PRESENTATIONS OF SOME FINITE GROUPS

A dissertation submitted for the degree of

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To the memory of my father

Declaration of originality

This dissertation consists entirely of my own work except as follows: the concept of "fabulous" groups is due to J.H.Conway, the results on finite Coxeter groups given in Section I.3 are well-known, Proposition M.2(iii) is the result of joint work with A.J.E.Ryba, specifically indicated parts of Chapter Y report on joint work with J.H.Conway and S.P.Norton, and Proposition O.1 is due to J.H.Conway and R.A.Parker.

Acknowledgements

I am indebted to my supervisor, Professor J.H.Conway, for inventing the concept of fabulous groups, and for his constant enthusiasm and helpful advice. I also thank Richard Parker, Robert Wilson, Simon Norton and Alex Ryba for many interesting and useful discussions.

I am grateful to Robert Courtney, Andrew Harris and Jeremy Hutson for help in implementing my coset enumerator on a VAX computer.

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Index of Notation

(a) Miscellaneous

n	a cyclic group of order n
p^n	an elementary abelian group of order p^n
p^{1+2n}	an extraspecial group of order p^{1+2n}
D_{2n}	the dihedral group of order 2n
A_n	the alternating group of degree n
S_n	the symmetric group of degree n
$A \times B$	the direct product of A and B
A^k	$A \times \dots \times A$ (k times)
$A.B$	an extension of A by B
$A{:}B$	a split extension of A by B
$A{}^{\cdot}B$	a non-split extension of A by B
A wr B	the wreath product of A and B
$P(X,Y,\dots)$	the "subpresentation" involving $X,Y,\dots$
$G(X,Y,\dots)$	the group presented by $P(X,Y,\dots)$
$ncl(X,Y,\dots)$	the normal closure of $\langle X,Y,\dots\rangle$
//	end of proof

(b) Simple groups of Lie type

The following symbols always denote <u>simple</u> groups:

$L_{n+1}(q)$	$= A_n(q)$
$U_{n+1}(q)$	$= {}^2A_n(q)$
$O_{2n+1}(q)$	$= B_n(q)$
$S_{2n}(q)$	$= C_n(q)$
$O_{2n}^+(q)$	$= D_n(q)$
$O_{2n}^-(q)$	$= {}^2D_n(q)$
	$G_2(q)$
	${}^2E_6(q)$

(c) Sporadic simple groups

M_{24}, M_{23}, M_{22}, M_{12}, M_{11}	the Mathieu groups
Co_1, Co_2, Co_3	the Conway groups
Fi_{24}', Fi_{23}, Fi_{22}	the Fischer groups
J_1, J_2, J_3, J_4	the Janko groups
Suz	the Suzuki group
McL	the McLaughlin group
HS	the Higman-Sims group
O'N	the O'Nan group
Ru	the Rudvalis group
Ly	the Lyons group
$M = F_1$	the Fischer-Griess "Monster"
$B = F_2$	the "Baby Monster"
$Th = F_3$	the Thompson group
$HN = F_5$	the Harada-Norton group
$He = F_7$	the Held group

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CHAPTER I

INTRODUCTION

In this thesis we give new presentations for a large number of the sporadic simple groups. These presentations, given by Coxeter graphs together with additional relations, illustrate a great deal of subgroup structure and give "pictures" for understanding these groups. Many of the presentations appear to be very natural.

We give the first published presentations (known to the author) for the Conway groups $2 \cdot Co_1$, Co_2, and Co_3, the Fischer groups Fi_{23} and Fi_{22}, the Suzuki group Suz, the McLaughlin group McL, and the Higman-Sims group HS. We also give new presentations for the Mathieu groups M_{24}, M_{23}, M_{22}, M_{12}, and M_{11}, the Janko groups J_2 and J_1, and the Held group He.

Also included are presentations for various covering groups and automorphism groups of these, and for related finite groups. In fact, if a group G we present has a proper covering group A.G, where $|A|$ is odd, we usually give a presentation for A.G; furthermore, our presentation can often be used to prove that A is the odd part of the Schur multiplier of G. We have thus verified some results about the Schur multipliers of various simple groups. (We note that computing Schur multipliers is often a delicate business.)

We also investigate generators and relations which hold in groups for which we are unable to obtain a presentation. For example, in Chapter Y, we investigate generators and relations for the MONSTER group M, and we obtain presentations for many groups related to M. We also give conjectured presentations for M, the BABY MONSTER group B, and $3{}^{\cdot}Fi_{24}$. This work uses S.P.Norton's "monogram" for M. Monograms and "hemigrams" are described in Chapter S, where we give a monogram and a hemigram for Co_1.

In Chapter O we give a presentation for a group having the O'Nan group O'N as an image. This is used to prove that M_{11} is a subgroup of O'N, a fact which was previously unknown.

In an appendix we give two infinite sequences of presentations for finite groups. One sequence is of presentations for the alternating groups A_n, while the other is of presentations for generalizations of the Weyl group of B_n.

The standard collection of presentations for discrete groups is [9]. Presentations for simple groups of order up to 1,000,000 are given in [3] and [5]. For presentations of some sporadic groups not covered here, see [10] for the Janko group J_3, [11] for the Lyons group Ly, and [2] for the Janko group J_4. Presentations for simple groups are important tools for the identification of simple groups.

1. Fabulous Presentations

The form of our presentations was inspired by Conway's definition of "fabulous" presentations, and the discovery by Conway and Parker of the elegant fabulous presentation for J_1 given in Proposition 0.1. Fabulous presentations generalize the presentations for Coxeter groups, and also the presentations for complex reflection groups generated by order 2 reflections (see [8]).

Definition Given a presentation

$$\langle A,B,\ldots \mid R_1=S_1,R_2=S_2,\ldots\rangle,$$

and a subset $\{X,Y,\ldots\}$ of $\{A,B,\ldots\}$, we define the subpresentation $P(X,Y,\ldots)$ to be the presentation

$$\left\langle X,Y,\ldots \;\middle|\; \begin{matrix}\text{those } R_i=S_i \text{ such that both } R_i \text{ and } S_i\\ \text{are words in } X,Y,\ldots \text{ only}\end{matrix}\right\rangle.$$

We define $G(X,Y,\ldots)$ to be the group presented by $P(X,Y,\ldots)$.

Definition We inductively define the notion of a fabulous presentation.

The presentation $\langle A,B,\ldots \mid \;\rangle$ is fabulous.

Suppose that $\langle A,B,\ldots \mid R_1=S_1,R_2=S_2,\ldots\rangle$ is a fabulous presentation, and $\{X,Y,\ldots\}$ is a subset of $\{A,B,\ldots\}$. If $G(X,Y,\ldots)$ has an abelian normal subgroup N such that the quotient by N is finite, and VW^{-1} is a word representing an element of N, then

$$\langle A,B,\ldots \mid R_1=S_1,R_2=S_2,\ldots,V=W\rangle$$

is a fabulous presentation. In this context, V=W is a fabulous relation.

A group is called fabulous if it has a fabulous presentation. The term fabulous is used because we Factor out an ABelian group Under a Little (finite) group. Clearly, we can use this process to factor out a finite soluble group under a finite group.

<u>Definition</u> A <u>Coxeter group</u> is a group having a presentation of the form

$$\langle X_1,\ldots,X_n \mid 1=(X_iX_j)^{k_{ij}},\ k_{ii}=1 \rangle.$$

Note that a Coxeter group is fabulous. We start with $\langle X_1,\ldots,X_n \mid \rangle$, and adjoin the fabulous relations $1=X_i^2$. Then we consider the subpresentations $\langle X_i,X_j \mid 1=X_i^2=X_j^2\rangle$ $(i\neq j)$. These present the infinite dihedral group, in which the infinite cyclic group $\langle X_iX_j\rangle$ is a normal subgroup. We may therefore adjoin the fabulous relations $1=(X_iX_j)^{k_{ij}}$.

A zero generator fabulous group is trivial, and a one generator fabulous group is cyclic. We now show that a non-cyclic finite fabulous group must be presented as a quotient of a Coxeter group. This plainly follows from:

Proposition I.1 Let G and H be non-trivial groups. Then the free product $G*H$ has a non-trivial abelian normal subgroup if and only if $|G| = |H| = 2$.

Proof Throughout, g's and h's represent non-identity elements of G and H respectively.

If $|G| = |H| = 2$, then $G*H$ is the infinite dihedral group, which has the infinite cyclic group $\langle gh \rangle$ as a normal subgroup.

Now suppose (wlog) that $|G|>2$, and that $G*H$ has an abelian normal subgroup K, with $1 \neq k \in K$. We may take k to be an element not in G or H, since neither G nor H is normal in $G*H$. If k is of the form $hg_1h_1g_2h_2\ldots$, then replace k by k^h. Now k cannot be of the form $g_1h_1\ldots g_nh_ng$, for then $k^hk \neq kk^h$. Thus, we may assume that k is of the form $g_1h_1\ldots g_nh_n$. Now let $(1\neq)\ g\neq g_1$ (we can do this since $|G|>2$). Then k^gk has syllable length at most 4n while kk^g has syllable length $4n+1$, contradicting the assumption that K is an abelian normal subgroup of $G*H$. //

Remark Not all groups generated by involutions are fabulous. Consider $G \cong 1/2(D_{2p} \times D_{2p}) \cong (p \times p):2$, where $p \geqslant 5$ is a prime. Suppose we try to derive a fabulous presentation for G. We are forced to start by specifying a set of generating involutions, and the orders of pairwise products of these involutions. These orders are always p. Thus, any subpresentation of the presentation so far derived is for the trivial group, the cyclic group of order 2, D_{2p}, or for a hyperbolic reflection group. But no hyperbolic reflection group can be abelian-by-finite. (Such groups have exponential growth, while abelian-by-finite groups have polynomial growth.) //

Most of our presentations are fabulous; however, on occasion we include an interesting non-fabulous presentation.

2. Notation

Our typical presentation is given as a presentation for a Coxeter group, together with additional relations. The presentation for the Coxeter group is given in the usual way by a Coxeter graph: each node represents a generator A and the relation $1=A^2$; if two generators A,B are unjoined then AB=BA (equivalently $1=(AB)^2$); if they are joined by an unlabelled edge then $1=(AB)^3$; and if joined by an edge with label n then $1=(AB)^n$. A presentation for a group G is usually given with capital letters representing generators. We then use the corresponding small letters for their images in a homomorphic image of G. Relations enclosed in square brackets are redundant.

We also make use of the bar convention for homomorphic images, so that $\overline{G}$ is some image of G, and the image of $X \in G$ in $\overline{G}$ is called $\overline{X}$. (An image of a group always means a homomorphic image of that group.)

We use ATLAS notation (see [6]) for group names. Single letter names are used for generically simple groups (with, for example, $L_3(4)$ and $O_8^+(3)$ denoting simple groups). Also:

n denotes the cyclic group of that order;

p^{1+2n} denotes an extraspecial group of that order;

D_{2n} is the dihedral group of order 2n;

A^k denotes $A \times \ldots \times A$ (k times).

A group (of shape) A.B is an arbitrary extension of A by B; that is, A is a normal subgroup and the quotient by A is B. Furthermore, A:B is a split extension of A by B, and $A^{\cdot}B$ is a non-split extension of A by B.

As a consequence of this notation, we make statements such as "the normal $2^2 = \ldots$" to mean "the normal four-group = ...". In this way, we often implicitly give a presentation for G, by giving generators for the normal subgroup N of an extension N.G.

We also use ATLAS names for conjugacy classes, in which elements of order n are labelled nA, nB, nC,... in descending order of their centralizer orders.

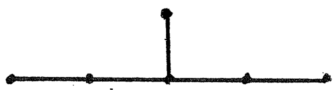
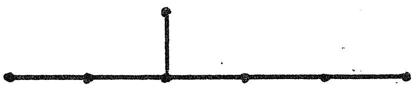
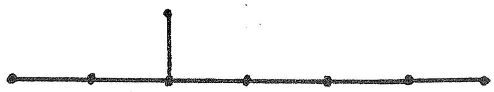
For a group G, $ncl_G(X,Y,\ldots)$ means the normal closure in G of $\langle X,Y,\ldots\rangle$. We omit the subscript when there is no danger of confusion.

3. Coxeter groups

For the reader's convenience we list the finite Coxeter groups. These are just the finite real reflection groups, which were enumerated by Coxeter in [7]. These groups have faithful irreducible real representations, so if a finite Coxeter group has a non-trivial centre, this centre has order 2. For each finite Coxeter group C with a non-trivial centre, the central involution is of the form $\pi^{h/2}$, where π is the product (in any order) of the generating involutions of C, and h is the Coxeter number of C.

In the following table, a Coxeter group C is given by its Coxeter graph, followed by names for C, followed by a generator of the centre of C if this is non-trivial. The notation is as follows: we assume that a diagram containing "..." has n nodes, W(X) denotes the Weyl group of X, and π is the product (in any order) of the nodes (generators) in the Coxeter graph.

The Finite Coxeter Groups

Coxeter Graph	Group Name(s)	Centre
	$W(A_n) \cong S_{n+1}$	no centre if $n \geqslant 2$
	$W(B_n) = W(C_n) \cong 2^n{:}S_n$	π^n
	$W(D_n) \cong 2^{n-1}{:}S_n$	π^{n-1} for n even $(n \geqslant 4)$
	$W(E_6) \cong U_4(2){:}2$	no centre
	$W(E_7) \cong 2 \times S_6(2)$	π^9
	$W(E_8) \cong 2 \cdot O_8^+(2){:}2$	π^{15}
	$W(F_4) \cong 2^{1+4}.(S_3 \times S_3)$	π^6
	D_{2r}	$\pi^{r/2}$ for r even
	$2 \times A_5$	π^5
	$2 \cdot (A_5 \times A_5){:}2$	π^{15}

Similarly there are the Euclidean Coxeter groups (see e.g. [9]) which are of the shape $\mathbb{Z}^n{:}C$, where C is a finite Coxeter group having an n node Coxeter graph. These Euclidean Coxeter groups can be used in fabulous presentations by adjoining a relation which makes the normal subgroup of translations have finite order. We have done little to investigate the use of quotients of Euclidean Coxeter groups in fabulous presentations, except for our work on the MONSTER monogram.

