PHYSICS PREPRINT

Non-commutative geometric quantization of photons and string field theory

Shahn Majid

5 August, 1988

Lyman Laboratory of Physics
Harvard University
Cambridge MA02138

Abstract

We sketch a formulation of second quantized photons based on non-commutative geometry. The formulation is locally identical to standard canonical quantization but extends to topologically non-trivial gauge fields, as for example in the complement of a knot. The formalism can also be considered for gluons, and is connected in a novel way with string field theory.

Recently, E. Witten has proposed[1] that the vacuum expectation value of Wilson loops in a Yang-Mills gauge theory with lagrangian $\mathcal{L} = AdA + \frac{2}{3}A^3$, is the Jones polynomial[2] knot invariant. Because of this revival of interest in the Chern-Simmons form in connection with knots, this may be a suitable time to draw attention to some related work of the author[3]. Some details of what follows are given in [3]. Concrete steps towards realising the relation with string field theory are in [13].

1. Non-commutative geometry Functions on a manifold M, with pointwise addition and multiplication, form a commutative C^* algebra, $C(M)$. Working with this is fully equivalent to working with M directly. However, often the algebra may be deformed to a non-commutative C^* algebra, which is not C(anything), but nevertheless behaves as if there is an underlying "non-commutative M"[4]. The main example is the non-commutative torus [5]. This is the C^* algebra generated by a pair of operators such that

$$u_1 u_2 = u_2 u_1 e^{2\pi\imath\alpha}$$

where α is a real parameter. As $\alpha \mapsto 0$, this becomes commutative and the C^* algebra becomes isomorphic by Fourier transform to the algebra of functions on a torus $S^1 \times S^1$. The case of α irrational has been extensively studied, within the framework of non-commutative geometry[5].

The above construction is known for a general abelian[6] (and with care, non-abelian) group G as follows. It will be termed "non-commutative geometric quantization of a particle with dual configuration space G". Given a bicharacter β on G, i.e. $\beta : G \times G \to U(1)$ a character in each argument, consider faithful operator representations U_1, U_2 of G such that

$$U_1(f)U_2(g) = U_2(g)U_1(f)\beta(f,g), \qquad \forall f,g \in G$$

The C^* algebra generated by these will be called "non-commutative $\hat{G}\times\hat{G}$" because as β becomes trivial, and for sufficiently nice groups G, it becomes isomorphic by Fourier transform to the algebra of functions on $\hat{G} \times \hat{G}$. Here $\hat{G}$ is the Pontryagin dual group of G (the group of characters)[8]12.1. For example $\hat{\boldsymbol{Z}} = S^1$ and $\hat{S}^1 = \boldsymbol{Z}$.

To see this, note that β defines a 2-cocycle, $\tilde{\beta}$ on $G \times G$,

$$\tilde{\beta}((f,g),(f',g')) = \beta(f',g)^{-1}, \qquad \forall (f,g),(f',g') \in G \times G.$$

Then there is an abstract C^* algebra, $C^*(G \times G, \tilde{\beta})$, associated to this cocycle. It is generated concretely by operators W obeying

$$W(z)W(z') = W(zz')\tilde{\beta}(z,z'), \qquad \forall z,z' \in G \times G.$$

It coincides with the one generated by the U_i through the relation $W(f,g) \equiv U_1(f)U_2(g)$. However, $C^*(G \times G; \tilde{\beta})$ can also be defined abstractly as a modification of the group convolution algebra. To do this, the points in $G \times G$ are equivalently thought of as

δ-functions on $G \times G$. The algebra of linear combinations of such functions is then a modification of convolution:

$$F * H(z) = \int_{G\times G} dz' F(z') H({z'}^{-1} z) \tilde{\beta}(z', {z'}^{-1} z).$$

The case $\tilde{\beta}$ trivial is the usual convolution C^* algebra, $C^*(G \times G)$. In the case of abelian and sufficiently nice $G \times G$, this is isomorphic by Fourier transform to the algebra of functions on $\hat{G} \times \hat{G}$[8]12.1.

