On balanced incomplete-block designs with repeated blocks

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In Memory of Jack van Lint

Abstract

Balanced incomplete-block designs (BIBDs) with repeated blocks are studied and constructed. We continue work initiated by van Lint and Ryser in 1972 and pursued by van Lint in 1973. We concentrate on constructing (v, b, r, k, λ)-BIBDs with repeated blocks, especially those with $\gcd(b, r, \lambda) = 1$ and $r \leq 20$. We obtain new bounds for the multiplicity of a block in terms of the parameters of a BIBD, and improvements to these bounds for a resolvable BIBD. This allows us to answer a question of van Lint about the sufficiency of certain conditions for the existence of a BIBD with repeated blocks.

1 Introduction

A balanced incomplete-block design (BIBD) with parameters (v, b, r, k, λ) may or may not have repeated blocks. We believe that the first published BIBDs with repeated blocks were those in series β_1 and series β_2 of Bose [3], but that paper was overlooked for some decades. To our knowledge, there has not been much study of necessary or sufficient conditions for the existence of a BIBD with repeated blocks and given parameters, nor on bounds for the multiplicity of a block in such a BIBD. For a (v, b, r, k, λ)-BIBD with m the maximum multiplicity of a block, Mann [23] proved in 1969 that $m \leq b/v$. In 1972, van Lint and Ryser [20] proved that in addition, if $m = b/v$, then m divides $\gcd(b, r, \lambda)$. They also gave constructions for BIBDs with repeated blocks, usually with $\gcd(b, r, \lambda) > 1$. In 1973, van Lint [19] considered tuples (v, b, r, k, λ) of positive integers satisfying $2 \leq k \leq v/2$, $\lambda(v-1) = r(k-1)$, $vr = bk$, $\lambda > 1$, $\gcd(b, r, \lambda) = 1$ and $b > 2v$, and asked whether for each such tuple (v, b, r, k, λ) there exists a BIBD with repeated blocks. He showed that this is indeed the case when $k \leq 4$, except possibly when $(v, k, \lambda) = (45, 4, 3)$ (for which we give an example with repeated blocks in section 4.1). Additionally in [19], van Lint tabulated all tuples (v, b, r, k, λ) satisfying his conditions with $v \leq 22$, and for many of these tuples constructed BIBDs with repeated blocks. In 1986, Rosa and Hoffman [31] completely determined, for each v for which there exists a $(v, b, r, 3, 2)$-BIBD, the set of numbers n for which there is such a BIBD with exactly n repeated blocks (see also [30]).

Recently, the present authors made an extensive catalogue of BIBDs with repeated blocks, with $\gcd(b, r, \lambda) = 1$ and $r \leq 20$. Many new BIBDs were found, and all gaps in van Lint's table when $r \leq 20$ have now been filled with examples of BIBDs with repeated blocks. In this paper, we give some constructions for BIBDs with repeated blocks and we present a summary of results from our catalogue.

We also prove that if D is a (v, b, r, k, λ)-BIBD and m is the multiplicity of a block of D, then, for every integer y,

$$m(k-y)(k-y-1) \leq (y+1)yb - 2ykr + k(k-1)\lambda,$$

and if D is resolvable then

$$m\left((y+1)yv/k - 2yk + k(k-1)\right) \leq (y+1)yb - 2ykr + k(k-1)\lambda.$$

This allows us to answer van Lint's question in the negative: for example, applying the first bound with $y = 1$ shows that there is no BIBD with repeated blocks and parameters $(35, 85, 17, 7, 3)$ (although there does exist a BIBD with these parameters [8, p.17]). At present, $(31, 93, 15, 5, 2)$ is the only parameter tuple in our catalogue for which the existence of a BIBD with repeated blocks is unsettled.

2 Definitions

We now define the key concepts used in this paper.

A *block design* is an ordered pair $(V, \mathcal{B})$, such that V is a finite non-empty set, whose elements are called *points*, and $\mathcal{B}$ is a finite non-empty multiset of nonempty subsets of V called *blocks*. It is important to note that $\mathcal{B}$ is a multiset, and we treat $\mathcal{B}$ as a list of the blocks, where repeats count, but order does not matter. The *multiplicity* of a block in $\mathcal{B}$ is the number of times it occurs in this list. (We remark, however, that as the blocks are sets, statisticians would call our block designs *binary* block designs (see [5]).) If each block has multiplicity 1 then we say the block design is *simple* [8]; otherwise we say that it has *repeated* blocks or is *non-simple*. A block design with repeated blocks has *multiplicity pattern* $a^i b^j \cdots$ if it has exactly a blocks with multiplicity i, exactly b blocks with multiplicity j, etc., and all other blocks have multiplicity 1. A *resolution* of a block design is a partition of its block multiset into submultisets called *parallel classes*, each of which forms a partition of the point-set.

Two block designs $D_1 = (V_1, \mathcal{B}_1)$ and $D_2 = (V_2, \mathcal{B}_2)$ are *isomorphic* if there is a bijection $\phi : V_1 \to V_2$ such that ϕ transforms $\mathcal{B}_1$ into $\mathcal{B}_2$ (*i.e.* when ϕ is applied to each element of each block in the list $\mathcal{B}_1$, we obtain the list $\mathcal{B}_2$ in some order). An *automorphism* of a block design D is an isomorphism from D to D. Thus, an automorphism of D is a permutation of the points of D which sends each block of D with multiplicity m to a block of D also with multiplicity m. The set of all automorphisms of D forms a group, the *automorphism group* $\mathrm{Aut}(D)$ of D. A block design D is *cyclic* if it has an automorphism which permutes all the points in a single cycle, and D is 1-*rotational* if it has an automorphism which fixes one point and permutes all the other points in a single cycle.