4. Some remarks on coset enumeration

Throughout this thesis we make use of computer coset enumeration. Coset enumeration is a fundamental computational group theory algorithm which is described in [1], [4], and [9]. Coset enumeration would ultimately determine the index of a subgroup H of a finitely presented group G when this index is finite, but there can be no general bound on the time or space required to do this. Thus, given a fixed amount of time and space, a coset enumeration may fail to give an answer, even when G is the trivial group! The design of a coset enumerator can significantly change the time and space (computer memory) required for a given enumeration (see [4]). Coset enumeration is also used to produce permutation generators of G on the cosets of H.

I programmed a modified version of the enumerator described in [4], using the FORTRAN language. One feature of my enumerator, suggested by R.A.Parker, is to perform a "lookahead" (see [4]) every time the number of cosets defined is twice the number defined after the last lookahead was completed. The enumerator, together with a PASCAL program to preprocess the input to the enumerator, and PASCAL programs to calculate with permutations, were used to perform all the computations described in this thesis. (Most of these calculations were done on the Cambridge IBM 3081.) The interested reader who has access to a coset enumerator (e.g. through the group theory language CAYLEY) should be able to verify most of my calculations.

The practical limit at present for computer coset enumeration is in the order of 100,000's of cosets. The largest number of cosets that I have enumerated is 196,560 (see Chapter L). My coset enumerator appears to work well with virtual store, which is necessary for very large enumerations.

5. On determining the group defined by a presentation

Finding a good presentation for a group G can be difficult, but given a candidate P for a presentation for G, we often use the following techniques to decide whether P presents G.

Where possible, we use coset enumeration to determine an upper bound n on the order of the group presented by P. Then we try to prove that the known group G, of order n, has generators satisfying P.

For the alternating groups and Mathieu groups we give explicit permutation generators satisfying the appropriate presentation. For many other groups we use theoretical arguments, based on subgroup structure, to find generators satisfying a given presentation.

For some groups we use the permutations resulting from a coset enumeration to prove that the group presented is simple of a given order, and then we can appeal to the classification of finite simple groups. (For example, the sporadic simple groups are characterized by their orders.) We use the following standard technique to show that some given permutation group G is simple.

Suppose that G acts transitively and faithfully on $\Omega = \{1,...,n\}$, and let $H = stab_G(1)$. If G acts imprimitively on Ω, then a union of orbits of H on Ω, including the orbit $\{1\}$, is a non-trivial proper imprimitivity block. In the cases we deal with, the orbit-lengths of H on Ω will imply that G is primitive. Thus, suppose that G is primitive on Ω, and that N is a proper normal subgroup of G. Suppose N is non-trivial. Then N is transitive, and if H has trivial intersection with N (e.g. if H is simple), then N is a regular minimal normal subgroup of G. But if n is not the order of a characterically simple group, we must conclude that N is trivial, and so G is simple.

6. On presentations of covering groups

The following propositions give some sufficient conditions for N.G to be a quotient of a presented group, when G is already known to be a quotient of that group.

Proposition I.2 Let C be the Coxeter group with presentation

$$\langle X_1,\ldots,X_n \mid 1=(X_iX_j)^{k_{ij}},\ k_{ii}=1 \rangle,$$

and let G be a group generated by involutions $x_1,\ldots,x_n$. Suppose that N is an odd order normal subgroup of G, and that all but at most one of $x_1,\ldots,x_n$ centralize N.

If G/N is an image of C by a homomorphism taking X_i to Nx_i, then G is an image of C by a homomorphism taking X_i to x_i $(i=1,\ldots,n)$.

Proof We show that the relations of C are satisfied by the involutions $x_1,\ldots,x_n$. Suppose not. Then for some i,j, $1=(X_iX_j)^{k_{ij}}$, but $t=(x_ix_j)^{k_{ij}}$, $1\neq t \in N$. In the dihedral group generated by involutions x_i,x_j, t is inverted by both x_i and x_j. Since t is of odd order, $t \neq t^{-1}$, contradicting the assumption that one of x_i,x_j must centralize N. //

Proposition I.3 Let K be the group having the presentation

$$\langle X_1,\ldots,X_n \mid R_1=S_1, R_2=S_2,\ldots \rangle,$$

and let $\overline{K}$ be an image of K, obtained by adjoining an additional relation $1=W(\overline{X}_1,\ldots,\overline{X}_n)$ to the presentation for K, where $1=W(X_1,\ldots,X_n)^p$ holds in K (p prime). Let G be a group generated by elements $x_1,\ldots,x_n$ which satisfy the presentation for K, and let N be a normal subgroup of G, such that p does not divide |N|.

If G/N is an image of $\overline{K}$ by a homomorphism taking $\overline{X}_i$ to Nx_i, then G is an image of $\overline{K}$ by a homomorphism taking $\overline{X}_i$ to x_i $(i=1,\ldots,n)$.

Proof Suppose not. Then $t = W(x_1,\ldots,x_n)$, where $1 \neq t \in N$. But then t would have order p, contradicting the assumption that p does not divide |N|. //

We often apply Proposition I.2, and then repeatedly apply Proposition I.3 with p=2, to be able to assert that a group "odd.G" is an image of some group X, given that G is an image of X.

Now suppose that G has a proper covering group A.G (i.e. A is central in A.G, and A is contained in (A.G)'); where |A| is odd. If G is a quotient of a group X presented by a Coxeter graph together with additional relations which successively kill elements of order 2, then A.G is an image of X. Moreover, if the above X is a group of shape B.G, where B is central in B.G, and |B| is odd, then B is the odd part of the Schur multiplier of G. (Since X is generated by involutions, and so has a commutator quotient of shape 2^k, B is contained in X'.)

7. References

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CHAPTER M

THE MATHIEU GROUPS

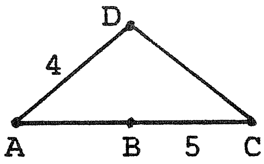
1. Introduction and preliminaries

In this chapter we give presentations for M_{11}, $2^{\cdot}M_{12}$, $2^{\cdot}M_{12}:2 \cong 2^{\cdot}Aut(M_{12})$, $3^{\cdot}M_{22}$, $3^{\cdot}M_{22}:2 \cong 3^{\cdot}Aut(M_{22})$, M_{23}, and M_{24}. The only other presentation of M_{24} of which the author is aware is due to Todd [3]. Todd derives presentations for M_{11}, M_{12}, and M_{22}, M_{23}, M_{24} by working up stabilizer chains, and he uses many short relations. Our presentations use fewer relations than Todd's do. The presentation for M_{24} in Proposition M.3 illustrates the maximal subgroups $2^4:A_8$, $M_{22}:2$, and M_{23}, while the presentation for M_{24} in Proposition M.4 is based on the maximal subgroup $M_{12}:2$. The reader who is not familiar with the maximal subgroups of M_{24} is referred to [2].

We start with presentations for $2^4:3^{\cdot}A_6$ and $2^{1+8}:3^{\cdot}A_7$. These presentations illustrate many relations used in defining $3^{\cdot}M_{22}$, M_{22}, and M_{23}, and in Chapter L, $3^{\cdot}McL$ and McL.

Proposition M.1

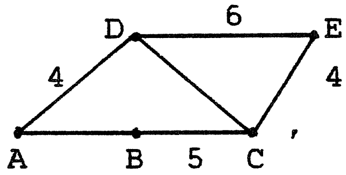
(i) $2^4{:}3^{\cdot}A_6 \cong (3 \times 2^4).A_6 \cong$

$\langle$ 

$, 1=(DAB)^3=(ABC)^5=(BCD)^5 \rangle$.

The central $3 = \langle (ABCD)^8 \rangle$; $2^4 \cong ncl((CDA)^3)$.

(ii) $2^{1+8}{:}3^{\cdot}A_7 \cong 3.2.2^4.2^4.A_7 \cong$

$\langle$ 

,

$A=(CE)^2$, $B=(DE)^3$, $1=(DAB)^3=(BCD)^5$, $[1=(ABC)^5]$ $\rangle$.

The central $3 = \langle (ABCD)^8 \rangle = \langle (BCDE)^8 \rangle = \langle (CDE)^{14} \rangle$;

$2^5 \cong ncl((BCDE)^{12}) \neq ncl((CDE)^{21}) \cong 2^5$;

$2^{1+8} \cong ncl((CDA)^3)$.

Proof The following permutations satisfy the presentation in (ii):

$a = (12)(45)$

$b = (12)(34)$

$c = (14)(25)$

$d = (34)(56)$

$e = (12)(67)$.

We observe that $\langle a,b,c,d \rangle \cong A_6$, and $\langle a,b,c,d,e \rangle \cong A_7$.

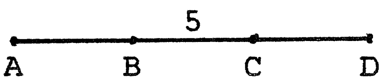
The results of this proposition now follow from various coset enumerations, and computer calculations using the resulting permutations. We omit the details. //

2. Presentations for the Mathieu groups

We start with presentations for $L_2(11)$, M_{11}, and $2^{\cdot}M_{12}$.

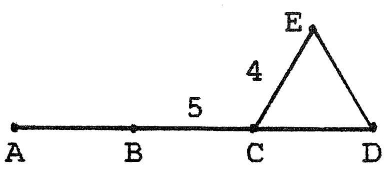
Proposition M.2

(i) $L_2(11) \cong$

$\langle$ 

, $1=(ABC)^5$, $[1=(BCD)^5]$ $\rangle$.

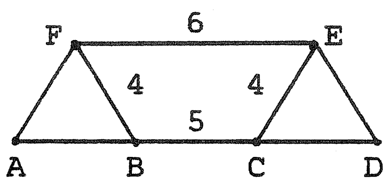
(ii) $M_{11} \cong$

$\langle$ 

,

$A=(CE)^2$, $[1=(ABC)^5=(BCD)^5]$ $\rangle$

(iii) (Ryba and Soicher) $2^{\cdot}M_{12} \cong$

$\langle$ 

, (1)

$A=(CE)^2$, $D=(BF)^2$, $1=(AFED)^3$ $\rangle$.

The central $2 = \langle (BCEF)^6 \rangle$.

Proof The following permutations on $\{\infty,0,1,2,3,4,5,6,7,8,9,X\}$ satisfy the presentation in (iii):

$$a = (08)(2X)(56)(79)$$

$$b = (08)(12)(37)(45)$$

$$c = (07)(14)(56)(89)$$

$$d = (08)(34)(57)(69)$$

$$e = (08)(14)(25)(6X)$$

$$f = (\infty X)(06)(57)(89).$$

The containment of these permutations in M_{12} can be checked on the MINIMOG (see [1]) using the "modulo 11" labelling given below:

0	3	∞	2
5	9	8	X
4	1	6	7

It is easy to check that a,b,c,d,e (fixing ∞) generate M_{11}, and a,b,c,d,e,f generate M_{12}.

We consider the presentation in (i), and enumerate 11 cosets of $\langle A,B,C\rangle$ ($\cong A_5$) in $\langle A,B,C,D\rangle$. Then we consider the presentation in (ii), and enumerate 12 cosets of $\langle A,B,C,D\rangle$ in $\langle A,B,C,D,E\rangle$. We remark that P(A,B,C,E) presents S_5, where $1=(ABC)^5$ is an implied relation. Results (i) and (ii) follow. For (iii), we enumerate 24 cosets of $\langle A,B,C,D,E\rangle$ in $\langle A,B,C,D,E,F\rangle$, and calculate that BCEF has order 12, while bcef has order 6. //

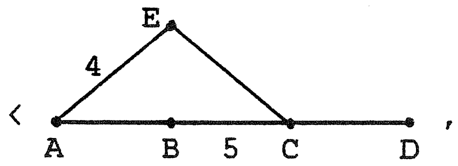
<u>Remarks</u> The triad {1,3,4} is fixed by $\langle a,c,d,e,f\rangle$, and in fact, $\langle a,c,d,e,f\rangle \cong M_9{:}S_3$, the triad stabilizer in M_{12}. Furthermore, P(A,C,D,E,F) in (1) is found to present $M_9{:}S_3$.

We can extend (1) to a presentation for $2^{\cdot}M_{12}:2$ by adjoining an involution which preserves the symmetry of (1), i.e. adjoin the generator T such that $1=T^2$, $A^T=D$, $B^T=C$, and $E^T=F$. Note that T interchanges two M_{11}'s not conjugate in $\langle A,\ldots,F\rangle$, and similarly interchanges two $M_9:S_3$'s.

Now we give presentations for the large Mathieu groups.

Proposition M.3

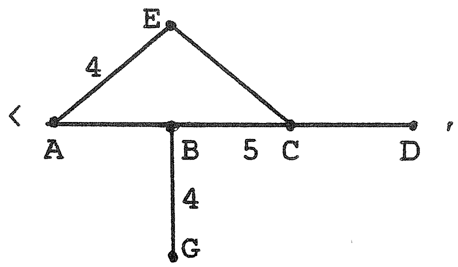
(i) $3^{\cdot}M_{22} \cong$



$1=(EAB)^3=(ABC)^5=(BCE)^5=(AECD)^4$, $[1=(BCD)^5]$ $\rangle$.

The central $3 = \langle (ABCE)^8 \rangle$.

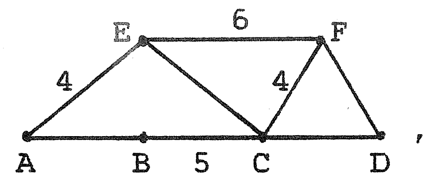
(ii) $3^{\cdot}M_{22}:2 \cong$



$E=(BG)^2$, $1=(ABC)^5=(AECD)^4$, $[1=(EAB)^3=(BCE)^5]$ $\rangle$.

The normal $3 = \langle (ABCE)^8 \rangle$.

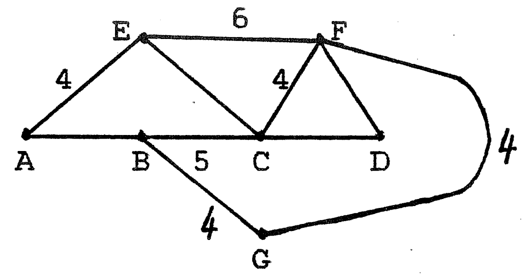
(iii) $M_{23} \cong$

$\langle$ 

,

$A=(CF)^2, B=(EF)^3, 1=(EAB)^3=(BCE)^5=(AECD)^4=(BCEF)^4, [1=(ABC)^5]\rangle$.