The example $G = \boldsymbol{Z}$ with $U_i(n) = u_i^n$, $\forall n \in \boldsymbol{Z}$ is the non-commutative torus. We now review a simple physical example. Let X be 3-space. Then case $G = Maps(X, \boldsymbol{R})$ (more precisely, square integrable functions) and $\beta(f, g) = e^{-\imath \int_X f(x) g(x) dx}$ gives the *Weyl algebra* of second-quantized scaler bosons[7]Chapter 5.2. For, writing

$$U_1(f) = e^{\imath \int_X \Phi(x) f(x)}, \quad U_2(f) = e^{\imath \int_X \Pi(x) f(x)}$$

for operator-valued fields Φ, Π, we see that the U_1, U_2 relations correspond to the usual equal time commutation relations, $[\Phi(x), \Pi(y)] = \imath \delta^{(3)}(x - y)$. One usually identifies $z \in G \times G = Maps(X, \boldsymbol{R}) \oplus Maps(X, \boldsymbol{R}) = Maps(X, \boldsymbol{C})$ and the W's are then called the *Weyl representation* of the canonical commutation relations[7]5.2.

The key point is that this construction works for general abelian groups G, not just linear spaces as in the scaler boson case. We now point out the group and bicharacter needed to obtain second-quantized photons by this construction. It is a group of strings or string fields with a Witten-type product law. Non-commutative geometry may also be relevant to the subject of fractional statistics[9].

2. The abelianized string group We continue to denote physical 3-space by X. Consider oriented loops in X with the product law shown in figure 1, i.e. wherever loops partially overlap with opposite orientation, erase the overlap. This defines a group ω[3] (it is essentially the space of integral 1-cochains in X.) Note that the order in which segments are traversed is not distinguished, and as a result, ω is abelian.

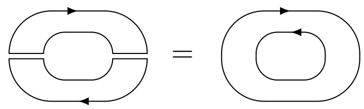

Figure 1: The "abelianized string group" product law. Parametrization is not distinguished.

Also note that this is necessarily a many-loop theory as the example demonstrates. The one-loop elements are a subset ω_0 (i.e. loops of one connected component.) One can identify ω with $Maps(\omega_0, \boldsymbol{Z})$.

On ω we define[3]

$$\text{link}_{\,\epsilon}(\sigma,\tau) = \int_{||\vec{\epsilon}||<\epsilon} d^3\vec{\epsilon}\, \text{link}\,(\sigma, \tau_{\vec{\epsilon}}), \qquad \forall \sigma, \tau \in \omega_0$$

Where "link " is the linking number when defined, $\epsilon > 0$ (sufficiently small), and $\tau_{\vec{\epsilon}}$ is τ displaced by $\vec{\epsilon}$ in X. This assumes a metric on X. This metric and ϵ are needed to regulate, c.f. "by point-splitting" so that link $_\epsilon$ is defined for almost all loops.

In particular, the key point for the final remark in section 4 is that link (σ, σ), the "self-linking number", is defined (the regulator ϵ can be removed in this case.) It is a real number, not a topological invariant (it depends on the metric.) In the case $X = \boldsymbol{R}^3$, it turns out to be simply given by Gauss's linking number formula[10], but now evaluated on the diagonal,

$$\text{link}\,(\sigma,\sigma) = \int ds \int dt \frac{(\sigma(s) - \sigma(t)) \cdot (\sigma'(s) \times \sigma'(t))}{|\sigma(s) - \sigma(t)|^{\frac{3}{2}}}$$

In the formula, curves can be taken parametrized by arc-length and $\sigma'(s)$ is the tangent at s. One can show[3] that link (σ, σ) vanishes for planar curves σ. The definition extends from ω_0 to all of ω additively. Then let

$$\beta(\sigma,\tau) = e^{-4\pi\imath\alpha \text{link}_{\,\epsilon}(\sigma,\tau)}$$

where α is a real parameter which we shall identify with the electromagnetic fine structure constant. The new contributions of this section are to point out the natural formulation of the "abelianized string group" ω, the novel regularization link $_\epsilon$ and the notion of "self-linking number".

