Classification of partial spreads in $\mathrm{PG}(4,2)$

Neil A. Gordon[a], Ron Shaw[b] and Leonard H. Soicher[c]

[a]Department of Computer Science, [b]Department of Mathematics,

University of Hull, Hull HU6 7RX, UK

n.a.gordon@hull.ac.uk, r.shaw@hull.ac.uk

[c]School of Mathematical Sciences, Queen Mary, University of London,

Mile End Road, London E1 4NS, UK

l.h.soicher@qmul.ac.uk

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Abstract

We consider partial spreads $\mathcal{S}_r$ of r (> 0) lines in $\mathrm{PG}(4,2)$, and obtain a complete classification, summarized in tables in appendix B. Under the action of $\mathrm{GL}(5,2)$ there are sixty-four distinct classes of partial spreads. The maximal partial spreads account for eight of these classes, one class of size $r = 5$, three of size 7 and four of size 9. Several of the non-maximal classes are not without interest, including a regulus-free $\mathcal{S}_8$ whose stabilizer $\cong 2^3{:}\mathrm{F}_{21}$, of order 168, acts 2-transitively upon the eight lines. Various invariants (regulus type, signature, profile, symplectic type) are employed to aid the allocation of a partial spread to its class. Even in cases of inequivalent partial spreads having the same invariants, we provide at least one way of distinguishing between them. Several examples are given of applications of the classification, involving flats external to the Grassmannian $G_{1,4,2}$ in $\mathrm{PG}(9,2)$, and also spreads of lines in $\mathrm{PG}(5,2)$.

Contents

1 Partial spreads in PG(4, 2) and their reguli

1.1 Introduction

We will be dealing with partial spreads $\mathcal{S} = \mathcal{S}_r = \{\lambda_1, \ldots, \lambda_r\}$ of $r\ (> 0)$ lines in PG(4, 2). (For partial spreads in PG(n, q) for other values of (n, q), see [1], [27] and the web page [26].) Because the field is GF(2) we will usually identify the 31 points of the projective space PG(4, 2) = $\mathbb{P}V(5, 2)$, acted upon by PGL(5, 2), with the corresponding non-zero vectors of the vector space $V_5 := V(5, 2)$, acted upon by GL(5, 2). We wish to classify all partial spreads in PG(4, 2) under the action of GL(5, 2). We will be especially interested in *maximal* partial spreads, that is in those which admit no extension to a larger partial spread. Two partial spreads which belong to the same GL(5, 2) orbit will be termed *equivalent.* Let us agree upon the following notation:

$\mathcal{S} = \mathcal{S}_r = \{\lambda_1, \ldots, \lambda_r\}$, a set of $r\ (> 0)$ mutually skew lines in PG(4, 2);
$\psi = \psi(\mathcal{S}_r) = \lambda_1 \cup \ldots \cup \lambda_r$, the underlying point-set; so $|\psi| = 3r$;
ψ^c = the complement of the subset $\psi \subset$ PG(4, 2);
$L(\psi) = \#\{$internal lines of $\psi\}$;
$L'(\psi) = \#\{$external lines of $\psi\}$ $(= L(\psi^c))$;
$\Theta(\psi)$ = orbit of ψ under the action of GL(5, 2);
$\Theta(\mathcal{S})$ = orbit (equivalence class) of $\mathcal{S}$ under the action of GL(5, 2);
$\mathcal{G}(\psi)$ = subgroup of GL(5, 2) which fixes ψ set-wise;
$\mathcal{G}(\mathcal{S})$ = subgroup of $\mathcal{G}(\psi)$ which permutes the elements λ_i of $\mathcal{S}$;

$\prec v_1, \ldots, v_r \succ =$ vector subspace of V_5 spanned by vectors $v_1, \ldots, v_r \in V_5$;
$\langle v_1, \ldots, v_r \rangle =$ projective subspace of PG(4, 2) generated by points $v_1, \ldots, v_r \in \mathrm{PG}(4,2)$.
$\sigma_{ij} = \langle \lambda_i, \lambda_j \rangle$, the solid (hyperplane, PG(3, 2)) generated by λ_i and λ_j;
$\tau_{ijk} =$ (a) transversal of the triple of lines $\{\lambda_i, \lambda_j, \lambda_k\} \subset \mathcal{S}_r$;
$\rho =$ a regulus in a PG(3, 2); $\rho^{\mathrm{opp}} =$ the opposite regulus;
$\sigma(\rho) = \sigma_{ijk} =$ the ambient PG(3, 2) of a regulus $\rho = \rho_{ijk} = \{\lambda_i, \lambda_j, \lambda_k\}$;
$N_r =$ the number of reguli contained in $\mathcal{S}_r$;
$\mathcal{H} =$ a hyperbolic quadric $\mathcal{H}_3$ in a PG(3, 2);
$\mathcal{P}_4 =$ a parabolic quadric (in PG(4, 2)).