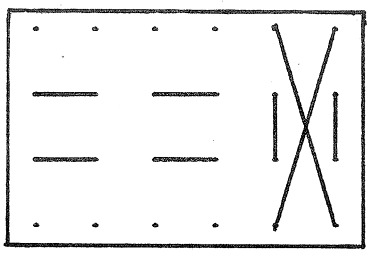
(iv) $M_{24} \cong$

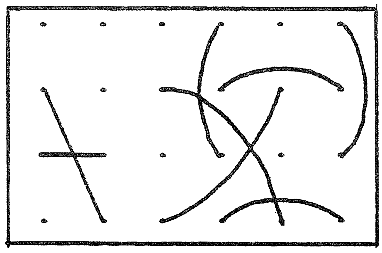
$\langle$ 

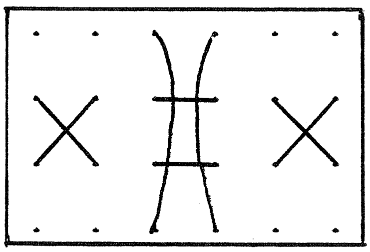
,

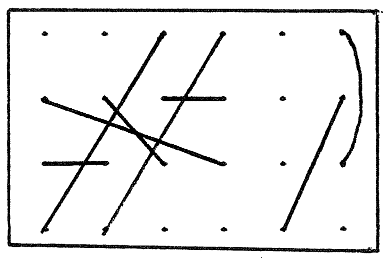
$A=(CF)^2,\ B=(EF)^3,\ E=(BG)^2,\ 1=(AECD)^4=(BCEF)^4=(BAEFG)^3\ \rangle$.

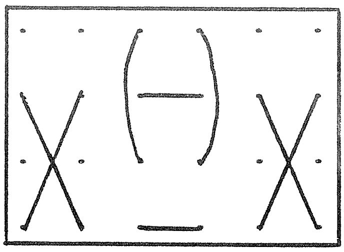
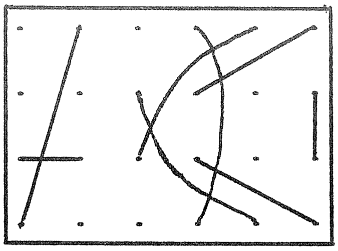
<u>Proof</u> Consider the following permutations in M_{24} (these are given in MOG notation (see [1]): this MOG differs from Curtis's original MOG by a left-right reflection).

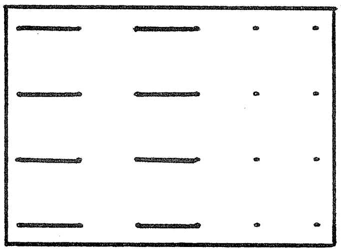
a = 

b = 

c = 

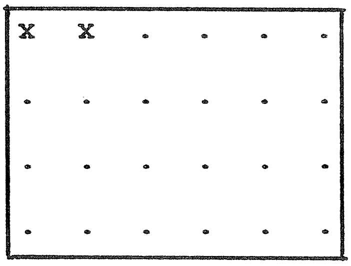
d = 

e =  f = 

g = 

Now a,...,g satisfy the presentation in (iv).

The duad



(2)

is fixed pointwise by ⟨a,b,c,d,e⟩, and setwise by ⟨a,b,c,d,e,g⟩. It follows that ⟨a,b,c,d,e⟩ $\cong M_{22}$, and ⟨a,b,c,d,e,g⟩ $\cong M_{22}{:}2$ (note the maximal $L_2(11)$ = ⟨a,b,c,d⟩ in ⟨a,b,c,d,e⟩). The results of Section I.6 show that $3^{\cdot}M_{22}$ is an image of the group presented in (i), and $3^{\cdot}M_{22}{:}2$ is an image of the group presented in (ii). Result (i) now follows by enumerating 77 cosets of ⟨A,B,C,E⟩ ($\cong 3.2^4.A_6$) in ⟨A,B,C,D,E⟩. In (ii), the relation $E=(BG)^2$ shows that $B^G=EB$. Thus G normalizes ⟨A,B,C,D,E⟩, and result (ii) follows.

Now $\langle a,\ldots,f\rangle$ fixes just one point of the duad (2), and thus $\langle a,\ldots,f\rangle \cong M_{23}$. To establish (iii), we enumerate 23 cosets of $\langle A,\ldots,E\rangle$ in $\langle A,\ldots,F\rangle$ (the relation $1=(ABCE)^8$ is implied).

The group $\langle a,\ldots,g\rangle$ is transitive, and so must be M_{24}. Result (iv) follows easily by coset enumeration. //

<u>Remark</u> The octad

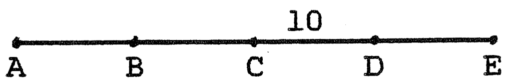
x	x	.	.	.	.
x	x	.	.	.	.
x	x	.	.	.	.
x	x	.	.	.	.

is fixed by $\langle a,b,c,e,f,g\rangle$, and it follows easily that $\langle a,b,c,e,f,g\rangle \cong 2^4{:}A_8$, the octad stabilizer in M_{24}. Coset enumeration shows that (in (iv)), $G(A,B,C,E,F,G) \cong 2^4{:}A_8$.

We now give a concise presentation for M_{24}, based on the maximal subgroup $M_{12}{:}2$.

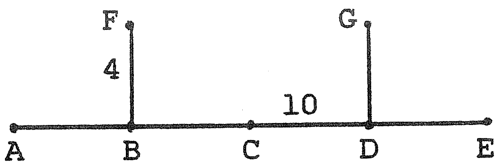
<u>Proposition M.4</u>

(i) In the presentation just below, $G(A,B,C,D) \cong (3 \times L_2(11)){:}2$, and $G(A,B,C,D,E) \cong (3 \times M_{12}){:}2$ (in both of these groups, the normal $3 = \langle (BCD)^{11}\rangle$).

$\langle$ 

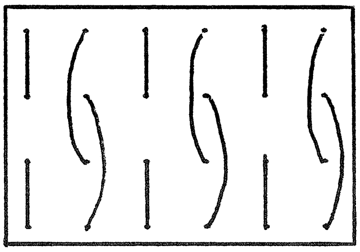
, $A=(CD)^5=(CDE)^5$ $\rangle$

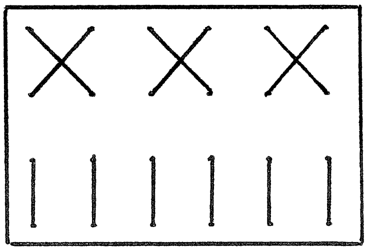
(ii) $M_{24} \cong$

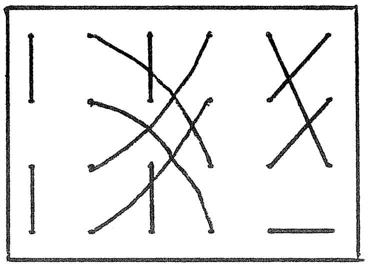
$\langle$  ,

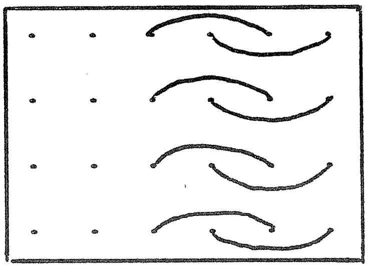
$A=(CD)^5=(CDE)^5$, $F=(CDG)^5$, $E=(BCF)^3$, $1=(ABF)^3$, $[1=(BCD)^{11}]$ $\rangle$.

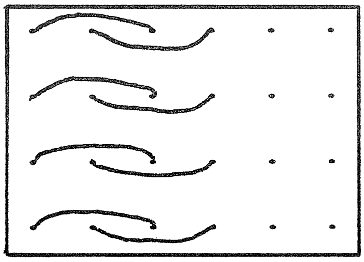
<u>Proof</u> The following permutations (in MOG notation) generate M_{24}, and satisfy the presentation in (ii):

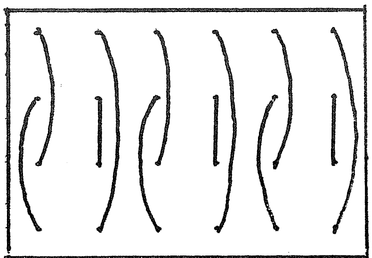
a = 

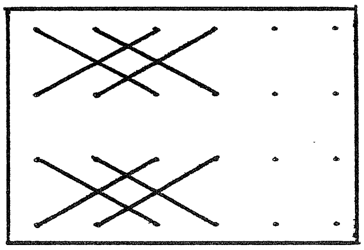
b = 

c = 

d = 

e = 

f = 

g = 

Moreover, a,b,c,d,e generate $M_{12}:2$, stabilizing the dodecad-pair:

x	x	x	x	x	x
.	.	.	.	.	.
x	.	x	.	x	.
.	x	.	x	.	x

It is now routine to prove (i) and (ii). We use coset enumeration, and the facts that any subgroup of $M_{12}:2$ of index 144 is isomorphic to $L_2(11):2$ and that M_{24} cannot be properly triply covered. //

3. References

1. J.H.Conway, Hexacode and tetracode - MOG and MINIMOG, in "Computational Group Theory", Academic Press, London, 1984.

2. R.T.Curtis, The maximal subgroups of M_{24}, Math. Proc. Cambridge Philos. Soc. 81 (1977), 185-192.

3. J.A.Todd, Abstract definitions for the Mathieu groups, Quart. J. Math. Oxford (2) 21 (1970), 421-424.

CHAPTER L

LEECH LATTICE GROUPS

1. Introduction

We now investigate various sublattice stabilizers in the automorphism group of the Leech lattice. In particular, we give presentations for:

the Conway groups $2{\cdot}Co_1$, Co_2 and Co_3,

the McLaughlin group McL (via its triple cover $3{\cdot}McL$),

the Higman-Sims group HS, and Aut(HS) $\cong$ HS:2.

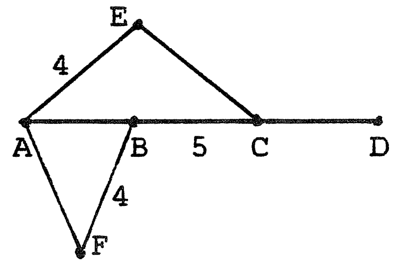
The Conway groups, McL, and HS are best described in terms of the Leech lattice (see [1] and [3]). The Leech lattice is the unique 24-dimensional even unimodular lattice which has no norm 2 vectors (see [2]). The automorphism group of the Leech lattice is $2{\cdot}Co_1$, which has order 8,315,553,613,086,720,000 = $2^{22}.3^9.5^4.7^2.11.13.23$. In this group, the stabilizer of a norm 4 vector is Co_2, of order 42,305,421,312,000 = $2^{18}.3^6.5^3.7.11.23$, and the stabilizer of a norm 6 vector is Co_3, of order 495,766,656,000 = $2^{10}.3^7.5^3.7.11.23$ (by vector we mean Leech lattice vector). The pointwise stabilizer of two norm 4 vectors which sum to a norm 6 vector is McL, which has order 898,128,000 = $2^7.3^6.5^3.7.11$, and the pointwise stabilizer of two norm 6 vectors which sum to a norm 4 vector is HS, which has order

$44,352,000 = 2^9.3^2.5^3.7.11.$

2. Presentations for HS and HS:2

Like M_{23}, HS and McL can be constructed by extending an $L_2(11)$ in M_{22} to an M_{11}. This is illustrated in the presentations which follow.

Propostition L.1 HS ≅

<  , (1)

$D=(BF)^2=(FAE)^3$, $1=(EAB)^3=(BCE)^5$,

$[1=(ABC)^5=(BCD)^5=(AECD)^4=(ABCE)^8]$ >.

Proof We first remark that

$W(F_4)/2 \cong$ <  ,

$D=(FAE)^3$, $[1=(FAEC)^6=(AECD)^4]$ >

The other implied relation that is not immediately clear is $1=(ABCE)^8$. Let G be the group presented in (1), and let $\overline{G}$ = G/ncl($(ABCE)^8$). We enumerate 100 cosets of $\langle\overline{A},\overline{B},\overline{C},\overline{D},\overline{E}\rangle$ (≅ M_{22}) in $\overline{G}$, and calculate that $\langle\overline{A},\overline{B},\overline{C},\overline{D},\overline{E}\rangle$ has orbit-lengths 1,22,77 on these 100 points. The Higman-Sims group is defined as the rank 3 group on 100 points, with point stabilizer M_{22}, and suborbit-lengths 1,22,77; therefore $\overline{G}$ ≅ HS. We then consider G, and enumerate 100 cosets of

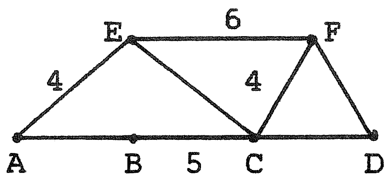
$\langle A,B,C,D,E\rangle$ in G. Since HS cannot be properly triply covered, this means that $\langle A,B,C,D,E\rangle \cong M_{22}$, not $3{\cdot}M_{22}$. Therefore, $G \cong \overline{G} \cong$ HS, and the proof of the proposition is complete. //

A presentation for HS:2 is obtained by adjoining an involution T to the presentation (1) above, such that T centralizes $\langle A,C,D,E,F\rangle$, and the relation $E=(BT)^2$ holds. We enumerate 100 cosets of $\langle A,B,C,D,E,T\rangle$ ($\cong M_{22}:2$) in $\langle A,B,C,D,E,F,T\rangle$ to ensure that these relations do not imply a collapse to the group of order 2. We remark that the relations $1=(EAB)^3$ and $1=(BCE)^5$ are redundant in this presentation for HS:2.

<u>3. Presentations for 3·McL and McL</u>

It is remarkable that we find a presentation for 3·McL by changing just one relation in our presentation for M_{23} given in Proposition M.3 (note also Proposition M.1).

<u>Proposition L.2</u> $3{\cdot}\mathrm{McL} \cong$

$\langle$ 

, (2)

$A=(CF)^2$, $B=(EF)^3$, $1=(EAB)^3=(BCE)^5=(AECD)^4=(CEF)^{21}$,
$[1=(ABC)^5=(BCD)^5]$ $\rangle$.

The central $3 = \langle (CEF)^7\rangle$.