3. Non-commutative $\mathcal{A}_1/\mathcal{G}$ is second-quantized photons We now apply, formally, the "quantization scheme" of section 1 to the dual configuration space $G = \omega$ of section 2. One may also take "string fields", $G = Maps(\omega_0, \boldsymbol{R})$ rather than integral linear combinations of the 1-loops ω_0. The scheme requires to find U_1, U_2 operator representations of ω such that

$$U_1(\sigma)U_2(\tau) = U_2(\tau)U_1(\sigma)e^{-4\pi\imath\alpha \text{link }_\epsilon(\sigma,\tau)}.$$

As ansatz we take

$$U_1(\sigma) = e^{\imath\int_\sigma A_1}, \qquad U_2(\sigma) = e^{\imath\int_\sigma A_2}, \qquad \forall\sigma \in \omega$$

for a pair of operator-valued gauge fields A_i. The corresponding commutation relations for these are

$$[\int_\sigma A_1, \int_\tau A_2] = 4\pi\imath\alpha \text{link }(\sigma,\tau), \qquad \forall\sigma,\tau \in \omega$$

If we identify A_1 as the three space components at a fixed time, of a gauge field A on $X \times \boldsymbol{R}$ in temporal gauge, and A_2 as a new connection on X such that $\nabla \times A_2 = E$, the electric field of A, then these commutation relations are equivalent to the usual equal-time commutation relations for second-quantized photons A, $[A_i(x), E_j(y)] = 4\pi\alpha\imath\delta_{ij}\delta^{(3)}(x-y)$. The presence of α here comes from our choice of geometric normalization for A such that the covariant derivative on spacetime is $\partial + A$. The normalization of E is then fixed by $E = \frac{1}{c}\frac{\partial}{\partial t}A$. This is necessary for Lorentz invariance since it ensures that both A_1 and A_2 are geometrical objects and at each t have 3-curvatures that fit together to give the 4-curvature of A. This is a basic assumption of the formulation. If A^{phys} and E^{phys} denote the usual normalizations, the relations are $A = \frac{e}{\hbar}A^{\text{phys}}$ and $E = \frac{e}{\hbar c}E^{\text{phys}}$ where $E^{\text{phys}} = \frac{\partial}{\partial t}A^{\text{phys}}$.

The U_i form of the gauge field commutation relations were discussed by [11] in a different context. In that context, quark confinement, the role of $e^{4\pi\imath\alpha}$ is played by an element $e^{2\pi\imath/n} \in \boldsymbol{Z}_N$ the center of $SU(N)$. In the present context there is no such integer n: instead a close look at the units above showed that $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$, the fine structure constant. Since linking number is usually an integer, it is clear that *some* parameter α, is inevitable, so it should not be surprising to learn, as we have shown, that α is the fine structure constant. In the real world this runs, but in some idealization of only quantized photons and infinitely massive static charges, it may have some interesting value. Experience with the non-commutative torus, section 1, suggests that the structure of the algebra, i.e. the non-commutative geometry, depends critically on α, for example on whether or not it is rational[5].

The dual connection was also discussed in [12]. Note that this formulation replaces a connection A on spacetime $X \times \boldsymbol{R}$ by a path in the space of pairs of connections A_i. At each fixed t, $A(t)$ is replaced by $A_i(t)$. Locally, this is the same, but globally it imposes constraints. In particular, although such a procedure is not manifestly Lorentz-invariant, it has the advantage that the flux of E through any space-like closed surface is quantized (completely independent of monopoles, since in our model, A_2 is itself a connection.) More precisely, $\frac{1}{e}\int_{S^2} E^{phys} = \frac{4\pi\hbar c}{e^2}\frac{1}{4\pi}\int_{S^2} E = \frac{1}{\alpha}q$ where $q \in \boldsymbol{Z}$. The fact that A_2 is itself a connection makes electrons into topological objects, "electropoles" (like monopoles). Like Dirac monopoles, they should be thought of as configurations of gauge fields in the complement of a point, i.e. active electric charge is an element of $H^2(\boldsymbol{R}^3 - \text{point}, \boldsymbol{Z}) = \boldsymbol{Z}$. One can[3] also allow gauge fields in the complement of other topological defects, such a knots. At large scales such a knot deleted from X would appear as a point source, but on smaller scales would have fine structure corresponding to more exotic gauge field configurations A_i.