For t a non-negative integer and K a set of positive integers, a t-*wise balanced design*, or t-(v, K, λ) *design*, is a block design $(V, \mathcal{B})$, with $v = |V| \geq t$, such that each block has size in K and each t-subset of V is contained in a constant number $\lambda > 0$ of blocks. If D is a t-(v, K, λ) design in which all blocks have constant size k, then D is a *t-design*, or a *t-(v,k,λ) design*.

The t-(v,k,λ) designs with $t=2$ and $k<v$ are of great importance to both statisticians and combinatorialists and are called *balanced incomplete-block designs* or BIBDs. In a 2-(v,k,λ) design, each point is in exactly $r := \lambda(v-1)/(k-1)$ blocks, and there are exactly $b := vr/k$ blocks (counting repeats). It is customary (when $k<v$) to say that such a design has *parameters* (v,b,r,k,λ), or is a (v,b,r,k,λ)-BIBD.

3 Bounding the multiplicity of a block

A celebrated theorem states that there exist simple non-trivial t-designs for all $t \geq 0$ [34]. (Here, *non-trivial* means that the block (multi)set does not consist of all k-subsets of the point-set). However, there has not been much study of necessary or sufficient conditions for the existence of a t-design with repeated blocks and given parameters, or bounds on the multiplicity of a block in such a design.

For a (v,b,r,k,λ)-BIBD with m the maximum multiplicity of a block, Mann [23] proves that $m \leq b/v$, and van Lint and Ryser [20] prove that in addition, if $m = b/v$, then m divides $\gcd(b,r,\lambda)$. In particular, if a (v,b,r,k,λ)-BIBD has repeated blocks, then $b \geq 2v$ and if $b=2v$ then $\gcd(b,r,\lambda)$ is even.

In [19], van Lint considers *primitive repetition designs* or PRDs, which are (v,b,r,k,λ)-BIBDs with repeated blocks and $\gcd(b,r,\lambda)=1$. Without loss of generality, we may assume that a PRD has $k \leq v/2$. This is because there is no PRD with $k=v-1$, and a BIBD with $v/2<k<v-1$ is a PRD if and only if its complement design is. Thus, the following conditions hold on the parameters of a PRD:

$$2 \leq k \leq v/2,\ \gcd(b,r,\lambda)=1,\ \lambda(v-1)=r(k-1),\ vr=bk,\ \lambda>1,\ b>2v. \tag{1}$$

After listing the tuples (v,b,r,k,λ) of positive integers satisfying (1) (as observed in [19] one need consider only $3 \leq k \leq v/2-1$), with $v \leq 22$, van Lint constructs PRDs for many of these parameter tuples. He then remarks that "A question which naturally comes up is whether for every set of allowable parameters, *i.e.* those satisfying [(1)], there is a PRD". We now show that the answer to this question is no for infinitely many parameter tuples. We start by proving a general result.

Theorem 3.1 *Let $D = (V, \mathcal{B})$ be a block design with b blocks, let $B \subseteq V$, and let $k = |B| \geq 2$. Suppose that B is contained in m blocks $B_1, \ldots, B_m$, and that there are (at least) d further blocks $B_{m+1}, \ldots, B_{m+d}$ disjoint from B. Further suppose that each element of B is contained in exactly r blocks and that every 2-subset of B is contained in exactly λ blocks. Then, for every integer y,*

$$m(k-y)(k-y-1) \leq (y+1)y(b-d) - 2ykr + k(k-1)\lambda, \tag{2}$$

with equality holding if and only if each block other than $B_1, \ldots, B_{m+d}$ intersects B in exactly y or $y+1$ points.

Proof. Let x_i denote the number of blocks other than $B_1, \ldots, B_{m+d}$ which intersect B in exactly i points ($i = 0, \ldots, k$). We have

$$\begin{aligned} \sum x_i &= b - m - d, \\ \sum i x_i &= k(r-m), \\ \sum i(i-1)x_i &= k(k-1)(\lambda - m), \end{aligned}$$

from which we see that, for every integer y,

$$\sum (i-y)(i-y-1)x_i = k(k-1)(\lambda-m) - 2yk(r-m) + (y+1)y(b-m-d). \tag{3}$$

Since the left-hand side of (3) is non-negative, we obtain (2). The inequality (2) is exact if and only if the left-hand side of (3) is zero, which holds if and only if all x_i are zero, except possibly x_y or x_{y+1}. ■

Corollary 3.2 *Let D be a (v, b, r, k, λ)-BIBD, and let m be the multiplicity of a block B of D. Then, for every integer y,*

$$m(k-y)(k-y-1) \leq (y+1)yb - 2ykr + k(k-1)\lambda, \tag{4}$$

with equality holding if and only if each block other than a copy of B intersects B in exactly y or $y+1$ points.

Proof. Apply Theorem 3.1 with $d = 0$. ■

Corollary 3.3 *Let D be a resolvable (v, b, r, k, λ)-BIBD, and let m be the multiplicity of a block B of D. Then, for every integer y,*

$$m\left((y+1)yv/k - 2yk + k(k-1)\right) \leq (y+1)yb - 2ykr + k(k-1)\lambda, \quad (5)$$

with equality holding if and only if each block, other than a block in a parallel class with a copy of B, intersects B in exactly y or $y+1$ points.

Proof. Apply Theorem 3.1 with $B_{m+1}, \ldots, B_{m+d}$ being the blocks in parallel classes containing copies of B, other than the copies of B, so that $d = m(v/k - 1)$. ∎

Now, let n be a positive integer, and let

$$U_n = ((3n+1)(2n+1), (6n+5)(2n+1), 6n+5, 3n+1, 3).$$

Then, if $(v, b, r, k, \lambda) = U_n$, the conditions (1) are satisfied. However, applying (4) with $y = 1$ to these parameters gives

$$m \leq \frac{15n-1}{9n-3},$$

and so, if $n > 1$, there are no non-simple BIBDs with parameters U_n. We observe that there do exist non-simple BIBDs with parameters $U_1 = (12, 33, 11, 4, 3)$ (see section 6.2.3), but applying (5) with $y = 1$ shows that there are no resolvable non-simple BIBDs with these parameters. (This was also deduced in [24].)