<u>Proof</u> Let G be the group presented in (2), and let $\overline{G} = G/\mathrm{ncl}((CEF)^7)$. We enumerate 2025 cosets of $\langle \overline{A},\overline{B},\overline{C},\overline{D},\overline{E}\rangle$

($\cong M_{22}$) in $\overline{G}$, and calculate that $\langle\overline{A},\overline{B},\overline{C},\overline{D},\overline{E}\rangle$ has orbit-lengths 1,330,462,1232 on these 2025 points. It follows that $\overline{G}$ acts primitively on these points, and since 2025 is not the order of a characteristically simple group, $\overline{G}$ is simple. Since McL is the unique simple group of order $2025.|M_{22}|$, we conclude that $\overline{G} \cong$ McL.

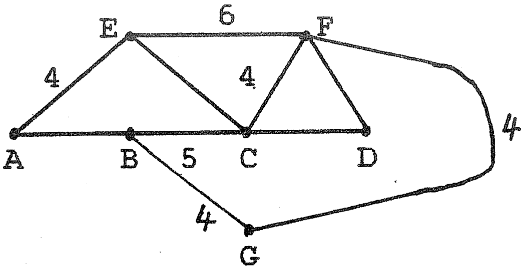
Thus, McL is a quotient of G, and by applying the results of Section I.6, we see that 3·McL is a quotient of G. We enumerate 2025 cosets of $\langle A,B,C,D,E\rangle$ in G to complete the proof. //

<u>Remark</u> The only difference between our presentations for M_{23} and McL is that the relation $1=(BCEF)^4$ of Proposition M.3(iii) is replaced by $1=(CEF)^7$. Moreover, in both of these presentations, $\langle A,B,C,E,F\rangle \cong 2^4{:}A_7$.

4. A presentation for Co_2

It is equally remarkable that a presentation for Co_2 can be found simply by changing one relation in the presentation for M_{24} given in Proposition M.3(iv). We change $1=(BCEF)^4$ to $1=(CEF)^7$, so that A,...,F now generate McL instead of M_{23}. We have:

Proposition L.3 $Co_2 \cong$

$\langle$ 

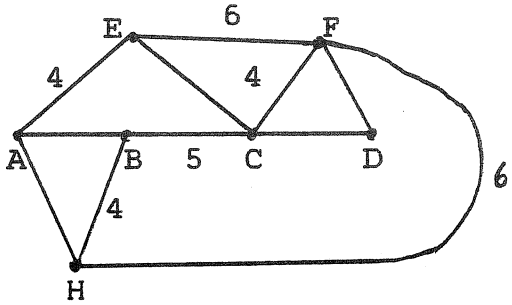
$A=(CF)^2,\ B=(EF)^3,\ E=(BG)^2,\ 1=(AECD)^4=(CEF)^7=(BAEFG)^3\ \rangle.$

Proof We enumerate 47,104 cosets of $\langle A,\ldots,F\rangle$ ($\cong$ McL) in $\langle A,\ldots,G\rangle$, and calculate the orbit-lengths 1, 275, 2025, 7128, 15400, 22275 of $\langle A,\ldots,F\rangle$ on these points. //

5. A presentation for Co_3

We construct Co_3 by extending an M_{11} in a HS to M_{12} (see Proposition M.2). The extending involution extends an M_{22} in HS to McL, as illustrated below.

Proposition L.4 $Co_3 \cong$

$\langle$ 

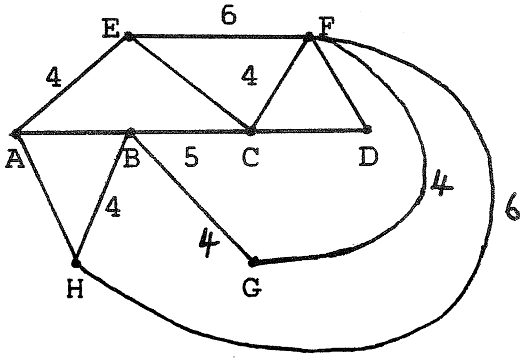
$A=(CF)^2,\ B=(EF)^3,\ D=(BH)^2=(HAE)^3,$

$1=(EAB)^3=(CBE)^5=(AHFD)^3=(CEF)^7,\ [1=(BCFH)^6]\ \rangle.$

Proof We enumerate 11,178 cosets of $\langle A,B,C,D,E,H\rangle$ ($\cong$ HS) in $\langle A,B,C,D,E,F,H\rangle$, and calculate the orbit-lengths 1, 352, 1100, 4125, 5600 of $\langle A,B,C,D,E,H\rangle$ on these points. //

6. A presentation for $2^{\cdot}Co_1$

Theorem L.5 $2^{\cdot}Co_1 \cong$

$\langle$ 

(3)

$A=(CF)^2$, $B=(EF)^3$, $D=(BH)^2=(HAE)^3$, $E=(BG)^2$,

$1=(AHFD)^3=(CEF)^7=(BAEFG)^3$ $\rangle$.

Proof We can construct $2^{\cdot}Co_1$ by extending a McL in Co_2 to a Co_3. Since Co_2 and Co_3 contain just one class of McL each, and Aut(McL) $\cong$ McL:2 is a subgroup of Co_3, there must exist involutions $a,\ldots,h$ generating $2^{\cdot}Co_1$, such that $a,\ldots,g$ satisfy $P(A,\ldots,G)$, and a,b,c,d,e,f,h satisfy $P(A,B,C,D,E,F,H)$ ($Co_2 \cong \langle a,\ldots,g\rangle$ and $Co_3 \cong \langle a,b,c,d,e,f,h\rangle$).

We now show that $gh=hg$, to prove that $2^{\cdot}Co_1$ is an image of the group presented in (3).

Let x be an involution in $2^{\cdot}Co_1$ such that x centralizes $\langle a,e,c,d\rangle$, and $e=(xb)^2$ holds (g and zg are such involutions, where $\langle z\rangle = Z(2^{\cdot}Co_1)$). Now x extends $\langle a,b,c,d,e\rangle \cong M_{22}$ to $M_{22}{:}2$. Since any M_{22} in Co_1 has trivial centralizer, we conclude that $N_{2\cdot Co_1}(a,b,c,d,e) \cong 2 \times M_{22}:2$. Therefore x is in M (say), one of the two $M_{22}:2$'s normalizing $\langle a,b,c,d,e\rangle$. We show that x is a uniquely determined element of M. Now x normalizes $\langle b,c,e\rangle \cong A_5$, and since any A_5 in $M_{22}:2$ has trivial centralizer, we conclude that x is in a uniquely determined S_5 in M. In this S_5, x centralizes $\langle c,e\rangle \cong S_3$; this specifies x uniquely in this S_5, and thus in M. We conclude that x is one of g,zg.

Now $\langle a,b,c,d,e,h\rangle \cong HS$ has normalizer $2 \times HS:2$ in $2^{\cdot}Co_1$, and we conclude that g,zg are precisely the elements x in $2^{\cdot}Co_1$ centralizing $\langle a,e,c,d,h\rangle$, and satisfying $e=(xb)^2$. (Note that in (3), $G(A,B,C,D,E,G,H) \cong HS:2$.) This implies that g commutes with h.

I used a VAX computer to enumerate 196,560 cosets of $\langle A,...,G\rangle$ in $\langle A,...,H\rangle$, to complete the proof. (This enumeration took approximately 9 megabytes of (virtual) store and 6 hours of CPU time. We note that the VAX is slower than the IBM 3081 by a factor of about 10.) //

7. References

1. J.H.Conway, A group of order 8,315,553,613,086,720,000, Bull. London Math. Soc. 1 (1969), 79-88.

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3. J.H.Conway, Three lectures on exceptional groups, in "Finite Simple Groups" (Powell and Higman, eds.), pp. 215-247, Academic Press, London, 1971.

4. R.A.Wilson, The maximal subgroups of Conway's group Co_1, J. Algebra 85 (1983), 144-165.

CHAPTER S

THE SUZUKI SEQUENCE GROUPS, AND A MONOGRAM AND A HEMIGRAM FOR Co_1

1. Introduction

The remarkable Suzuki sequence of normalizers inside Conway's largest sporadic simple group Co_1 was discovered by J.G.Thompson (see [1]). The centralizer of a 3D-element in Co_1 has shape $A_9 \times 3$. If we normalize a descending chain of A_n's embedded naturally in this A_9, we obtain the Suzuki sequence of normalizers:

$$NA_9 \cong A_9 \times S_3$$
$$NA_8 \cong A_8 \times S_4$$
$$NA_7 \cong (A_7 \times L_3(2)):2$$
$$NA_6 \cong (A_6 \times U_3(3)):2$$
$$NA_5 \cong (A_5 \times J_2):2$$
$$NA_4 \cong (A_4 \times G_2(4)):2$$
$$NA_3 \cong 3^{\cdot}Suz:2$$
$$NA_2 \cong Co_1.$$

(The NA_k notation is due to R.A.Wilson.)

The main result of this chapter is a very simple and natural sequence of presentations which illustrates the Suzuki sequence inside Co_1. The groups presented in this sequence are: S_3, S_4, $L_3(2):2$, $U_3(3):2$, $J_2:2$, $G_2(4):2$, $3^{\cdot}Suz:2$, a group having $Co_1 \times 2$ as image, and then two groups having Co_1 wr 2 as image.

Let P be a presentation for a group C; then P is called a diagram for a group G, if G is an image of C. If C is of shape H:2, and G is an image of H, then P is called a monogram for G. The term monogram was invented by S.P.Norton, and is used because "a monogram is half a diagram". Our diagrams for Co_1 wr 2 are monograms for Co_1. In Chapter Y we investigate relations that hold in the MONSTER by using Norton's monogram for the MONSTER.

Also in this chapter, we derive another configuration of Co_1-involutions, and in the process obtain new presentations for J_2, $G_2(4)$, $3^{\cdot}Suz:2$, and $3^2{\cdot}U_4(3):2^2$. The diagrams for these groups embed in a hemigram for Co_1 (a hemigram for G presents a group of shape $H:2^2$, where G is an image of H). Also embedded in this hemigram are monograms for J_2, $G_2(4)$, $3^{\cdot}Suz:2$, and Co_1. At the end of this chapter, we give a presentation for $3^{\cdot}Suz$.

We make use of published information on the subgroup structure of J_2 (see [2]), $G_2(4)$ (see [3]), Suz (see [4]), and Co_1 (see [5]).

Monograms and hemigrams often display more group structure than an associated diagram; this will be illustrated later in the chapter.

Remark Any semi-direct product A:B of groups A and B embeds naturally in A wr B by using the map ϕ defined below (we assume that B is finite to avoid problems of definition).

In A:B, fix a normal A, and a complement B = $\{b_1=1, b_2, \ldots, b_n\}$. For $a \in A$ define $\phi(a) \in A^n$ to be $(a^{b_1}, \ldots, a^{b_n})$, and for $b \in B$ define $\phi(b)$ to be the permutation which takes $(a_1, \ldots, a_n) \in A^n$ to $(a_{\pi(1)}, \ldots, a_{\pi(n)})$, where the permutation π is defined by $bb_i = b_{\pi(i)}$. It is easy to check that ϕ extends to an isomorphism of A:B into A wr B.

Now suppose that G is an image of A, with kernel K. Then there is a natural homomorphic image of A:B in G wr B, i.e. $\phi(A:B)/(\phi(A) \cap K^n)$, where K^n embeds in the obvious way in A^n.

Thus, if P is a monogram for G, then it is a diagram for a subgroup of G wr 2, and if P is a hemigram for G, then P is a diagram for a subgroup of G wr 2^2.

2. A monogram for Co_1

Consider the presentation:

$$\langle \; X_{10} - X_9 - X_8 \overset{8}{-} X_7 - X_6 - X_5 - X_4 - X_3 - X_2 - X_1 - X_0 \qquad (1)$$

$$X_{10}=(X_7X_8)^4, \; 1=(X_6X_7X_8X_9)^8 \; \rangle,$$

and let $G_k = G(X_k,\ldots,X_{10})$. We now prove:

Theorem S.1

(i) $G_9 \cong S_3$

(ii) $G_8 \cong S_4$

(iii) $G_7 \cong L_3(2):2$

(iv) $G_6 \cong U_3(3):2$

(omitting the relation $1=(X_6X_7X_8X_9)^8$ gives $(3 \times U_3(3)):2$)

(v) $G_5 \cong J_2:2$

(vi) $G_4 \cong G_2(4):2$

(vii) $G_3 \cong 3^{\cdot}Suz:2$

the normal $3 = \langle (X_3X_4X_5X_6X_7X_8X_7X_8X_9)^{13} \rangle$

(viii) $Co_1 \times 2$ is an image of G_2

(ix) Co_1 wr 2 is an image of G_1 and of G_0

Proof We first find involutions $x_2,\ldots,x_{10}$ in Co_1 which satisfy $P(X_2,\ldots,X_{10})$.

Fix a 3D-element in Co_1, and let A be the A_9 centralizing this element. Now fix a labelling of the points $\{1,\ldots,9\}$ on which A acts naturally, and define

$$x_k = (1\ 2)(k+1\ k+2) \quad (2 \leqslant k \leqslant 7).$$

Then $x_2, \ldots, x_k$ generate an S_k in A, and they satisfy $P(X_2, \ldots, X_k)$ $(2 \leqslant k \leqslant 7)$.

Let NA_k denote the Co_1-normalizer of the A_k in A which fixes all the points not in $\{1, \ldots, k\}$ $(2 \leqslant k \leqslant 9)$. Let x_9, x_{10} be involutions in Co_1 which generate the S_3 centralizing A, and let x_8 be the involution which extends $\langle x_9, x_{10} \rangle$ to the S_4 centralizing the A_8 in NA_8, such that x_8, x_9, x_{10} satisfy $P(X_8, X_9, X_{10})$. Now, for k=2,...,9, let

$$H_k = \langle x_k, \ldots, x_{10} \rangle.$$

Observe that H_k centralizes $\langle x_2, \ldots, x_{k-2} \rangle \cong S_{k-2}$ $(k \geqslant 4)$, and H_k normalizes (123) if $k \geqslant 3$; therefore H_k is contained in $NA_k, NA_{k-1}, \ldots, NA_2$. We have:

$$H_7 \leqslant L_3(2){:}2$$
$$H_6 \leqslant U_3(3){:}2$$
$$H_5 \leqslant J_2{:}2$$
$$H_4 \leqslant G_2(4){:}2$$
$$H_3 \leqslant 3{\cdot}Suz{:}2$$
$$H_2 \leqslant Co_1.$$

We will prove equality in all these cases by using the fact that H_k is not contained in NA_{k+1} $(k \leqslant 8)$.