Note that the presence of the factor $\frac{1}{\alpha}$ in the quantization of physical electric flux is an intrinsic limitation of this formulation. It indicates that the non-commutative geometric quantization of pure photons and massive static charges is too constrained. Presumably it will have to be modified to include quantized spinor fields [3]. The present reduced formulation does however, suffice to make the point in section 4 where we indicate a novel way to relate such models to string field theory.

The new contribution of this section has been to show how this formulation of second-quantized photons arises naturally in the context of non-commutative geometry and to observe the resulting implications for QED in the limit of massive static charges. Formally, what was done was to replace $G = Maps(X, \boldsymbol{R})$ with the coincidence cocycle (scaler bosons) by $G = Maps(\omega_0, \boldsymbol{Z})$ or $Maps(\omega_0, \boldsymbol{R})$ with the linking-number cocycle (vector bosons.) The dual of G is now, morally speaking, the group, denoted $\mathcal{A}_1/\mathcal{G}$, of all $U(1)$-connections-with-bundles on X modulo gauge transformations. Here $A \in \mathcal{A}$ denotes a choice of bundle and connection on it, the "1" reminds us that we are looking at the $U(1)$ case, and $\mathcal{G}$ denotes the groups of gauge transformations. It is evident that each connection-with-bundle, A, is a character on ω by $\sigma \mapsto e^{\imath \int_\sigma A}$ and that the character is the same for another connection that differs only by a gauge transform. It turns out that this space is a group[3]. The group structure is defined by pointwise addition of connections *and* multiplication of the bundle transition functions[3]. Thus in the limit as $\alpha \mapsto 0$, we recover, morally speaking, functions on $\mathcal{A}_1/\mathcal{G} \times \mathcal{A}_1/\mathcal{G}$. *Thus second quantized photons should be thought of as "non-commutative complexified $\mathcal{A}_1/\mathcal{G}$" and may be studied as such, in analogy with the non-commutative torus.* Here "complexified $\mathcal{A}_1/\mathcal{G}$" refers to the pair of connections on space X, or a connection A on spacetime at fixed t.

4. Relation with string field theory The construction in section 3 can also be carried out dually, with $G = \mathcal{A}_1/\mathcal{G}$. We therefore seek operators

$$U_1(A)U_2(A') = U_2(A')U_1(A)\beta(A, A').$$

Here an example of a bicharacter on $\mathcal{A}_1/\mathcal{G}$ is $\beta(A, A') = e^{-\imath\alpha' \int_X AdA'}$ based on the Chern - Simmons 3-form . One can also consider other quadratic Lagrangians, and the non-abelian case. The ansatz for the U_i is now of the form

$$U_i(A) = e^{\imath \sum_{\sigma \in \omega_0} \Psi_i(\sigma) \int_\sigma A}$$

for operator-valued string fields Ψ_i obeying certain commutation relations. As the correct commutation relations for string field theory are not known, this approach may help to single out interesting ones according to the choice of β. However, the main advantage of this approach to string field theory is the following, which we describe in some detail.

When G is a locally compact or finite abelian group, one has a precise Pontryagin theorem, $\hat{\hat{G}}\cong G$, and the Fourier transform connecting functions on G to ones on $\hat{G}$ and vice-versa[8]12.1. Therefore such a string theory could be related by a "functional Fourier transform" to the second - quantized photons of section 3. This opens up the possibility if a much simpler relation, than usual, between a suitable string field theory in 3+1 dimensions and the corresponding quantum gauge field theories. To describe it, first consider the case of trivial β. Then $C^*(G\times G)$ is essentially a Hopf algebra. This has a natural Hopf algebra dual, $C^*(G\times G)^* \approx C^*(\hat{G}\times\hat{G})$. Here the Hopf algebra dual is defined on the space of linear maps from $C^*(G\times G)$ to $\mathbb{C}$ (c.f. "states" in the quantum theory) and the approximate isomorphism is given by a "functional Fourier transform" as follows: if $\tilde{\Psi}[\chi_1,\chi_2]\in C^*(\hat{G}\times\hat{G})$ then the sense in which it is a linear map on $C^*(G\times G)$ is that it sends $\Psi[g,h]\in C^*(G\times G)$ to

$$\sum_{(g,h)\in G\times G}\ \sum_{(\chi_1,\chi_2)\in\hat{G}\times\hat{G}} \Psi[g,h]\tilde{\Psi}[\chi_1,\chi_2]\chi_1(g)\chi_2(h).$$