3.1 Remarks

We originally stated and proved Theorem 3.1 for the (very useful) special case of $y = 1$. Peter J. Cameron then generalized our result to the given Theorem 3.1. We thank him for allowing us to include this improvement. As remarked by Cameron, values of y greater than 1 may also be useful. For example, for $(v, b, r, k, \lambda) = (40, 130, 39, 12, 11)$, the inequality (4) with $y = 1$ gives $m \leq 7$, with $y = 2$ gives $m \leq 4$, and with $y = 3$ gives $m \leq 2$ (whereas the Mann, van Lint and Ryser bound gives $m \leq 3$). For $(v, b, r, k, \lambda) = (24, 69, 23, 8, 7)$, the inequality (5) with $y = 2$ shows that there is no resolvable non-simple BIBD with these parameters.

4 Constructions for t-designs with repeated blocks

McSorley and Soicher [22] give a straightforward construction (the $*$-*construction*) which produces a t-design from a t-wise balanced design. Their method is based on the systematic substitution of each block of size k_i in the t-wise balanced design by an appropriate multiple of the trivial t-$(k_i, k, \binom{k_i-t}{k-t})$ design. More precisely, the input to the $*$-construction consists of positive integers t and k, and a t-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design $D = (V, \mathcal{B})$, with all block-sizes k_i occurring in D, and $1 \le t \le k \le k_1 < k_2 < \cdots < k_s$. The output is a t-$(v, k, n\lambda)$ design, $D^* = D^*(t, k)$, where

$$n = \mathrm{lcm}\left(\binom{k_i - t}{k - t} : 1 \le i \le s\right).$$

The point set of D^* is that of D, and the block multiset of D^* consists of, for each $i = 1, \ldots, s$ and each block $B \in \mathcal{B}$ of size k_i (including repeats), exactly $n/\binom{k_i-t}{k-t}$ copies of every k-subset of B. It is shown in [22] that $\mathrm{Aut}(D) \subseteq \mathrm{Aut}(D^*)$, and that if $\lambda = 1$ and $t < k$, then $\mathrm{Aut}(D) = \mathrm{Aut}(D^*)$.

We make some observations:

- if $t < k$ then the output t-$(v, k, n\lambda)$ design D^* will have repeated blocks *unless* both $s = 1$ (*i.e.* the input D is a t-design) and no two blocks of D agree in k or more points;
- $n \ge \binom{k_s-t}{k-t}$, and in particular, if $t < k < k_s$, then $n \ge k_s - t$;
- if $k = k_1$ then the maximum multiplicity of a block of D^* is at least n;
- if $t \ge 2$ and $k = k_1 < v$, then D^* has at least nv blocks (by the Mann inequality).

Fortunately, t-wise balanced designs appear to exist in profusion. Many can be made by the removal of points (and possibly blocks) from a t-design, or by the judicious addition of points (and possibly blocks) to a t-design. We shall see both of these methods in action below, and in section 6.

Construction 4.1 We recall the #-construction of [22]. Let $T = (X, \mathcal{B})$ be a t-(v, k, λ) design with $1 \le t < k < v$, and let $x \in X$. Let D be the block design with point-set $X \setminus \{x\}$ and whose block-list is obtained by taking $\mathcal{B}$ and removing x from every block containing it. Then D is a t-$(v-1, \{k-1, k\}, \lambda)$ design with both block-sizes occurring, and $D^*(t, k-1)$ is a t-$(v-1, k-1, (k-t)\lambda)$ design, having repeated blocks if $t < k-1$. This t-design is denoted by $T^{\#}(t, x)$, or simply $T^{\#}(x)$ if $t = 2$.

For example, let T be a $(19, 19, 9, 9, 4)$-BIBD. There are just six such designs (up to isomorphism), all available online [11]. Then, for each point x of T, $T^{\#}(x)$ is a $(18, 153, 68, 8, 28)$-BIBD, having multiplicity pattern 9^7.

Construction 4.2 We describe a construction we call the $+$-*construction*. Let $T = (X, \mathcal{B})$ be a t-(v, k, λ) design with $1 \le t \le k < v$, such that there is a submultiset $\mathcal{B}'$ of $\mathcal{B}$ with $(X, \mathcal{B}')$ a $(t-1)$-(v, k, λ) design (for example, if T is a 2-$(v, k, 1)$ design then $\mathcal{B}'$ must be a parallel class). Now let ∞ be a new point, let $Y = X \cup \{\infty\}$, and let $\mathcal{C}$ be the block-list obtained by taking $\mathcal{B}$ and inserting ∞ into every block in $\mathcal{B}'$. Then $D := (Y, \mathcal{C})$ is a t-$(v+1, \{k, k+1\}, \lambda)$ design with both block-sizes occurring, and $D^*(t, k)$ is a t-$(v+1, k, (k+1-t)\lambda)$ design, having repeated blocks if $t < k$. We denote this design by $T^+(t, \mathcal{B}')$, or simply $T^+(\mathcal{B}')$ if $t = 2$.

4.1 Solving the last remaining case for $k = 4$

van Lint [19] reports that $v = 45$ is the last remaining case for $k = 4$ where it is unknown whether there exists a BIBD with repeated blocks whose parameters satisfy (1). Such a BIBD would have parameters $(45, 495, 44, 4, 3)$, and we construct such a BIBD with repeated blocks, making use of the $*$-construction, as follows. Start with a resolvable $(40, 130, 13, 4, 1)$-BIBD (at least two such exist [4]), choosing five parallel classes $P_1, \dots, P_5$ in a resolution, and adding five new points $\infty_1, \dots, \infty_5$ to the point set, with ∞_i also being added to each of the ten blocks in P_i $(i = 1, \dots, 5)$. Then add in the block $(\infty_1, \dots, \infty_5)$. The result is a 2-$(45, \{4, 5\}, 1)$ design D, and $D^*(2, 4)$ is the required $(45, 495, 44, 4, 3)$-BIBD, having multiplicity pattern 80^3.