We now consider $H_7 \leqslant L_3(2){:}2$. An element x of H_7 is in the outer half of this $L_3(2){:}2$ if and only if x inverts (123), which indeed $x_7 = (12)(89)$ does. In $L_3(2)$ there are two classes of maximal S_4's which are interchanged by the outer automorphism. We conclude that $H_7 = \langle x_7, x_8, x_9, x_{10} \rangle \cong L_3(2){:}2$. Now x_7x_8 is in the outer half of H_7, and so has order 2, 6, or 8. Since $C_{H_7}(x_7) \cong D_{12}$ and $C_{H_7}(x_{10}) \cong D_{16}$, it follows that $1=(x_7x_8)^8$, $x_{10}=(x_7x_8)^4$.

Now (i) and (ii) are well known results, and (iii) is proved by coset enumeration.

Since $L_3(2){:}2$ is maximal in $U_3(3){:}2$, $H_6 \cong U_3(3){:}2$. Coset enumeration is now used to prove (iv). Note that $1=(x_6x_7x_8x_9)^8$ must hold, for otherwise H_6 would have a normal subgroup of order 3. We have now shown that $x_2, \ldots, x_{10}$ satisfy $P(X_2, \ldots, X_{10})$.

Since $U_3(3){:}2$ is maximal in $J_2{:}2$, and $J_2{:}2$ is maximal in $G_2(4){:}2$, we have that $H_5 \cong J_2{:}2$ and $H_6 \cong G_2(4){:}2$. Since $G_2(4){:}2$ is maximal in Suz:2, and H_3 is not $(3 \times G_2(4)){:}2$ (for then H_3 would be contained in NA_4), we have that $H_3 \cong 3{\cdot}Suz{:}2$. Results (v), (vi), and (vii) now follow by using coset enumeration.

Since $3{\cdot}Suz{:}2$ is maximal in Co_1, $H_2 \cong Co_1$. Result (viii) follows since the commutator quotient of G_2 has order 2.

We now give generators $\overline{X}_0, \ldots, \overline{X}_{10}$ of Co_1 wr 2 which satisfy $P(X_0, \ldots, X_{10})$. We take Co_1 wr 2 to be

$$\langle (x,y) \in Co_1 \times Co_1,\ \tau \mid 1=\tau^2,\ (x,y)^\tau=(y,x) \rangle.$$

Define

$$\overline{X}_i = \begin{cases} (x_i, x_i) & (10 \geq i \geq 8) \\ (x_i, x_i)\tau & (7 \geq i \geq 2) \end{cases}$$

$$\overline{X}_1 = ((123),(132))\tau$$

$$\overline{X}_0 = \tau$$

The relations of (1) are easily checked, and so Co_1 wr 2 is an image of G_1 and of G_0. Note that $\overline{X}_0 = \tau$ is contained in $\langle \overline{X}_2, \ldots, \overline{X}_{10} \rangle$. //

Remark Since the central product of two $2{\cdot}Co_1$'s has a diagonal subgroup isomorphic to Co_1, it is easy to see that, in fact, $2{\cdot}(Co_1 \times Co_1):2$ is an image of G_1 and of G_0.

Now Co_1 is an image of the commutator subgroup of G_0, and we can clearly see the Suzuki sequence of normalizers by looking at the intersections with G_0' of certain subgroups of G_0. Let

$$C_k = G(X_0, \ldots, X_{k-2}, X_k, \ldots, X_{10}) \cap G_0' \qquad (k>0)$$

Then $C_k \cong NA_k$ for $k=4,5,\ldots,9$, and $C_3 \cong (3 \times 3{\cdot}Suz):2$.

Conjecture We conjecture that $G_2 \cong Co_1 \times 2$. We have checked that $G_2/ncl((X_3X_4X_5X_6X_7X_8X_7X_8X_9)^{13})$ has order 2, which it must have for this conjecture to be true. We also remark that if $G_2 \cong Co_1 \times 2$, then the central involution would be $((X_3X_4X_5X_6X_7X_8X_7X_8X_9)^{13}X_2)^3$.

3. A hemigram for Co_1

We start with a presentation which is only slightly different from $P(X_6,\ldots,X_{10})$ in (1).

<u>Proposition S.2</u> $2^{\cdot}(S_3 \times J_2) \cong$

$\langle$ 

$, A=(CD)^4, [1=(ABC)^5] \rangle.$ (2)

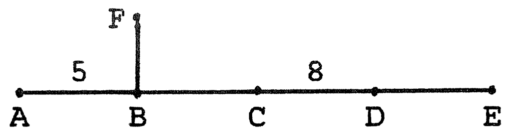
The normal $6 = \langle (BCDE)^{10} \rangle$; $2^{\cdot}S_3 \cong ncl(((BCDCD)^5E)^3)$.

<u>Proof</u> We first show that J_2 is an image of $G = G(A,B,C,D,E)$ in (2).

It is easy to show that $P(A,B,C,D)$ presents $3^{\cdot}PGL_2(9)$, where $1=(ABC)^5$ is an implied relation. We thus start with involutions a,b,c,d in J_2, generating $N(3A) \cong 3^{\cdot}PGL_2(9)$, and satifying $P(A,B,C,D)$ in (2). There is a unique class of A_5 in $3^{\cdot}PGL_2(9)$, and this A_5 has J_2-centralizer A_4. Thus there exists a 2^2 in J_2 centralizing $\langle a,b,c \rangle \cong A_5$. We now remark that $C(a,b) \cong A_5$; this A_5 contains the 2^2 above and the involution d. In any A_5, given a 2^2 subgroup V, and an involution z not in V, then the product of z with precisely one of the involutions in V has order 3. Thus there exists an involution e in J_2, such that e centralizes $\langle a,b,c \rangle$, and $1=(de)^3$ holds. Since $3^{\cdot}PGL_2(9)$ is maximal in J_2, and no involution in $3^{\cdot}PGL_2(9)$ centralizes an A_5 therein, we conclude that $\langle a,b,c,d,e \rangle \cong J_2$, and that J_2 is an image of G.

Note that $G/\mathrm{ncl}(A,B,C) \cong S_3$; thus $S_3 \times J_2$ is an image of G. Coset enumeration shows that $|G| = 12.|J_2|$, and we conclude that $G \cong 2^{\cdot}(S_3 \times J_2)$ (the extension is non-split, for otherwise G would have a 2^2 as a quotient). The other statements of this proposition follow easily by calculation. //

<u>Proposition S.3</u> Let G =

$\langle$  ,

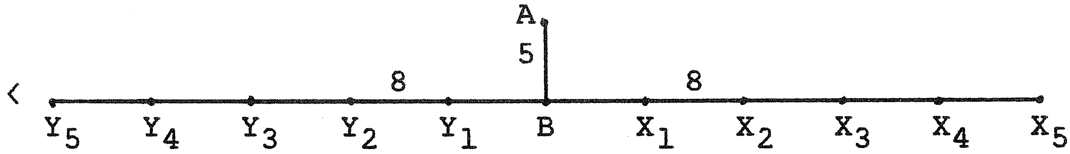
$A=(CD)^4,\ 1=(ABF)^5=(ABCF)^5,\ [1=(ABC)^5]\ \rangle$,

and let $H = \mathrm{ncl}_{\langle A,B,C,D,E\rangle}(((BCDCD)^5E)^3)$.
Then H is normal in G, and $G/H \cong G_2(4)$.

<u>Proof</u> We enumerate 416 cosets of $\langle A,B,C,D,E\rangle$ in G, and denote by $\overline{G}$ the permutation group image of G acting on these 416 points. We calculate that $1 = ((\overline{BCDCD})^5\overline{E})^3$, and that $\langle \overline{A},\overline{B},\overline{C},\overline{D},\overline{E}\rangle \cong J_2$ has orbit-lengths 1,100,315 on the 416 points. The result follows. //

By continuing in an essentially routine manner, we can prove:

Theorem S.4 Let G_{nm} be the the group $G(A,B,X_1,\ldots,X_n,Y_1,\ldots,Y_m)$ in the presentation below:



$A=(X_1X_2)^4=(Y_1Y_2)^4$, $1=(BX_1X_2X_3)^{10}=(BY_1Y_2Y_3)^{10}$,

$1=(ABX_1Y_1)^5=(ABX_1)^5=(ABY_1)^5$ >.

Also, let $\alpha = (BX_1X_2X_1X_2)^5$, $\beta = (BY_1Y_2Y_1Y_2)^5$, $\sigma = (\alpha X_3)^3$, and $\tau = (\beta Y_3)^3$. Then:

$G_{00} \cong D_{10}$

$G_{10} \cong A_5$

$G_{20} \cong 3^{\cdot}PGL_2(9)$; normal $3 = \langle\alpha\rangle$

$G_{30} \cong J_2 \times 2$; central $2 = \langle\sigma\rangle$

J_2 wr 2 is an image of G_{40}, G_{50}

$G_{11} \cong 2^4{:}A_5 \cong M_{20}$

$G_{21} \cong 3^{\cdot}L_3(4){:}2$; normal $3 = \langle\alpha\rangle$

$G_{31} \cong G_2(4) \times 2$; central $2 = \langle\sigma\rangle$

$G_2(4)$ wr 2 is an image of G_{41}, G_{51}

$G_{22} \cong 3^{2}{\cdot}U_4(3){:}2^2$; normal $3^2 = \langle\alpha,\beta\rangle$

$G_{32} \cong 3^{\cdot}Suz{:}2 \times 2$; normal $6 = \langle\beta,\sigma\rangle$

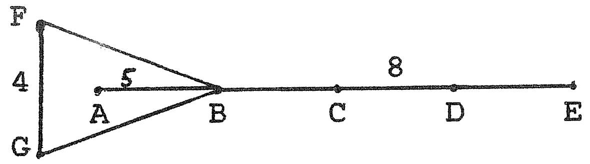
$((3^{\cdot}Suz \times 3^{\cdot}Suz){:}2){:}2$ is an image of G_{42}, G_{52}

$Co_1 \times 2^2$ is an image of G_{33}; central $2^2 = \langle\sigma,\tau\rangle$

$(Co_1 \text{ wr } 2) \times 2$ is an image of G_{43}, G_{53}; central $2 = \langle\tau\rangle$

Co_1 wr 2^2 is an image of G_{44}, G_{54}, G_{55}. //

Finally, from Theorem S.4 it follows that $3^{\cdot}Suz \cong$

$\langle$  ,

$A=(FG)^2=(CD)^4,\ 1=(ABCF)^5=(ABCG)^5=((BCDCD)^5E)^3 \rangle$,

in which the central 3 = $\langle (BFG)^5 \rangle$.

4. References

1. J.H.Conway, Three lectures on exceptional groups, in "Finite Simple Groups" (Powell and Higman, eds.), pp. 215-247, Academic Press, London, 1971.

2. L.Finkelstein and A.Rudvalis, Maximal subgroups of the Hall-Janko-Wales group, J. Algebra 24 (1973), 486-493.

3. R.A.Wilson, The quaternionic lattice for $2G_2(4)$ and its maximal subgroups, J. Algebra 77 (1982), 449-466.

4. R.A.Wilson, The complex Leech lattice and maximal subgroups of the Suzuki group, J. Algebra 84 (1983), 151-188.

5. R.A.Wilson, The maximal subgroups of Conway's group Co_1, J. Algebra 85 (1983), 144-165.

CHAPTER Y

ON NORTON'S MONSTROUS MONOGRAM

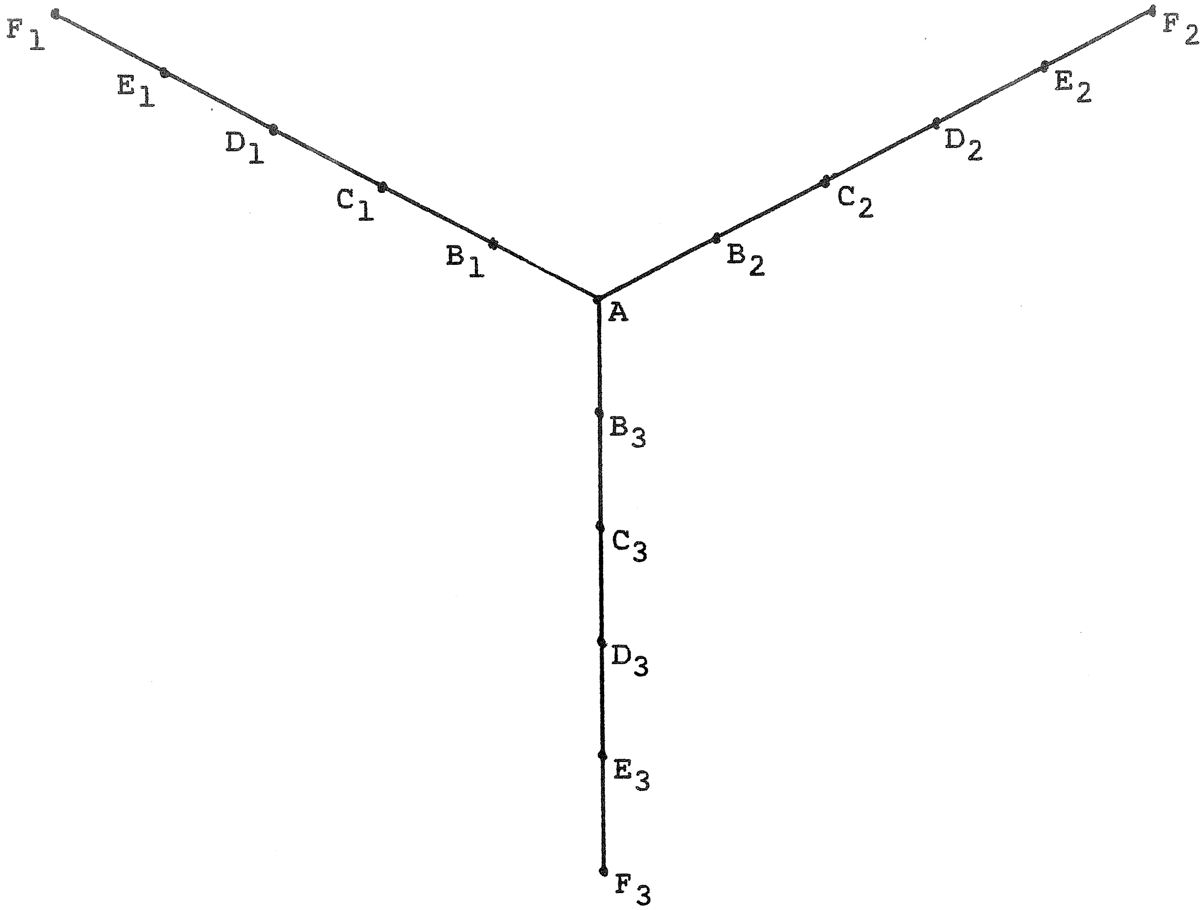
Recall the discussion of monograms in Chapter S: a monogram for a group G is a presentation for a group H:2 such that G is an image of H.