Let us record here the following well-known facts concerning a hyperbolic quadric $\mathcal{H} \subset \sigma$, where σ denotes the ambient PG(3, 2) of $\mathcal{H}$. The 6-set $\sigma \diagdown \mathcal{H}$ consists of a pair μ, μ' of skew lines, and if $\mu = \{a, b, c\}$ and $\mu' = \{a', b', c'\}$ then the nine points of the quadric $\mathcal{H}$ may be displayed as a 3×3 array

$$\mathcal{H} = \begin{pmatrix} a+a' & b+b' & c+c' \\ b+c' & c+a' & a+b' \\ c+b' & a+c' & b+a' \end{pmatrix}. \tag{1.1}$$

Recall that $\mathcal{H}$ has six generators ($L(\mathcal{H}) = 6$), one set of three generators constituting the regulus ρ whose lines are given by the three rows of (1.1), and the remaining three generators constituting the *opposite* regulus given by the three columns $\kappa = \rho^{\mathrm{opp}}$ of (1.1). Moreover the underlying set of *any* regulus in a PG(3, 2) is a hyperbolic quadric.

A *transversal* of the triple $\{\lambda_i, \lambda_j, \lambda_k\} \subset \mathcal{S}$ is a line τ which meets each of the three lines $\lambda_i, \lambda_j, \lambda_k$; it will sometimes also be referred to as a transversal of $\mathcal{S}$. For $\psi = \psi(\mathcal{S}_r)$, with $r \geq 3$, it should be noted that there will always exist transversals which, along with the r lines $\lambda_i \in \mathcal{S}_r$, will contribute to the number $L(\psi)$ of internal lines. This is so because each three members of $\mathcal{S}$ will possess (since we are in projective dimension 4) at least one transversal, which latter (since we work over GF(2)) will lie completely in ψ. (See lemma 1.3 below.) Given a line $\lambda = \{p_1, p_2, p_3\} \in \mathcal{S}$, suppose that the point p_i lies on $m_i - 1$ transversals, and hence on m_i of the $L(\psi)$ lines of $\psi(\mathcal{S})$. We record this information as profile$(\lambda) = (m_1, m_2, m_3)$. Of course $m_i \geq 1$; usually we order the points of λ so that $m_1 \leq m_2 \leq m_3$. A line $\lambda \in \mathcal{S}$ for which $m_1 = m_2 = m_3$ will be said to be *balanced*; otherwise we say it is *unbalanced,* or also (m_1, m_2, m_3)-*unbalanced*. The *profile* of a partial spread $\mathcal{S}_r$ is obtained by combining together the r profiles of its constituent lines. For example a statement profile$(\mathcal{S}_7) = (5,6,7)^2(6,6,8)^2(6,7,7)^2(7,7,8)$ means that two lines of $\mathcal{S}_7$ have profile $(5,6,7)$, two have profile $(6,6,8)$, two have profile $(6,7,7)$ and one line of $\mathcal{S}_7$ has profile $(7,7,8)$. The slightly more refined version of the profile used in tables B.2a and B.2b is explained in section B.3.2.

A partial spread $\mathcal{S}_r$ is *transitive* if $\mathcal{G}(\mathcal{S}_r)$ is transitive on $\{\lambda_1, \ldots, \lambda_r\}$. If $\mathcal{S}_r$ is transitive then each of its lines must have the same profile, and so the profile of $\mathcal{S}_r$ has to be of the form $(m_1, m_2, m_3)^r$, where the integers $m_i > 0$ satisfy $(m_1 + m_2 + m_3)r = 3L$. Amongst the transitive partial spreads $\mathcal{S}_r$ are the *cyclic* ones, for which $\mathcal{G}(\mathcal{S}_r)$ contains a cyclic subgroup $\cong Z_r$ which is transitive on $\{\lambda_1, \ldots, \lambda_r\}$.

The *signature* of a point-set ψ will be as used in [7]. For example, signature(ψ) $=$ $(15, 23, 4)(2^1 12^5 1^7)(2^0 12^2 1^4)$ conveys the information that the point-set ψ contains ($|\psi| =$) 15 points, ($L =$) 23 lines and 4 planes; moreover each of 2 of the 15 points lies on just 1 of the 23 lines, each of 12 of the points lies on 5 of the lines, and 1 of the points lies on 7 of the lines; also 2 of the points lie on none of the 4 planes, 12 of the points lie on 2 of the planes, and 1 of the points lies on all 4 of the planes.