5 More constructions for BIBDs with repeated blocks and $k = 4$

We now give some constructions that we have found useful for writing down non-simple BIBDs with $k = 4$ and $r \le 20$. These BIBDs are obtained from certain cyclic "starter" designs, and in practice, for modest values of r, we have found that initial blocks for starter designs of the required types can be written down by inspection, although we are not making claims about their existence in general.

The non-simple BIBDs constructed in this section can be obtained by cyclic generation modulo x from given initial blocks, where $x = v - 1$ or $v - 4$. In some instances, an initial block generates only y blocks where $y = x/s$ for some $s > 1$. Then only a partial cycle (PC) of generated blocks is needed, which we denote by appending the suffix $_{PCy}$ to the initial block.

As is often the custom, we write blocks in round brackets rather than set brackets, and write B^i for a block B repeated i times.

5.1 Non-simple 2-$(v, 4, 2)$ designs, $v \equiv 7 \pmod{12}$

Construction 5.1 for $v > 7$.
Write $v = 6i + 1$, and denote the points by $0, 1, \ldots, v-2, \infty$. Write $\mathcal{S} = \{0, 1, \ldots, (v-3)/2\} \setminus \{(v-1)/3\}$. Obtain a set of distinct initial blocks B_j ($j = 1, 2, \ldots, i-1$), each containing 4 distinct points from $0, 1, \ldots, v-2$, whose differences modulo $v-1$ include each element of $\mathcal{S}$ exactly twice save that, for some x, the differences x and $(v-1)/2 - x$ each occur just once. Then the following is a 1-rotational 2-$(v, 4, 2)$ design with the multiplicity pattern $((v-1)/3)^2$:

$$\left.\begin{array}{lllll} B_1,\ B_2,\ \ldots\ ,\ B_{i-1} & & & & \\ (0 & x & (v-1)/2 & x+(v-1)/2)_{\mathrm{PC}[(v-1)/2]} & \\ (0 & (v-1)/3 & 2(v-1)/3 & \infty &)^2_{\mathrm{PC}[(v-1)/3]} \end{array}\right\} \bmod (v-1)$$

For example, we obtain the following designs:
2-$(19, 4, 2)$ with $i = 3$, $x = 2$:

$$(0\ \ 1\ \ 3\ \ 8)\ (0\ \ 1\ \ 4\ \ 14)\ (0\ \ 2\ \ 9\ \ 11)_{\mathrm{PC9}}\ (0\ \ 6\ \ 12\ \ \infty)^2_{\mathrm{PC6}} \bmod 18$$

2-$(31, 4, 2)$ with $i = 5$, $x = 4$:

$$\left.\begin{array}{l} (0\ \ 1\ \ 4\ \ 9)\ (0\ \ 1\ \ 7\ \ 9)\ (0\ \ 2\ \ 13\ \ 16)\ (0\ \ 6\ \ 13\ \ 18) \\ (0\ \ 4\ \ 15\ \ 19)_{\mathrm{PC15}}\ (0\ \ 10\ \ 20\ \ \infty)^2_{\mathrm{PC10}} \end{array}\right\} \bmod 30$$

5.2 Non-simple 2-$(v, 4, 3)$ designs, $v \equiv 12$ or $20 \pmod{24}$

Construction 5.2 for $v > 12$.
Write $v = 8i + 4$ and denote the points by $0, 1, \ldots, v-5, \infty_1, \infty_2, \infty_3, \infty_4$.

For the points $0, 1, \ldots, v-5$, take a cyclic 2-$(v-4, \{3,4\}, 3)$ design of the following form, where the blocks B_j $(j = 1, 2, \ldots, 2i-3)$ are distinct and of size 4, and the blocks A_j $(j = 1, 2, 3, 4)$ are each of size 3:

$$\left.\begin{array}{l} B_1,\ B_2,\ \ldots\ ,\ B_{2i-3} \\ A_1,\ A_2,\ A_3,\ A_4 \\ (0\ \ x\ \ 4i\ \ x+4i)_{\mathrm{PC}4i}\ (0\ \ 2i\ \ 4i\ \ 6i)_{\mathrm{PC}2i} \end{array}\right\} \bmod (v-4)$$

Then, if $x \neq 2i$ or $6i$, a 2-$(v, 4, 3)$ design with multiplicity pattern 1^3 is obtained by inserting ∞_j in A_j $(j = 1, 2, 3, 4)$ and appending three copies of the block $(\infty_1\ \ \infty_2\ \ \infty_3\ \ \infty_4)$. If $x = 2i$, then

$$(0\ \ x\ \ 4i\ \ x+4i)_{\mathrm{PC}4i} \;=\; (0\ \ 2i\ \ 4i\ \ 6i)^2_{\mathrm{PC}2i}$$

so the introduction of the elements ∞_j gives us a 2-$(v, 4, 3)$ design with the multiplicity pattern $(2i+1)^3$.

The two variants of the construction give, for example, the 2-$(20, 4, 3)$ designs

(a)

$$\left.\begin{array}{l} (0\ \ 1\ \ 3\ \ 12) \\ (0\ \ 1\ \ 7\ \ \infty_1)\ (0\ \ 2\ \ 5\ \ \infty_2)\ (0\ \ 2\ \ 5\ \ \infty_3)\ (0\ \ 6\ \ 12\ \ \infty_4) \\ (0\ \ 1\ \ 8\ \ 9)_{\mathrm{PC}8}\ (0\ \ 4\ \ 8\ \ 12)_{\mathrm{PC}4} \end{array}\right\} \bmod 16$$
$$(\infty_1\ \ \infty_2\ \ \infty_3\ \ \infty_4)^3$$

and

(b)

$$\left.\begin{array}{l} (0\ \ 2\ \ 5\ \ 11) \\ (3\ \ 4\ \ 10\ \ \infty_1)\ (7\ \ 13\ \ 14\ \ \infty_2)\ (1\ \ 12\ \ 15\ \ \infty_3)\ (6\ \ 8\ \ 9\ \ \infty_4) \\ (0\ \ 4\ \ 8\ \ 12)^3_{\mathrm{PC}4} \end{array}\right\} \bmod 16$$
$$(\infty_1\ \ \infty_2\ \ \infty_3\ \ \infty_4)^3$$