Several years ago, S.P.Norton discovered a monogram for the MONSTER group M, of order

$$808017424794512875886459904961710757005754368000000000$$
$$= 2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71,$$

but this result has not yet been published. It is hoped that details of this work will appear in a forthcoming article by Conway, Norton, and me.

Norton proved that M wr 2 is a homomorphic image of the Coxeter group defined on the next page.



In this Coxeter group, define C_{pqr} to be the group generated by A and

the first p terms of B_1, C_1, D_1, E_1, F_1

the first q terms of B_2, C_2, D_2, E_2, F_2

the first r terms of B_3, C_3, D_3, E_3, F_3.

As is the usual practice, we shall use corresponding small letters to denote the M wr 2-images of elements of C_{555}. In addition, we denote the M wr 2-image of C_{pqr} by Y_{pqr}. Norton, Conway and I have determined the structure of the Y_{pqr}, as shown in the table of Section 3. (As will be seen later, $Y_{4qr} = Y_{5qr}$ if $q>1$ and $r>0$.)

1. Relations that hold in M wr 2

We are interested in determining relations which hold in $Y_{555} \cong$ M wr 2 in addition to the Coxeter relations of C_{555}, and in the implications of these relations.

Consider $Y_{421} \cong O_8^+(2){:}2$. In this group there is a unique involution centralizing $2 \times O_7(2) \cong Y_{321}$. We conclude that

$$f_1 = \text{central involution of } Y_{321} = (ab_1b_2b_3c_1c_2d_1)^9.$$

Similarly, we have the six relations:

$$f_i = (ab_ib_jb_kc_ic_jd_i)^9 = f_{ij} \text{ (say)} \quad (\{i,j,k\}=\{1,2,3\})$$

In particular, $f_{ij}=f_{ik}$, which implies that $1=[f_{ij},c_k]$.

In C_{555} define the six involutions

$$F_{ij} = (AB_iB_jB_kC_iC_jD_i)^9 \quad (\{i,j,k\}=\{1,2,3\}).$$

Coset enumeration shows that

$$C_{322}/\text{ncl}(F_{12}F_{13}) = C_{322}/\text{ncl}([F_{12},C_3]) \cong 2 \times 3{\cdot}O_7(3),$$

where the central 3 is the image of $\langle(AB_1C_1AB_2C_2AB_3C_3)^{10}\rangle$. In C_{555} define

$$S = (AB_1C_1AB_2C_2AB_3C_3)^{10}.$$

Now in $Y_{322} \cong 2 \times O_7(3)$, the image of S is the identity. Coset enumeration proves that

$$C_{322}/\text{ncl}(S,F_{12}F_{13}) = C_{322}/\text{ncl}(S) \cong 2 \times O_7(3).$$

Further coset enumeration shows that

$$C_{422}/ncl(S,F_{12}F_{13}) = C_{422}/ncl(S) = C_{422}/ncl(F_{12}F_{13})$$
$$\cong 2^{\cdot}O_8^+(3):2 \cong 2.Y_{422}$$

(The central involution is the image of $(F_{12}E_1)^3$.)

Let

$$C_S = C_{444}/ncl(S), \text{ and}$$

$$C_F = C_{555}/ncl(F_1F_{12}, F_2F_{23}, F_3F_{31}).$$

(In C_F, the relation $F_i=F_{ij}$ is equivalent to $F_i=F_{ik}$.)

The discussion so far shows that M wr 2 is an image of both C_F and C_S, and that C_F is an image of C_S. We now show that C_S is an image of C_F, and so $C_S \cong C_F$. This gives some evidence for our conjecture that $C_S \cong$ M wr 2.

In $C_S = \overline{C}_{444}$ (say), $\overline{F}_{ij}=\overline{F}_{ik}$. Define

$$\overline{F}_i = \overline{F}_{ij} \; (=\overline{F}_{ik}).$$

Now $\overline{F}_i$ commutes with all the generators of C_S commuting with $\overline{F}_{ij}$ or $\overline{F}_{ik}$, and so $\overline{F}_i$ commutes with all the images of the (Coxeter graph) generators of C_{444}, except for $\overline{E}_i$. Also, $\overline{F}_i$ commutes with $\overline{F}_j$, since $\overline{F}_j = \overline{F}_{ji}$ is a word in elements which commute with $\overline{F}_i$.

We now show that $1 = (\overline{E}_1\overline{F}_1)^3$, and then by symmetry $1 = (\overline{E}_2\overline{F}_2)^3 = (\overline{E}_3\overline{F}_3)^3$. Coset enumeration shows that

$$C_{431}/ncl(F_{21}F_{23}) \cong 2 \times O_9(2),$$

wherein the image of $(F_{12}E_1)^3$ is the identity. We conclude that $1 = (\bar{E}_i\bar{F}_i)^3$, from which follows the result that $C_S \cong C_F$.

2. The table of Y_{pqr}

We have performed further coset enumerations to determine presentations for many of the Y_{pqr} and for certain covering groups of them. In the table which follows, the "relations" column gives relations that complete (or conjecturally complete (indicated by "?")) the Coxeter relations of the relevant C_{pqr} to a presentation for the appropriate group. The "centre" column gives generators for the centre of the group, and we name elements by their preimages in C_{pqr}. This table also appears in the ATLAS of Finite Groups [1].

First, define the words

$$P = (AB_1B_2B_3C_1D_1E_1F_1)^7$$
$$Q = (F_{12}E_1)^3 \text{ or } (AB_1B_2B_3C_1C_2D_1E_1)^{15}$$
$$R = [F_{12},D_2] \text{ or } [F_{21},D_1]$$
$$S = (AB_1C_1AB_2C_2AB_3C_3)^{10}$$
$$T = (AB_1C_1B_2C_2B_3C_3)^{18}$$
$$U_i = F_{ij}F_{ik} \text{ or } [F_{ij},C_k] \text{ or } [F_{ik},C_j] .$$

Then:

Group	Structure	Relations	Centre
Y_{mn0}	S_{m+n+2}	none	
Y_{111}	$2^3:S_4$	none	$(AB_1B_2B_3)^3$
Y_{211}	$2^4:S_5$	none	
Y_{311}	$2^5:S_6$	none	$(AB_1B_2B_3C_1D_1)^5$
Y_{411}	$2^6:S_7$	none	
Y_{511}	$2^6:S_8$	P=1	
Y_{221}	$O_6^-(2):2 \cong O_5(3):2$	none	
Y_{321}	$O_7(2) \times 2$	none	F_{12}
Y_{421}	$O_8^+(2):2$	Q=1	
Y_{331}	$2^2.2^6.O_7(2)$	R=1	F_{12}, F_{21}
Y_{431}	$O_9(2) \times 2$	R=1	F_{21}
Y_{441}	$O_{10}^-(2):2$	R=1	
Y_{222}	$3^5:O_5(3):2$	S=1	
Y_{322}	$O_7(3) \times 2$	S=1	F_{12}
Y_{422}	$O_8^+(3):2$	Q=S=1	
Y_{332}	$2^2.Fi_{22}$	S=1	F_{12}, F_{21}
Y_{432}	$2 \times Fi_{23}$	S=1	F_{21}
Y_{442}	$3{\cdot}Fi_{24} = 3{\cdot}Fi_{24}':2$	S=1 ?	
Y_{333}	$2^3.{}^2E_6(2)$	S=1 ?	F_{12}, F_{23}, F_{31}
Y_{433}	$2^2.B$	S=1 ?	F_{21}, F_{31}
Y_{443}	$2 \times M$	S=1 ?	F_{31}
Y_{444}	M wr 2	S=1 ?	

(continued on next page)

Group	Structure	Relations	Centre
$2.Y_{511}$	$2^7{:}S_8$	none	P
$2.Y_{421}$	$2{\cdot}O_8^+(2){:}2$	none	Q
$3.Y_{222}$	$3^6{:}O_5(3){:}2$	T=1	S
$3.Y_{322}$	$3{\cdot}O_7(3) \times 2$	$U_1=1$	F_{12}, S
$2.Y_{422}$	$2{\cdot}O_8^+(3){:}2$	S=1	Q
$3.Y_{332}$	$(2^2 \times 3).Fi_{22}$	$U_1=U_2=1$	F_{12}, F_{21}, S
$3.Y_{333}$	$(2^3 \times 3).{}^2E_6(2)$	$U_1=U_2=U_3=1$?	F_{12}, F_{23}, F_{31}, S

3. The projective plane

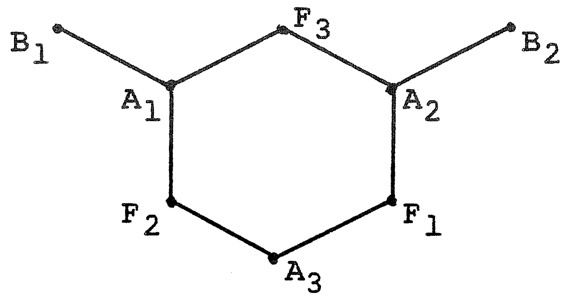
Norton showed that redundant generators can be added to Y_{555} to obtain a Coxeter graph of 26 nodes a, z_i, a_i, b_i, c_i, d_i, e_i, f_i, g_i, f (i = 1,2,3) in which the joins are from

a to b_i, b_j, b_k, f	z_i to a_i, c_j, c_k, e_i
a_i to z_i, b_i, f_j, f_k	b_i to a , a_i, c_i, g_i
c_i to z_j, z_k, b_i, d_i	d_i to c_i, e_i, g_j, g_k
e_i to z_i, d_i, f_i, f	f_i to a_j, a_k, e_i, g_i
g_i to b_i, d_j, d_k, f_i	f to a , e_i, e_j, e_k

where {i,j,k} = {1,2,3}. Abstractly, this is the incidence graph of the 13 points (z_i,b_i,d_i,f_i,f) and 13 lines (a,a_i,c_i,e_i,g_i) of the projective plane of order 3. (The notation for the 26 nodes is due to Conway.) Norton's construction of the 26 node graph shows that all automorphisms of the graph (i.e. elements of $L_3(3){:}2$) are actually in M wr 2.

The ATLAS [1,pp.232-233] reports on further work by Conway, Norton, and me to determine groups generated by subsets of the 26 nodes, and also to find presentations for these groups.

For example, we have shown that $2 \times 2^2 \cdot U_6(2) \cong$

$\langle$ 

$, 1=(A_1F_1A_2F_2A_3F_3)^4 \rangle,$

where the central $2^3 = \langle (A_1A_2A_3B_1B_2F_1F_3)^9, (A_1A_2B_1F_1F_2F_3)^5, (A_1A_2B_2F_1F_2F_3)^5 \rangle$.

4. References

1. J.H.Conway, R.T.Curtis, S.P.Norton, R.A.Parker and R.A.Wilson, "An ATLAS of Finite Groups", Oxford Univ. Press, London/New York, to appear, 1985.

CHAPTER O

THE O'NAN GROUP

1. Introduction

The O'Nan sporadic simple group O'N of order 460,815,505,920 = $2^9.3^4.5.7^3.11.19.31$ has, until recently, been a relatively unknown group. In this chapter we derive a concise presentation of a group having the triple cover $3{\cdot}O'N$ as a homomorphic image. This presentation illuminates the internal structure of O'N, and in particular provides a proof that M_{11} is a subgroup. We then expand this presentation so that an outer automorphism of $3{\cdot}O'N$ is a symmetry of the presentation, and we find an expression for a generator of the centre of $3{\cdot}O'N$. We then obtain a presentation for a group having $3{\cdot}O'N{:}2$ as an image, and from that we derive a presentation for a group of shape $12{\cdot}L_3(4){:}2$.

Since this work was done it has been proved [3] that $3{\cdot}O'N$ has a 45-dimensional $\mathbb{F}_7$-representation. This representation is built in terms of "M_{11} coordinates", and our presentation (2) is used as a conceptual tool in this construction.

2. A Presentation for J_1

Given that O'N contains a J_1 centralizing an outer involution (see [1] and [2]), it can be shown by hand that there is a unique class of J_1 in O'N, and that J_1 is a maximal subgroup of O'N (see [4]). We now give a presentation for J_1.

<u>Proposition 0.1</u> (Conway and Parker) $J_1 \cong$

$\langle$  , (1)

$A=(CDE)^5,\ 1=(ABC)^5,\ [1=(BCD)^5]\ \rangle$

<u>Proof</u> J_1 can be constructed by extending an A_5 in $L_2(11)$ to $2\times A_5$. Thus there exist involutions a,b,c,d,e generating J_1 such that a,b,c,d satisfy P(A,B,C,D), and a,b,c,e satisfy P(A,B,C,E). Thus, to show that J_1 is a homomorphic image of the group presented in (1), we need to show that $1=(de)^5$ and $a=(cde)^5$. There is one class of 3-elements in J_1, and $C_{J_1}(3) \cong 3\times D_{10}$. Thus involutions d and e (centralizing ab) are in a D_{10}, and so $1=(de)^5$. From the Coxeter graph relations satisfied by c,d,e, it follows that $\langle c,d,e\rangle$ is a quotient of $2\times A_5$. Furthermore, since $D_{10} \leqslant \langle c,d,e\rangle$, we see that $\langle c,d,e\rangle \cong A_5$ or $2\times A_5$. If $\langle c,d,e\rangle \cong 2\times A_5$ then its central involution is $(cde)^5$. If $\langle c,d,e\rangle \cong A_5$ then $1=(cde)^5$; but then a coset enumeration would show that $\langle a,b,c,d,e\rangle \cong 1$, a contradiction. Now the centralizer of an involution in J_1 is $2\times A_5$, and $\langle c,d,e\rangle$ centralizes a; therefore $a=(cde)^5$. The result now follows by enumerating the 266 cosets of $\langle A,B,C,D\rangle$ in $\langle A,B,C,D,E\rangle$. //

3. The main results

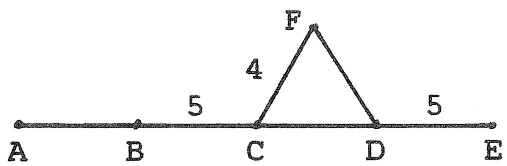
We first prove a little lemma about A_6.