In the present case $G=\omega$ and morally speaking, $\hat{G}=\mathcal{A}_1/\mathcal{G}$ and $\chi(g)$ is the holonomy pairing of the form $e^{\imath\int_\sigma A}$. In this way $C^*(\omega\times\omega)$ and $C^*(\mathcal{A}_1/\mathcal{G}\times\mathcal{A}_1/\mathcal{G})$ are essentially in duality.

We now consider the quantum theory, i.e with β non-trivial. If the above arguments went through, one would have $C^*(\omega\times\omega;\mathrm{link}\ )$ (i.e. the algebra if observables for photons according to section 3) in duality with $C^*(\mathcal{A}_1/\mathcal{G}\times\mathcal{A}_1/\mathcal{G};\beta)$ (i.e. the algebra of observables of a suitable string field theory.) A problem arises however, that $C^*(G\times G;\beta)$ for non-trivial β is no longer a Hopf algebra. This has been solved by the author in the case of G a finite-dimensional Lie group[13] by modifying the commutation relations. This variant of the construction of section 1 involves introducing non -

commutative - geometric curvature on the phase space in such a way as to maintain the Hopf algebra structure. The construction also involves replacing the role of $\hat{G}$ by G^*, a "matching group"[13]. This construction works well for either abelian or non-abelian G. In the latter case the necessary matching equations turn out to be a group generalization of the Classical Yang-Baxter Equations (CYBE) on the Lie algebra of G. Thus for every solution of the CYBE one obtains a "matching group" G^*. The resulting Hopf algebra, denoted $C^*(G) \bowtie C(G^*)$, plays the role of $C^*(G \times G; \beta)$ with the choice of "matching group" (e.g. choice of solution of the CYBE) playing the role of choice of β. The resulting mathematical theorem is[13]

$$(C^*(G) \bowtie C(G^*))^* \approx C^*(G^*) \bowtie C(G).$$

Physically in this variant of section 1, the dynamics of the system is also encoded (in the Hopf algebra structure) and is classically that of motion along geodesics on a homogeneous space[13]. To apply this formalism to string theory and gauge theory we would need to find "matching groups" to ω and $\mathcal{A}_1/\mathcal{G}$ such that the resulting dynamics in ω and $\mathcal{A}_1/\mathcal{G}$ are as desired physically in the classical limits. There are approaches to string theory based on the geometry of "loop groups" and their quotients. Moreover, because of the connection with Yang-Baxter equations, and the known connection between the (quantum) Yang-Baxter equations and crossing symmetry, one sees that in such a formulation the Hopf algebra duality described above is connected with the fundamental string theory duality [14]. Note that the Hopf algebras $C^*(G) \bowtie C(G^*)$ are different from those previously connected with the CYBE[15], where Hopf algebra duality does not play a role.

It should be stressed, however, that in the present case, the groups $\mathcal{A}_1/\mathcal{G}$ and ω are highly infinite-dimensional and non-compact. So far in this programme, only the case of finite dimensional G has been made precise. Further details are in [13].

It may be appropriate to end with the following related speculative remark. Consider a $U(1)$ gauge theory in three dimensions with "topological mass term"[16] and suppose for the moment that the functional integral in some situation is dominated by "vortex states" A_σ along loops (i.e. curvature a δ-function pointing along the loop.) Then the action on these states reduces to

$$1/4\int_X ||F(A_\sigma)||^2 - 4\pi\alpha'\int_X A_\sigma dA_\sigma = 1/2\delta(0)\int_\sigma ds\sigma'(s)\cdot\sigma'(s) - 4\pi\alpha'\text{link }(\sigma,\sigma).$$

This could arise as some three-dimensional limit of a 3+1 string theory, *the Polyakov string action, with "string mass term" based on the self-linking number* link (σ, σ) introduced in section 2.