5.3 Non-simple 2-$(v, 4, 3)$ designs, $v \equiv 5$ or $9 \pmod{12}$

Construction 5.3 for $v > 9$.
Write $v = 4i+1$ and denote the points by $0, 1, \ldots, v-5, \infty_1, \infty_2, \infty_3, \infty_4$. For the points $0, 1, \ldots, v-5$, take a cyclic 2-$(v-4, 4, 3)$ design with at least $v-4$ pairs of repeated blocks (*i.e.* with at least one intitial block repeated), which can thus be written

$$\left.\begin{array}{llll} B_1, & B_2, & \dots, & B_{i-3} \\ (e & f & g & h)^2 \end{array}\right\} \bmod (v-4)$$

where the blocks $B_1, B_2, \dots, B_{i-3}$ may or may not all be distinct. Then a 2-$(v, 4, 3)$ design with at least a triple of repeated blocks is

$$\left.\begin{array}{l} B_1,\ B_2,\ \dots,\ B_{i-3} \\ (e\ \ f\ \ g\ \ \infty_1)\ (e\ \ f\ \ h\ \ \infty_2)\ (e\ \ g\ \ h\ \ \infty_3)\ (f\ \ g\ \ h\ \ \infty_4) \\ (\infty_1\ \ \infty_2\ \ \infty_3\ \ \infty_4)^3 \end{array}\right\} \bmod (v-4)$$

For example, we obtain the following designs:
2-$(17, 4, 3)$, with multiplicity pattern 1^3:

$$\left.\begin{array}{l} (1\ \ 2\ \ 4\ \ 10) \\ (1\ \ 2\ \ 4\ \ \infty_1)\ (1\ \ 2\ \ 10\ \ \infty_2)\ (1\ \ 4\ \ 10\ \ \infty_3)\ (2\ \ 4\ \ 10\ \ \infty_4) \\ (\infty_1\ \ \infty_2\ \ \infty_3\ \ \infty_4)^3 \end{array}\right\} \bmod 13$$

2-$(21, 4, 3)$, with multiplicity pattern 1^3:

$$\left.\begin{array}{l} (0\ \ 2\ \ 6\ \ 9)\ (0\ \ 1\ \ 3\ \ 15) \\ (0\ \ 1\ \ 6\ \ \infty_1)\ (0\ \ 1\ \ 10\ \ \infty_2)\ (0\ \ 6\ \ 10\ \ \infty_3)\ (1\ \ 6\ \ 10\ \ \infty_4) \\ (\infty_1\ \ \infty_2\ \ \infty_3\ \ \infty_4)^3 \end{array}\right\} \bmod 17$$

A fruitful variant of this construction, producing further designs, is available if the set of differences (mod $(v-4)$) provided by the repeated initial block $(e \quad f \quad g \quad h)$ is identical to that provided by the pair of blocks $(a \quad c \quad d)$ and $(f \quad g \quad h)$, for some a, c and d. In the final design, the initial blocks each containing just one of the points ∞_i can then be taken to be

$$(a\ \ c\ \ d\ \ \infty_1)\ (a\ \ c\ \ d\ \ \infty_2)\ (f\ \ g\ \ h\ \ \infty_3)\ (f\ \ g\ \ h\ \ \infty_4) \quad \bmod (v-4)\ .$$

In the above examples, these four blocks can then be taken to be

$$(5\ \ 6\ \ 9\ \ \infty_1)\ (5\ \ 6\ \ 9\ \ \infty_2)\ (2\ \ 4\ \ 10\ \ \infty_3)\ (2\ \ 4\ \ 10\ \ \infty_4) \qquad \bmod 13$$

and

$$(4\ \ 5\ \ 11\ \ \infty_1)\ (4\ \ 5\ \ 11\ \ \infty_2)\ (1\ \ 6\ \ 10\ \ \infty_3)\ (1\ \ 6\ \ 10\ \ \infty_4) \quad \bmod 17$$

respectively.

6 A catalogue of BIBDs with repeated blocks

Recently, the present authors made an extensive catalogue of BIBDs with repeated blocks, whose parameters satisfy (1) and have $r \le 20$, including many previously unknown BIBDs. The cyclic and the 1-rotational such examples with $v \le 22$, and certain other new examples with repeated blocks, were generated using the DESIGN package [32] for GAP [13]. In particular, the DESIGN package was used to construct and classify (up to isomorphism) BIBDs invariant under given groups constructed by GAP or stored in a GAP library, and DESIGN was also used to determine various properties of given BIBDs, such as resolvability. The nauty package [21] was used directly and indirectly for determining the automorphism groups of designs and for isomorphism testing of designs. Some further new examples of BIBDs with repeated blocks were found using a (modified) program of Krčadinac [18] implementing a "tabu search" for BIBDs, with the results filtered for isomorphism using pynauty [10]. Use was also made of the pydesign package [9] for combinatorial and statistical design theory. More new examples with large automorphism groups were constructed using the $*$-construction of [22], discussed in section 4, and further examples came from the constructions of section 5.

All gaps in van Lint's table when $r \le 20$ are now filled with examples of BIBDs with repeated blocks. Indeed, the only parameter tuple in our catalogue for which the existence of a BIBD with repeated blocks is unknown is (31,93,15,5,2). In addition, we do not know of any resolvable BIBDs with repeated blocks and parameters (35,119,17,5,2) or (20,76,19,5,4).