Lemma 0.2 Let $V = \langle x,y \rangle$ be a four-group, and z be an involution, both contained in an A_6. If $\langle x,z \rangle \cong D_{10}$ then $\langle V,z \rangle \cong A_5$.

Proof A_6 contains just two classes of four-groups, and these classes are interchanged by an automorphism extending A_6 to $PGL_2(9)$. Thus, we may assume that V fixes two points in a degree 6 permutation representation of the A_6. But $\langle x,z \rangle \cong D_{10}$ must fix one of these two points, and so $\langle V,z \rangle \cong A_5$. //

We now prove:

Theorem 0.3 $3^{\cdot}O'N$ is a homomorphic image of

$\langle$ 

, (2)

$A=(CF)^2=(CDE)^5,\ 1=(DEF)^5,\ [1=(ABC)^5=(BCD)^5]\ \rangle.$

Proof We show that O'N is an image. It then follows from Section I.6 that $3^{\cdot}O'N$ is an image.

In (2), A,B,C,D,E satisfy a presentation for J_1. Now J_1 is a maximal subgroup of O'N, so we may assume there are involutions a,b,c,d,e in O'N satisfying P(A,B,C,D,E). From

Wilson's analysis of A_5's in O'N (see [4]) it follows that a $2 \times A_5$ in a J_1 may be extended to a $2 \times S_5$ in O'N. Thus ⟨a,b,c⟩ extends to an S_5 commuting with e, and so (as $Aut(A_5) \cong S_5$) there must exist an involution f in O'N having the required relations with a,b,c,e (P(A,B,C,F) presents S_5, where the relation $1=(ABC)^5$ is implied).

In O'N there is one class of 3-elements, and $C_{O'N}(3) \cong 3^2 \times A_6$. Since all involutions in $3^2 \times A_6$ are in the A_6, we conclude that d,e,f (centralizing ab) are in an A_6. From Lemma 0.2 we see that, in fact, $\langle d,e,f \rangle \cong A_5$. It follows that df has order n = 3 or 5. The order n=3 is the desired result, and in this case, since $\langle d,e,f \rangle \cong A_5$, we must have $1=(def)^5$. If n=5 we replace f by f' = ef which satisfies the equivalent relations that f does with a,b,c,e. But in $\langle d,e,f \rangle \cong A_5$, if df has order 5, then df' has order 3. //

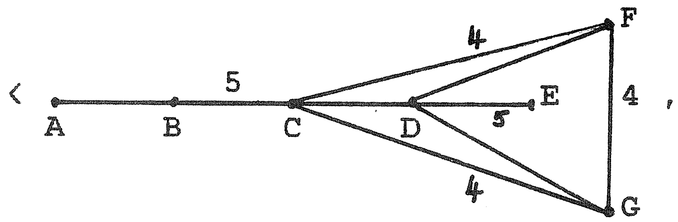
<u>Corollary 0.4</u> $M_{11} \leqslant$ O'N.

<u>Proof</u> We have exhibited involutions a,b,c,d,f in O'N satisfying a presentation for M_{11}. //

This result has been used in Wilson's enumeration [4] of the maximal subgroups of O'N. In fact, Wilson deduces that there are two classes of (maximal) M_{11} in O'N, and that these are interchanged by the outer automorphism. Note that we have used Wilson's analysis of J_1's and A_5's in O'N. This analysis is independent of our results, and also independent of the results of [3].

Now let a,b,c,d,e,f be involutions in $3^{\cdot}O'N$ satisfying the presentation (2), and let h be an outer involution in $3^{\cdot}O'N{:}2$ centralizing $\langle a,b,c,d,e\rangle \cong J_1$. We are interested in determining relations involving $g = f^h$. Since h is an automorphism of $\langle a,\ldots,f\rangle$, and h fixes $\langle a,\ldots,e\rangle$, the relations of f^h with a,b,c,d,e must be equivalent to those of f; in particular f^h centralizes $\langle a,b,e\rangle$. Now the centralizer of a non-central 3-element in $3^{\cdot}O'N$ is $3 \times 3^{\cdot}A_6$ (see [2]). It follows that $\langle d,e,f,f^h\rangle \cong 3^{\cdot}A_6$. There is one class of involutions in $3^{\cdot}A_6$ and the centralizer of an involution therein is $3 \times D_8$. It follows that $1=(ff^h)^4$ and $e=(ff^h)^2$. The central 3 in the $3^{\cdot}A_6$ and hence in the $3^{\cdot}O'N$ is $(dff^h)^5$. We have thus proved the following:

<u>Theorem 0.5</u> $3^{\cdot}O'N$ is a image of

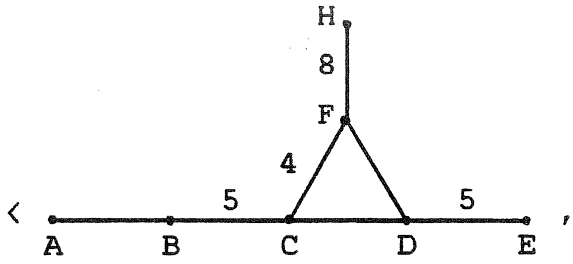


(3)

$A=(CF)^2=(CG)^2=(CDE)^5$, $E=(FG)^2$, $1=[C,(DFG)^5]$,

$[1=(DEF)^5=(DEG)^5]$ $\rangle$.

Equivalently, $3 \cdot O'N:2$ is an image of



(4)

$A=(CF)^2=(CDE)^5$, $E=(FH)^4$, $1=[C,(DFHFH)^5]$,

$[1=(DEF)^5]$ ⟩. //

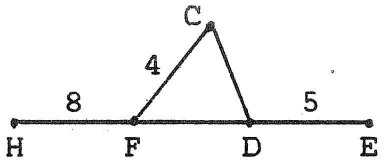
Let X be the group presented in (3). A coset enumeration shows that ⟨A,...,F⟩ has index 1 in X; therefore X is a quotient of the group presented in (2).

Now let $\overline{X} = X/\langle (DFG)^5 \rangle$ ($(DFG)^5$ is clearly central in X). We have that $X \cong 3 \cdot O'N$ if and only if $\overline{X} \cong O'N$. I have recently made an attempt to prove that $\overline{X} \cong O'N$. Using the representation of $3 \cdot O'N$ described in [3], I verified that if $\overline{X} \cong O'N$, then $L_3(7){:}2 \cong L = \langle \overline{A},\overline{B},\overline{E},\overline{F},\overline{G}, \overline{EDEFDEDE}, \overline{CFDEDEDGDC} \rangle$. However, an attempted coset enumeration of L in $\overline{X}$, in which space was allowed for 300,000 cosets to be defined, ran out of space instead of giving the hoped-for index of 122,760.

4. A presentation for $12 \cdot L_3(4){:}2$

As a bonus we have the following:

<u>Proposition 0.6</u> $12{\cdot}L_3(4){:}2$ ($\leqslant 3{\cdot}O'N{:}2$) is isomorphic to

$\langle$ 

, (5)

$(CF)^2=(CDE)^5,\ E=(FH)^4,\ 1=[C,(DFHFH)^5],\ [1=(DEF)^5]\ \rangle$.

The normal $12 = \langle (DFHFH)^5(CDEF)^5\rangle$. The quotient group of shape $L_3(4){:}2$ is $L_3(4)$ extended by the transpose-inverse automorphism. (This is the automorphism 2_3 in the ATLAS notation.)

<u>Proof</u> Let a,c,d,e,f,h be involutions in $3{\cdot}O'N{:}2$ satisfying P(A,C,D,E,F,H) in (4). Then $\langle c,d,e,f,h\rangle \leqslant C_{3\cdot O'N:2}(a) \cong 12{\cdot}L_3(4).2^2$.

Let $g = f^h$. By abelianizing the relations in (3) satisfied by c,d,e,f,g, we see that $\langle c,d,e,f,g\rangle$ is perfect, and hence a subgroup of $12{\cdot}L_3(4)$. Since $\langle d,e,f,g\rangle \cong 3{\cdot}A_6$, and any A_6 in $L_3(4)$ is maximal, we have that $\langle c,d,e,f,g\rangle \cong 12{\cdot}L_3(4)$. Now h extends $\langle d,e,f,g\rangle$ to $3{\cdot}PGL_2(9)$, and h commutes with c. It follows that $\langle c,d,e,f,h\rangle \cong 12{\cdot}L_3(4){:}2$.

Let X be the group defined by (5). Coset enumerations show that $|X| \leqslant 12.|L_3(4)|.2$ and that $\langle (CDEF)^5\rangle$ is a normal subgroup of X of order 4. The result follows. //

5. References

1. S.Andrilli, On the uniqueness of O'Nan's sporadic simple group, Thesis, Rutgers, 1980.

2. M.E.O'Nan, Some evidence for the existence of a new simple group, Proc. London Math. Soc. 32 (1976), 421-479.

3. A.J.E.Ryba, A new construction of the O'Nan simple group, submitted to J. Algebra.

4. R.A.Wilson, The maximal subgroups of the O'Nan group, to appear in J. Algebra.

5. S.Yoshiara, The maximal subgroups of the O'Nan group, Preprint, Tokyo, 1984.

<u>Note</u> Several months after the work of this chapter was completed (except for the attempt to prove that (3) presents $3 \cdot O'N$), R.A.Wilson received a copy of [5], which shows that Yoshiara independently proved that $M_{11} \leqslant O'N$.

CHAPTER H

THE HELD GROUP

1. Introduction

The Held sporadic simple group, He, of order 4,030,387,200 = $2^{10}.3^3.5^2.7^3.17$, was first constructed by G.Higman and J.McKay. To accomplish this, Higman derived a presentation for Aut(He) ≅ He:2, based on the (then conjectured) subgroup $S_4(4):4$ of index 2058 (see [3]). Cannon and Havas [2] used the Reidemeister-Schreier process to obtain a presentation for He from Higman's presentation for He:2.

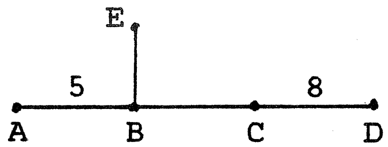
In this chapter we give a new concise presentation for He:2. This presentation illustrates the subgroups C(2A) ≅ $2^2 \cdot L_3(4):2^2$ and N(3A) = C(2C) ≅ 2 × $3 \cdot S_7$. (Throughout this chapter, C(X) and N(X) mean $C_{He:2}(X)$ and $N_{He:2}(X)$, respectively.) We then immediately derive a presentation for He from our presentation for He:2.

We make use of the information on the subgroup structure of He given in [1].

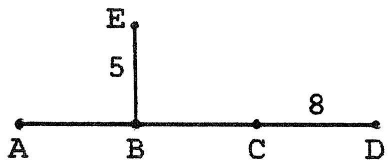
2. A presentation for $2^2 \cdot L_3(4):2_3$

Up to isomorphism, there is a unique group T (say) of shape $2^2 \cdot L_3(4):2_3$, where the subgroup of shape $2^2 \cdot L_3(4)$ is perfect ($L_3(4):2_3$ is the ATLAS notation for $L_3(4)$ extended by the transpose-inverse automorphism). We now derive a presentation for $T \leqslant C(2A)$.

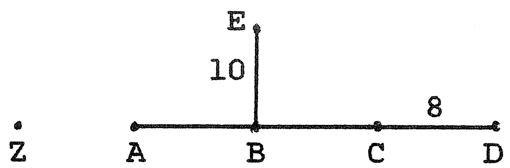
It follows from Proposition 0.6 that $3 \cdot L_3(4):2_3 \cong$

$\langle$  , $A=(CD)^4$, $1=(ABE)^5=(ABCE)^5$ $\rangle$.

By substituting $E^{(BA)^2}$ for B, ACE for C, and leaving A, D, and E as above, it follows straightforwardly that $3 \cdot L_3(4):2_3 \cong$

$\langle$  , $A=(CD)^4$, $1=(ABE)^5=(BCE)^5$ $\rangle$,

in which the normal $3 = \langle (EBCDCD)^4 \rangle$. It is then easy to show that $T \cong 2^2 \cdot L_3(4):2_3 \cong$

$\langle$  ,

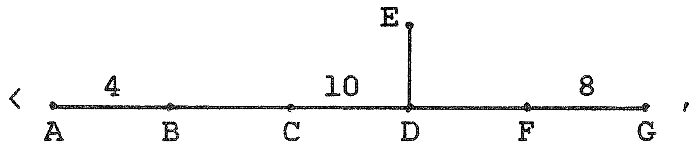
$Z=(BE)^5$, $A=(CD)^4$, $1=(ABE)^5=(EBCDCD)^4$ $\rangle$.

The normal $2^2 = \langle Z, (BCE)^5 \rangle$.

3. Presentations for He:2 and He

We now derive a presentation for He:2.

Proposition H.1 He:2 $\cong$



(1)

$$A=(CD)^5,\ E=(ABC)^3=(FG)^4,\ 1=(CDFGFG)^4,\ [1=(CDE)^5]\ \rangle.$$

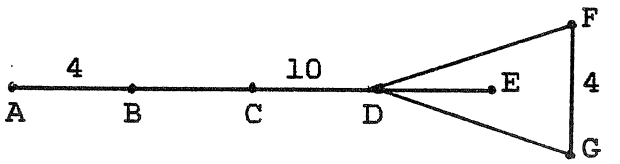
Proof We start with involutions a,c,d,e,f,g in He:2, generating $2^2{\cdot}L_3(4){:}2_3$, and satisfying P(A,C,D,E,F,G) in (1). We shall show that $C(d,e,f,g) \cong D_8$, in which a is not central, and then we will extend $\langle a \rangle$ to this D_8.

Now $PGL_2(7) \cong \langle d,e,f,g \rangle = P$ (say), and let L be the subgroup $L_2(7)$ of P. Then $P \leqslant N(L) = S{\times}P_1$, where $S_4 \cong S \leqslant He$, and $P_1 \cong PGL_2(7)$. It follows that S contains $C(P) \cong S_4$, D_8, or 2^2. Let V be the unique non-normal 2^2 in S containing a. From the 2-local structure of He, we see that $V = O_2(C(a))$. If $C(P) \cong S_4$ or 2^2, then C(P) contains V; but since g is a 2_3-involution in C(a), g interchanges the two involutions other than a in V. Therefore V is not contained in C(P), and so $C(P) \cong D_8$ in which a is not central.