The Yang-Mills case is similar but more complex, and there are some indications in this case that such vortex states could dominate[17]. It is proposed to formulate string theory with this self-linking number as the basis for a "string mass term". There should be no conformal anomaly now, indeed the "mass term" only makes sense in three space dimensions.

Strings, "Chern-Simmons-like forms" and non-commutative geometry were previously connected in a very different context in the paper [18]. See also [19]. One may attempt to generalize the construction of section 3 to gluons. The "abelianized string group ω" introduced in section 2 must now be replaced by a non-abelian version. It is significant that in the variant of section 1 that has been described, it is the non-abelian G that makes contact with the CYBE.

References

[1] E. Witten, Proc. Int. Congress of Math. Physics (Swansea,1988) to appear.

[2] V.F.R. Jones, Bull. Amer. Math. Soc. **12** (1985) 103-111; V.G. Turaev, Inv. Math. **92** (1988) 527-553.

[3] S. Majid, "Non-commutative $\mathcal{A}_1/\mathcal{G}$: a new approach to quantization of photons" preprint (unpublished) (1986).

[4] A. Connes, Publ. Math. IHES **62** (1986)

[5] M.A. Rieffel, Pac. J. Math **93** (1981) 415; A. Connes, C.R. Acad. Sc. Paris **290** (1980) 599-604; M. A. Rieffel, Can. J. Math. vol XL, No2 (1988) 257-338; A. Connes and M. A. Rieffel, Contemporary Math **62** (1987) 237-266; J. Cuntz, G.A. Elliot, F.M. Goodman and P.E.T. Jorgensen, C.R. Math. Rep. Acad. Sci. Canada **7** (1985), 189-194; S. Disney, G.A. Elliot, A. Kumjan and I. Raeburn, C.R. Math. Rep. Acad. Sci. Canada **7** (1985), 137-141; G.A. Elliot, Operator algebras and group representations 1 (Pitman, London, 1984),157-184.

[6] J. Slawny, CMP **24** (1972) 151-170'

[7] O. Bratteli and D. W. Robinson, Vol II Springer-Verlag (1979) chapter 5.2.

[8] A.A. Kirillov, Springer-Verlag (Heidelberg) (1979).

[9] F. Wilczek and A. Zee, Phys. Rev. Let **51** (1983) 2250-2; A. Zee, Phys. Lett. **135B** (1984) 307.

[10] D. Rolfsen, Publish or Perish (Berkeley) (1976).

[11] G. 'tHooft, "On the phase transition towards permanent quark confinement" Nuc. Phys. **B138** (1978) 1.

[12] S. Mandelstam, Phys.Rev. **D19** (1979) 2391.

[13] S. Majid, J. Class. Quantum Gravity, to appear (1988) and Harvard preprint HUTMP-87/B215-L; S. Majid, submitted to J. Algebra and Harvard preprint HUTMP-87/B216-L; S. Majid, submitted to Pac. J. Math and Harvard preprint HUTMP-88/B220; S. Majid, Proc.Int.Symp. on Spacetime Symmetries, Univ. Maryland (1988).

[14] A. Belavin, A. Polyakov and A.B. Zamalodchikov Nuc. Phys. **B241** (1984) 333; G. Moore and N. Seiberg Princeton preprint IASSNS-HEP-88/18 (1988).

[15] V.G. Drinfeld Proc. Inc. Conf. Math. (Berkeley, 1986); M. Jimbo Lecture notes in Physics **246**, Springer - Verlag (1985).

[16] R. Jackiw and S. Templeton, Phys.Rev. D **23** (1981) 2291; J. Schonfeld, Nuc. Phys. **B185** (1981) 157; W. Seigel, Nuc. Phys. **B156** (1979) 135.

[17] R. Mawhinney, Harvard PhD thesis (1987); G. Mack and V.B. Petrova, Ann. Phys. **125** (1980) 117.

[18] E. Witten, Nuc.Phys. B268 (1986) 253-294.

[19] L. Crane and C. Gomez, Phys. Lett. **179B** (1986) 43.