6.1 On the condition $\gcd(b, r, \lambda) = 1$

Given $(v, b_i, r_i, k, \lambda_i)$-BIBDs D_i $(i = 1, 2)$ on the same point set, we may construct a BIBD which has the same point-set as D_1 and D_2, and whose block list is obtained by concatenating those of D_1 and D_2. Denote this BIBD by $D_1 + D_2$. Note that there always exists a permutation ϕ of the points of D_2 which maps some block of D_2 to some block of D_1, in which case $D_1 + \phi(D_2)$ has repeated blocks and parameters $(v, b_1 + b_2, r_1 + r_2, k, \lambda_1 + \lambda_2)$. A special case of this is the *double* of D_1, that is, $D_1 + D_1$.

As observed in [19], if D is a (v, b, r, k, λ)-BIBD with $\gcd(b, r, \lambda) = 1$ then there is no positive integer $\lambda' < \lambda$ such that both $\lambda'(v - 1)/(k - 1)$ and $\lambda' v(v - 1)/(k(k - 1))$ are integers. In particular, D cannot be of the form

$D_1 + D_2$ (where D_1 and D_2 are BIBDs on the same point set as D). Thus, BIBDs with repeated blocks and $\gcd(b, r, \lambda) = 1$ are of particular interest. However, as noted in [19], the condition $\gcd(b, r, \lambda) = 1$ does exclude some potentially interesting cases, such as small "quasi-multiples" of non-existent designs. (For some quasi-doubles with repeated blocks, see [35] and [17].)

6.2 The parameters with $\gcd(b, r, \lambda) = 1$ and $r \leq 20$

We now give all parameter tuples (v, b, r, k, λ) satisfying (1) with $r \leq 20$, in lexicographic order of (r, k, λ). For each, we summarize what we know about BIBDs with repeated blocks and those parameters. For $(v, b, r, 3, 2)$-BIBDs, the reader is additionally referred to Rosa and Hoffman's important work [31].

The cyclic and the 1-rotational (v, b, r, k, λ)-BIBDs with $v \leq 22$, $r \leq 20$, $k \leq v/2$, and $\gcd(b, r, \lambda) = 1$ (together with many other BIBDs) are available online in [11], together with many of their combinatorial, group-theoretical and statistical properties, in a machine and human readable XML format [6], as part of the `DesignTheory.org` project [2].

6.2.1 (10,30,9,3,2)

For these parameters, there is no cyclic BIBD, and the unique 1-rotational BIBD is simple. However, all BIBDs with these parameters are known [7, 12], and are available online [11]. Precisely 566 of these 960 BIBDs (up to isomorphism) have repeated blocks, with multiplicity patterns:

$$(1^2)^{346},\ (2^2)^{142},\ (3^2)^{53},\ (4^2)^{15},\ (5^2)^{4},\ (6^2)^{4},\ (7^2)^{1},\ (9^2)^{1}$$

(*i.e.* there are 346 BIBDs having multiplicity pattern 1^2, 142 with multiplicity pattern 2^2, and so on).

The 2-(10,3,2) design given by Parker [26] has multiplicity pattern 1^2. The unique 2-(10,3,2) design with multiplicity pattern 9^2 (whose automorphism group has order 108) can be constructed as $T^+(P)$, where T is the affine plane of order 3 (the unique 2-(9,3,1) design) and P is any parallel class of T. This design is given by Hanani [14], van Lint[19] and Hedayat and Hwang [15].

6.2.2 (12,44,11,3,2)

These BIBDs have been completely classified by Östergård [25].

Precisely 4 of the 9 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, and these all have multiplicity pattern 4^2. Moreover, 3 of these 4 BIBDs with repeated blocks are resolvable. One of the resolvable designs has automorphism group of order 144; this design is given by Taylor and Carr [33].

An unresolvable 2-(12,3,2) design with multiplicity pattern 4^2 and automorphism group of order 432 can be constructed as $T^{\#}(x)$, where T is the projective plane plane of order 3 (the unique 2-(13,4,1) design) and x is any point of this plane.

One of the 2-(12,3,2) designs with multiplicity pattern 16^2 has automorphism group of order 576; this design is given by Preece [27].

6.2.3 (12,33,11,4,3)

This is the tuple U_1 of section 3, where it is shown that there is no resolvable non-simple BIBD with these parameters. (This is also shown by Morales and Velarde in [24], where all resolvable BIBDs with these parameters are classified.)

All 10 of the cyclic or 1-rotational BIBDs with these parameters are simple.

van Lint [19, pp.305–306] gives a design with multiplicity pattern 1^2 and trivial automorphism group. (The first line of the matrix B_4 on page 306 of [19] should read 0 1 0 0 .) Making use of a GAP program of Alexander Hulpke to construct permutation groups of low order, we found other non-simple BIBDs for this parameter tuple, in particular a design D with multiplicity pattern 6^2. The automorphism group of D is

$$A := \langle (2,3)(5,6)(8,9)(10,11),\ (1,2)(4,5)(7,8)(10,11) \rangle$$

of order 6, and the blocks of D are:

(0 1 2 3)	(0 1 4 7)	(0 1 10 11)
(0 2 5 8)	(0 2 10 11)	(0 3 6 9)
(0 3 10 11)	(0 4 5 9)	(0 4 6 8)
(0 5 6 7)	(0 7 8 9)	(1 2 4 7)
(1 2 5 8)	(1 3 4 7)	(1 3 6 9)
$(1\ 5\ 9\ 10)^2$	$(1\ 6\ 8\ 11)^2$	(2 3 5 8)
(2 3 6 9)	$(2\ 4\ 9\ 11)^2$	$(2\ 6\ 7\ 10)^2$
$(3\ 4\ 8\ 10)^2$	$(3\ 5\ 7\ 11)^2$	(4 5 6 10)
(4 5 6 11)	(7 8 9 10)	(7 8 9 11)

There are two further $(12, 33, 11, 4, 3)$-BIBDS with repeated blocks and automorphism group A. Both of these have multiplicity pattern 3^2.