Thus there exists an involution b in He such that ab has order 4, and b centralizes $\langle d,e,f,g \rangle$. Now a,b,c,d,e are in the $3{\cdot}S_7$ in He centralizing g. By simple calculations in S_7, we see that the relations $1=(bc)^3$ and $e=(abc)^3$ are forced to hold in the $3^{\cdot}S_7$, and that a,b,c,d,e must generate $3^{\cdot}S_7$. Since $3^{\cdot}S_7$ is maximal in He, it follows that $\langle a,\ldots,f\rangle \cong$ He, and $\langle a,\ldots,g\rangle \cong$ He:2. Since we have shown that a,...,g satisfy the presentation (1), He:2 is an image of the group presented in (1).

By coset enumeration, we see that P(A,B,C,D,E) in (1) presents $3^{\cdot}S_7$, and that the relations $1=(CDE)^5$ and $C^{(AB)^2}=AEC$ are implied. We conclude that $(AB)^2$ normalizes $\langle A,C,D,E,F,G\rangle$, and that $\langle A,C,D,E,F,G,(AB)^2\rangle \cong 2^2{\cdot}L_3(4){:}2^2$. The proposition follows by enumerating 24,990 cosets of $\langle A,C,D,E,F,G,(AB)^2\rangle$ in $\langle A,\ldots,G\rangle$. //

<u>Corollary H.2</u> He $\cong$

$\langle$  , (2)

$A=(CD)^5,\ E=(ABC)^3=(FG)^2,\ 1=(CDFG)^4,\ [1=(CDE)^5]\ \rangle$.

<u>Proof</u> Let X be the group presented in (2). It follows that the group presented in (1) is of shape X:2. But the group presented in (1) is He:2. //

We remark that the proof of Proposition H.1 could be used to derive a new proof of the existence and uniqueness of He, and that this proof would not depend on the existence of a subgroup $S_4(4)$ of He.

4. References

1. G.Butler, The maximal subgroups of the sporadic simple group of Held, J. Algebra 69 (1981), 67-81.

2. J.J.Cannon and G.Havas, Defining relations for the Held-Higman-Thompson simple group, Bull. Austral. Math. Soc. 11 (1974), 43-46.

3. J.McKay, Computing with finite simple groups, in "Proceedings, Second International Conference on the Theory of Groups, Canberra, 1973," pp. 448-452, Lecture Notes in Mathematics No. 372, Springer-Verlag, New York/Berlin, 1974.

APPENDIX

SOME INFINITE SEQUENCES OF PRESENTATIONS FOR FINITE GROUPS

1. Introduction

In this appendix we give three infinite sequences of presentations.

The first is a sequence of presentations for:

$$A_5,\ 3^{\cdot}A_6,\ 3^{\cdot}A_7,\ A_8,\ A_9,\ \ldots,\ A_n,\ \ldots\ .$$

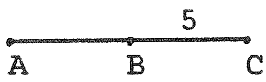
These are the first presentations (known to the author) which present A_n in terms of generating involutions. By applying the results of Section I.6, these presentations yield a new proof that the odd part of the Schur multiplier of A_n ($n \geq 5$) is trivial, unless $n = 6$ or 7, in which case it is 3. A presentation for A_n in terms of generating 3-cycles can be found in [1].

The second sequence of presentations gives a generalization of the Weyl group of B_n. Usually, the Weyl group of B_n, $W(B_n)$, is thought of in terms of its shape $2^n{:}S_n$. Instead, we consider its shape to be $2^{n-1}.(S_n \times 2)$. We then extend $W(B_{n+1})$ in a natural way to give presentations for groups of shape $2^{nm}.(S_{n+1} \times S_{m+1})$.

We conjecture that the third sequence of presentations is for the groups $2^n{:}L_n(2)$ $(n\geqslant 2)$. This has been verified for n up to 7.

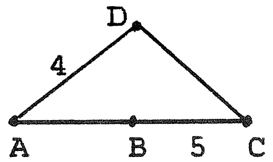
2. Presentations for the alternating groups

It is well-known that

$A_5 \cong \langle$ 

$,\ 1=(ABC)^5 \rangle.$

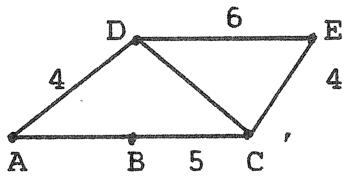
It is easy to show that

$3{\cdot}A_6 \cong \langle$ 

,

$1=(CDA)^3=(DAB)^3,\ [1=(ABC)^5=(BCD)^5]\ \rangle,$

and

$3{\cdot}A_7 \cong \langle$ 

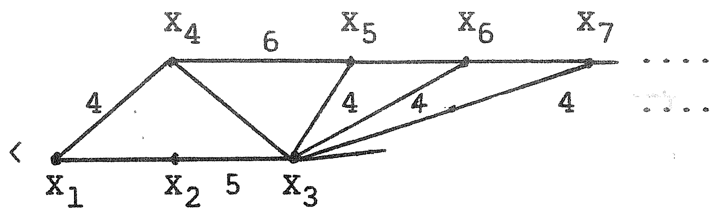
,

$A=(CE)^2,\ B=(DE)^3,\ 1=(CDA)^3=(DAB)^3\ \rangle.$

(In both cases the central $3 = \langle (ABCD)^4\rangle$.)

We now prove:

Theorem A.1 Let $n \geqslant 8$, and let G_n be $G(X_1,\ldots,X_{n-2})$ in the presentation below.



(1)

$X_2=(X_4X_5)^3,\ X_1=(X_3X_5)^2=(X_3X_6)^2=(X_3X_7)^2=\ldots\ \rangle.$

Then $G_n \cong A_n$. (The relations $1 = (X_3X_4X_1)^3 = (X_4X_1X_2)^3 = (X_1X_2X_3X_4)^4$ are implied.)

Proof The following permutations generate A_n and satisfy $P(X_1,\ldots,X_{n-2})$ $(n\geq 8)$:

$$x_1 = (1\ 2)(4\ 5)$$
$$x_2 = (1\ 2)(3\ 4)$$
$$x_3 = (1\ 4)(2\ 5)$$
$$x_4 = (3\ 4)(5\ 6)$$
$$x_k = (1\ 2)(k+1\ k+2) \quad (k=5,6,\ldots,n-2).$$

A coset enumeration shows that $\langle X_1,X_2,X_3,X_5\rangle$ $(\cong S_5)$ has index 168 in G_8, and so $G_8 \cong A_8$.

Now consider the following elements C_i of G_n:

$$C_n = 1$$
$$C_k = C_{k+1}X_{k-1} \quad (k=n-1,n-2,\ldots,5)$$
$$C_4 = C_5X_1$$
$$C_3 = C_5X_1X_2$$
$$C_2 = C_5X_3$$
$$C_1 = C_5X_3X_1.$$

(In the image A_n of G_n, $n^{C_i} = i$.) Define the coset H_i to be

$\langle X_1,\ldots,X_{n-3}\rangle C_i$. We assume inductively that $G_{n-1} \cong A_{n-1}$, and show that G_n acts on $\{H_1,\ldots,H_n\}$ by right multiplication $(n \geqslant 9)$.

It is easy to see that X_{n-2} interchanges H_n and H_{n-1}, fixes $H_{n-2}, H_{n-3},\ldots,H_3$, and interchanges H_2 and H_1.

Now let X be in $\{X_1,\ldots,X_{n-3}\}$, and $1 \leqslant i \leqslant n-1$. Then X fixes H_n, and from the inductive hypothesis

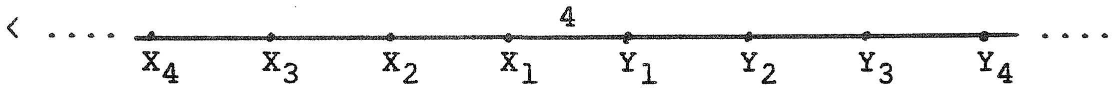
$$C_i X = X_{n-2}\ T\ X_{n-2} C_j$$

for some T in $\langle X_1,\ldots,X_{n-4}\rangle$, and $1 \leqslant j \leqslant n-1$. But X_{n-2} normalizes $\langle X_1,\ldots,X_{n-4}\rangle$, and so

$$H_i X = H_j \quad .//$$

1. A generalization of the Weyl group of B_n

Let B_{nm} $(n,m \geqslant 1)$ be $G(X_1,\ldots,X_n,Y_1,\ldots,Y_m)$ in the presentation below:



$1=(X_2X_1Y_1Y_2)^6\ \rangle$. (2)

Note that $B_{nm} \cong B_{mn}$, $B_{n-1,1} \cong W(B_n)$, and $B_{22} \cong W(F_4)/\text{centre}$.

<u>Theorem A.2</u> B_{nm} has shape $2^{nm}.(S_{n+1} \times S_{m+1})$.

<u>Outline of Proof</u> We first define a permutation group P_{nm} whose generators satisfy (2), and show that $|P_{nm}| \geqslant$

$2^{nm}(n+1)!(m+1)!$. Next, we show that $ncl_{B_{nm}}((X_1Y_1)^2)$ is a subgroup of 2^{nm}. This completes the proof as it is clear from (2) that $B_{nm}/ncl((X_1Y_1)^2) \cong S_{n+1} \times S_{m+1}$.

Proof Consider the set

$$\Omega = \{(S,j) \mid S\subseteq\{1,2,\ldots,n+1\},\ 1\leq j\leq m+1,\ |S| \text{ odd iff } j=1\},$$

and let P_{nm} acting on Ω (by .) be generated by involutions $x_1,\ldots,x_n$, $y_1,\ldots,y_m$ defined by:

$(S,j).x_i = (S^{(i\ i+1)},j)$

(the action on sets is the natural one);

$(S,j).y_i = (S,j^{(i\ i+1)})$ if $i>1$ or $j\neq 1,2$,

$(S+\{1\},j^{(1\ 2)})$ otherwise

(+ is symmetric difference).

It is straightforward to show that these generators satisfy (2).

Now P_{n1} is of the shape $2^{n+1}{:}S_{n+1}$, and we assume inductively that

$$|P_{n,m-1}| \geqslant 2^{n(m-1)}(n+1)!m! \quad (m>1).$$

In P_{nm}, $H = \langle x_1,\ldots,x_n,y_1,\ldots,y_{m-1}\rangle$ is a subgroup of stab(({},m+1)), and H acts on $\{(S,j) \in \Omega \mid j<m+1\}$ as $P_{n,m-1}$. Since P_{nm} is transitive on Ω, it follows that

$$|P_{n\ m}| \geqslant |P_{n,m-1}|2^n(m+1) \geqslant 2^{nm}(n+1)!(m+1)!\ .$$

In B_{nm} let

$$Z_{ij} = ((X_1Y_1)^2)^{X_2\ldots X_iY_2\ldots Y_j} \quad (i=1,\ldots,n,\ j=1,\ldots,m),$$

where a product of the form $A_2\ldots A_1$ is to be interpreted as the identity. From properties of $W(B_n)$, and since X_i commutes with Y_j unless $i=j=1$, we see that $\langle X_1,\ldots,X_n\rangle$ normalizes

$$\langle Z_{1j},\ldots,Z_{nj}\rangle \quad (j=1,\ldots,m),$$

and $\langle Y_1,\ldots,Y_m\rangle$ normalizes

$$\langle Z_{i1},\ldots,Z_{im}\rangle \quad (i=1,\ldots,n).$$

Thus $ncl_{B_{nm}}((X_1Y_1)^2) =$

$$\langle Z_{ij} \mid 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant m\rangle = T_{nm} \text{ (say)}.$$

We show that T_{nm} is abelian, and since T_{nm} is generated by nm involutions, it is a subgroup of 2^{nm}. We may assume that $n,m\geqslant 2$.

A simple calculation in $B_{22} \cong W(F_4)/\text{centre} \cong P_{22}$ shows that T_{22} is abelian. We assume that $T_{n,m-1}$ is abelian for a fixed n, and $m\geqslant 3$, and show that T_{nm} is abelian. This proves that $T_{2\ m} \cong T_{m\ 2}$ $(m\geqslant 2)$ is abelian, and then this shows that T_{nm} is abelian for $n,m\geqslant 2$.

Thus consider B_{nm} $(m\geqslant 3)$, and assume that $T_{n,m-1}$ is abelian. Then

$$\langle Z_{ij} \mid 1\leqslant i\leqslant n,\ 1\leqslant j\leqslant m-1\rangle$$

is abelian, and it is also easy to see that

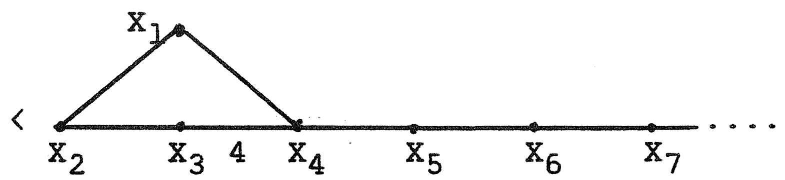
$\langle Z_{ij} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ j \neq m-1 \rangle$

is abelian. Now let U,V be in $\{Z_{i,m-2} \mid 1 \leqslant i \leqslant n\}$, and let $A=Y_{m-1}$ and $B=Y_m$. We need only to show that U^A commutes with V^{AB} to complete the proof. We have:

$$\begin{aligned} AUABAVAB &= AUBABVAB = ABUAVBAB = ABU(AVA)BA \\ &= AB(AVA)UBA \quad \text{(by inductive hypothesis)} \\ &= BABVAUBA = BAVBABUA = BAVABAUA\ .\ // \end{aligned}$$

4. Presentations for $2^n{:}L_n(2)$?

Let G_n be $G(X_1,\ldots,X_{n+1})$ in the presentation below:



$1=(X_1X_3X_4)^3=(X_2X_3X_4X_5)^6\ \rangle$

Conjecture We conjecture that for $n \geqslant 2$, $G_n \cong 2^n{:}L_n(2)$, and $L_n(2)$ is obtained by adjoining the relation $1=(X_1X_3X_2)^2$ if $n=2$, $1=(X_1X_3X_2X_4X_5\ldots X_{n+1})^n$ if $n > 2$. We have performed coset enumerations which verify this for $n \leqslant 7$.

5. References

1. R.D.Carmichael, "Introduction to the Theory of Groups of Finite Order", Dover, Reprint, 1956.