor that an R-homomorphism maps submodules to submodules.)

(b) If M is a Noetherian R-module and $\varphi : M \to K$ is an R-homomorphism, prove that $M\varphi$ is Noetherian.

(c) If $\varphi : R \to S$ is a ring homomorphism, explain briefly how an S-module M can be made into an R-module. Assume φ is surjective. Show that, if R is a left Noetherian ring, then S is left Noetherian.

(d) Give an example to show that a submodule of a finitely generated module need not be finitely generated.

6. (a) If M is an R-module, explain how $\mathrm{End}_R(M)$ can be made into a ring. Prove that R is isomorphic to $\mathrm{End}_R({}_R R)$ as rings.

Explain what is meant by a simple module. If M is a simple R-module, prove that $\mathrm{End}_R(M)$ is a division ring.

(b) Define the centre $Z(R)$ of a ring R, and show that it is a subring of R. If R is a division ring, prove that $Z(R)$ is a field. If $R = \prod_{i \in I} R_i$ is a product of rings, show that $Z(R) = \prod_{i \in I} Z(R_i)$.

(c) State the Artin-Wedderburn Theorem. If R is a semisimple ring, prove that $Z(R)$ is a finite direct product of fields. (You may assume $Z(R^{n \times n}) \cong Z(R)$ for any ring R.)

[End of examination paper]