6.2.4 (19,57,12,4,2)

Precisely 3 of the 22 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, and these all have multiplicity pattern 6^2.

6.2.5 (22,77,14,4,2)

Precisely 3 of the 43 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, and these all have multiplicity pattern 7^2. One of them, given by Bose [3], Hanani [14] and van Lint [19], has automorphism group of order 126.

6.2.6 (15,35,14,6,5)

All 8 of the cyclic or 1-rotational BIBDs with these parameters are simple.

Preece [28] gives a BIBD with multiplicity pattern 1^2 and with automorphism group of order 12.

6.2.7 (16,80,15,3,2)

Precisely 9 of the 122 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(15^2)^3,\ (16^2)^3,\ (30^2)^2,\ (32^2)^1.$$

The last of these is given by Preece [27] and van Lint [19].

A 2-(16,3,2) design with multiplicity pattern 30^2 can be constructed as $T^+(P)$, where T is a 2-(15,3,1) design (an STS(15)) having at least one parallel class P. In particular, if T is the 2-(15,3,1) design consisting of the points and lines of the projective space $PG(3, 2)$, and P is any parallel class (*i.e.* spread of the projective space), then $T^+(P)$ has an automorphism group of order 360, and is isomorphic to a BIBD in a family constructed many years ago by David H. Rees [29].

6.2.8 (11,55,15,3,3)

Precisely 8 of the 21 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(10^2)^2,\ (11^2)^2,\ (5^3)^2,\ (10^2 5^3)^2.$$

In [22], a 2-(11,3,3) design with multiplicity pattern 15^3 and automorphism group isomorphic to Sym(5) is constructed by applying the $*$-construction to the unique 2-$(11, \{3, 5\}, 1)$ design.

6.2.9 (31,93,15,5,2)

We do not know whether there exists a BIBD with these parameters and repeated blocks. However, we can show that there is no such cyclic or 1-rotational BIBD.

6.2.10 (16,48,15,5,4)

All of the 294 cyclic or 1-rotational BIBDs with these parameters are simple.

Using our version of the tabu search program of Krčadinac [18] we found some BIBDs with these parameters, multiplicity pattern 1^2 and trivial automorphism group. We give the blocks of one of these:

(0 1 2 10 13)	$(0\ 1\ 4\ 9\ 11)^2$	
(0 1 6 10 12)	(0 2 3 14 15)	(0 2 5 11 14)
(0 2 7 8 13)	(0 3 4 7 14)	(0 3 5 7 12)
(0 3 8 10 12)	(0 4 5 10 15)	(0 5 6 8 14)
(0 6 7 11 12)	(0 6 9 13 15)	(0 8 9 13 15)
(1 2 4 7 8)	(1 2 5 12 14)	(1 2 8 9 12)
(1 3 5 6 15)	(1 3 6 8 10)	(1 3 7 11 15)
(1 3 10 13 14)	(1 4 7 14 15)	(1 5 6 8 11)
(1 5 7 9 13)	(1 12 13 14 15)	(2 3 4 6 13)
(2 3 6 9 15)	(2 3 7 8 9)	(2 4 5 10 13)
(2 4 6 10 11)	(2 5 11 12 15)	(2 6 9 12 14)
(2 7 10 11 15)	(3 4 5 12 13)	(3 4 8 11 12)
(3 5 9 11 13)	(3 9 10 11 14)	(4 5 8 9 14)
(4 6 7 12 13)	(4 6 8 14 15)	(4 9 10 12 15)
(5 6 7 9 10)	(5 7 8 10 15)	(6 7 11 13 14)
(7 9 10 12 14)	(8 10 11 13 14)	(8 11 12 13 15)

6.2.11 (13,39,15,5,5)

Just one of the 76 cyclic or 1-rotational BIBDs with these parameters has repeated blocks, with multiplicity pattern 12^2. This BIBD, which has an automorphism group of order 240, is constructed as part of an infinite family in [1], and is given by van Lint [19].

6.2.12 (26,65,15,6,3)

In a BIBD with these parameters and repeated blocks, each repeated block has multiplicity 2, and since equality is achieved in (4) when $b = 65$, $r = 15$, $k = 6$, $\lambda = 3$, $m = 2$ and $y = 1$, each block in a repeated pair must meet each block not in this pair in 1 or 2 points (48 blocks in one point and 15 in two points). Using this information, it was straightforward to discover the following BIBD which is invariant under a $C_5 \times C_5$ and has multiplicity pattern 5^2:

$$(00\ \ 10\ \ 20\ \ 30\ \ 40\ \ \infty)^2\ \ \text{mod}\ (–,5)$$
$$(00\ \ 01\ \ 02\ \ 03\ \ 04\ \ \infty)\ \ \text{mod}\ (5,–)$$
$$(00\ \ 10\ \ 22\ \ 23\ \ 41\ \ 44)\ (00\ \ 20\ \ 32\ \ 33\ \ 41\ \ 44)\ \ \text{mod}\ (5,5)$$

The automorphism group of this BIBD has order 100.

For these parameters, we can show that there is no cyclic or 1-rotational BIBD with repeated blocks.

6.2.13 (17,68,16,4,3)

Precisely 4 of the 542 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(16^2)^2,\ (17^2)^2.$$

A 2-(17,4,3) design with multiplicity pattern 16^3 and automorphism group of order 1152 can be constructed as $T^+(P)$, where T is the affine plane of order 4 (the unique 2-(16,4,1) design) and P is any parallel class of T.

6.2.14 (18,102,17,3,2)

Precisely 12 of the 186 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(17^2)^{10},\ (34^2)^2.$$

Three of these BIBDs with repeated blocks are resolvable, and each of these has multiplicity pattern 17^2.

A 2-(18,3,2) design with multiplicity pattern 21^2 can be constructed as follows. Start with a with a resolvable 2-(15,3,1) design and a resolution (*i.e.* a KTS(15)), choosing three parallel classes P_1, P_2, P_3 in this resolution, and adding three new points $\infty_1, \infty_2, \infty_3$ to the point set, with ∞_i also being added to each of the five blocks in P_i ($i = 1, 2, 3$). Then add in the block ($\infty_1\ \infty_2\ \infty_3$). The result is a 2-$(18, \{3, 4\}, 1)$ design D, and $D^*(2, 3)$ is the required 2-$(18, 3, 2)$ design.

6.2.15 (35,119,17,5,2)

Bose [3] gives a BIBD with multiplicity pattern 7^2 and automorphism group of order 210.

We do not know of any resolvable non-simple BIBD for this parameter tuple.

6.2.16 (18,51,17,6,5)

All of the 582 cyclic or 1-rotational BIBDs with these parameters are simple.

John [16] gives a resolvable BIBD with multiplicity pattern 18^2 and automorphism group of order 2160. (In line -3 of page 641 of [16], the second value 10 is a misprint for 16.) He gives an explicit resolution whose stabiliser in the automorphism group of the design has order 360.

Making use of the GAP library of transitive permutation groups, we found an unresolvable non-simple BIBD D with these parameters, having multiplicity pattern 9^2 and automorphism group

$$A := \left\langle \begin{array}{l} (0,14)(1,4)(2,6)(3,5)(7,12)(8,11)(9,10)(13,16)(15,17), \\ (0,4,12)(5,10,16)(6,11,17)(7,8,9)(13,15,14) \end{array} \right\rangle$$

of order 54. The blocks of D are are obtained by concatenating the A-orbits of (0 1 2 4 7 16) (giving 27 blocks), $(0\ 2\ 6\ 8\ 11\ 14)^2$ (giving 9 pairs of repeated blocks), and (0 4 5 6 16 17) (giving 6 blocks).

6.2.17 (35,85,17,7,3)

This is the tuple U_2 of section 3, where it is shown that there is no BIBD with these parameters and repeated blocks.

6.2.18 (19,57,18,6,5)

Precisely 3 of the 1535 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, all with multiplicity pattern 19^2 and automorphism group of order 57.

6.2.19 (20,95,19,4,3)

Precisely 129 of the 10040 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(19^2)^{76},\ (20^2)^{10},\ (5^3)^{36},\ (5^3 20^2)^6,\ (25^3)^1.$$

Three of these designs are given by, respectively, Bose [3], Preece [28] and van Lint [19]. Moreover, 5 of these BIBDs with repeated blocks are resolvable, and these have multiplicity patterns:

$$(19^2)^1,\ (5^3)^1,\ (5^3 20^2)^2,\ (25^3)^1.$$

The resolvable BIBD with multiplicity pattern 19^2 has an automorphism group of order 19, and can be written as:

$$(1\ \ 2\ \ 4\ \ 8)^2\ (0\ \ 2\ \ 6\ \ 11)\ (0\ \ 3\ \ 10\ \ 11)\ (0\ \ 5\ \ 10\ \ \infty)\ \ \bmod 19$$

Its unique resolution can be obtained by cyclic generation modulo 19 of the parallel class:

$$[\ (1\ \ 2\ \ 4\ \ 8)\ (12\ \ 13\ \ 15\ \ 0)\ (3\ \ 5\ \ 9\ \ 14)\ (7\ \ 10\ \ 17\ \ 18)\ (6\ \ 11\ \ 16\ \ \infty)\ ]$$

The resolvable BIBD with multiplicity pattern 25^3 has an automorphism group of order 800, and can be written as:

$$(0\ \ 1\ \ 3\ \ 14)^3\ (0\ \ 5\ \ 10\ \ 15)^3_{\mathrm{PC5}}\ (0\ \ 4\ \ 8\ \ 12)\ \ \bmod 20$$

An unresolvable 2-(20,4,3) design with multiplicity pattern 5^3 and automorphism group of order 5760 can be constructed as $T^{\#}(x)$, where T is the projective plane plane of order 4 (the unique 2-(21,5,1) design) and x is any point of this plane.

6.2.20 (20,76,19,5,4)

Precisely 28 of the 10067 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, all with multiplicity pattern 19^2. None of these BIBDs with repeated blocks is resolvable, nor do we know any other resolvable non-simple BIBD for this parameter tuple.

6.2.21 (31,155,20,4,2)

The BIBDs with these parameters invariant under a group of order 93 are available online [11]. Just 2 of these 43 (cyclic) BIBDs have repeated blocks; both have multiplicity pattern 31^2.

6.2.22 (21,105,20,4,3)

Precisely 259 of the 26320 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(20^2)^{66},\ (21^2)^{192},\ (20^3)^1.$$

The 1-rotational BIBD with multiplicity pattern 20^3 (which has an automorphism group of order 80) and some of its "near resolutions" are constructed in an extended example in the introduction to the DESIGN package documentation [32]. Use is made of the $*$-construction of [22].

6.2.23 (11,55,20,4,6)

Precisely 90 of the 348 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(5^2)^6,\ (10^2)^{53},\ (11^2)^{31}.$$

Two of the designs with multiplicity pattern 11^2 are given by, respectively, Preece [28] and van Lint [19].

In [22], it is reported that there are four 2-$(11, \{4, 5\}, 2)$ designs where both block sizes occur, and these, using the $*$-construction, lead to four 2-(11,4,6) designs, each with multiplicity pattern 10^3, and respective automorphism group sizes 6, 8, 12, 120.

A 2-$(11, \{4, 5\}, 2)$ design with both block sizes occurring may be obtained by starting with a (symmetric) 2-$(16, 6, 2)$ design S, taking a 5-set Y of points on a block B, deleting B, and then removing the points in Y from the point-set and from each remaining block of S.

6.2.24 (17,68,20,5,5)

Precisely 49 of the 7260 cyclic or 1-rotational BIBDs with these parameters have repeated blocks, with multiplicity patterns:

$$(8^2)^{18},\ (16^2)^3,\ (17^2)^{26},\ (24^2)^2.$$

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