We will find very useful the somewhat surprising fact that the sum $L(\psi) + L'(\psi)$ depends only upon $|\psi|$, and not upon the orbit $\Theta(\psi)$ of ψ:

Lemma 1.1 *For* any *point-set* $\psi \subset \mathrm{PG}(4,2)$ *let* $k = k(\psi) = |\psi| - 15$. *Then*

$$L(\psi) + L'(\psi) = 35 + k(k-1)/2. \tag{1.2}$$

In particular (1.2) holds in the cases $\psi = \psi(\mathcal{S}_r)$, $k = 3r - 15$, *and then the values of* $L(\psi) + L'(\psi)$ *are as follows*

$$\begin{array}{rccccccccc} r = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ L + L' = & 113 & 80 & 56 & 41 & 35 & 38 & 50 & 71 & 101 \end{array}. \tag{1.3}$$

Proof. Set $n = 4$ in the $\mathrm{PG}(n,2)$ result [24, Theorem 1.1]. ■

By part (ii) of the next lemma there was no need to include the value $r = 10$ in (1.3).

Lemma 1.2 *(i) If* $\psi = \psi(\mathcal{S}_r)$ *then* $\sum_{x\in\psi} x = 0$ *and* $\sum_{x\in\psi^c} x = 0$.

(ii) Partial spreads of 10 *lines do not exist in* $\mathrm{PG}(4,2)$.

(iii) For a partial spread of 9 *lines the complementary 4-set* ψ^c *is a (plane$\setminus$line).*

Proof. (i) This holds since $\sum_{x\in\lambda} x = 0$ for each $\lambda \in \mathcal{S}_r$, and since $\sum_{x\in\mathrm{PG}(4,2)} x = 0$.
(ii) From $\sum_{x\in\psi^c} x = 0$ it follows that ψ^c cannot be a single point.
(iii) This follows since $\sum_{x\in\psi^c} x = 0$. ■

1.2 Reguli and reguli patterns

A partial spread $\mathcal{S}_r = \{\lambda_1, \ldots, \lambda_r\}$ of r (≥ 3) lines in $\mathrm{PG}(4,2)$ will be termed *regulus-free,* or also *non-degenerate,* whenever $\langle\lambda_i, \lambda_j, \lambda_k\rangle = \mathrm{PG}(4,2)$ holds for each triple of lines of $\mathcal{S}_r$. In the degenerate cases there is at least one triple such that $\langle\lambda_i, \lambda_j, \lambda_k\rangle = \mathrm{PG}(3,2)$, that is such that $\{\lambda_i, \lambda_j, \lambda_k\}$ is a regulus. Take note that *if* S_r *contains a regulus* ρ *then another partial spread* S_r^* *can be obtained by replacing* ρ *by* ρ^{opp}. In this connection it should be borne in mind that although the subgroup $\mathrm{O}^+(4,2)$ of $\mathrm{GL}(4,2)$ which preserves a quadric $\mathcal{H} = \psi(\rho) = \psi(\rho^{\mathrm{opp}})$ lying in a $\mathrm{PG}(3,2)$ contains elements which interchange ρ and ρ^{opp}, it does *not* follow thereby that S_r and S_r^* are equivalent.

Lemma 1.3 *For an* $\mathcal{S}_r$ *in* $\mathrm{PG}(4,2)$ *the number* N_r *of reguli of* $\mathcal{S}_r$ *and the total number* L *of internal lines of* $\psi(\mathcal{S}_r)$ *are related by*

$$r + \binom{r}{3} + 2N_r = L. \tag{1.4}$$

Proof. The r members of $\mathcal{S}_r$ give rise to $\binom{r}{3}$ triples of lines. Each non-degenerate triple has a unique transversal, while each triple which is a regulus has 3 transversals (the members of the opposite regulus). ■

Given a partial spread $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$, let R_{ijk}, for distinct i,j,k, carry the meaning that the triple of lines $\{\lambda_i,\lambda_j,\lambda_k\} \subset \mathcal{S}_r$ is a regulus. It will turn out, see corollary 5.2, that the only *reguli patterns* which can arise are those listed in the second column of the following table 1.

Table 1 The possible reguli patterns

N	Regulus Pattern	Type	Types for $\mathcal{S}_r\diagdown\{\lambda\}$
0	regulus-free	O	O
1	R_{123}	I	I (if $r > 3$), O
2	R_{123}, R_{456}	II	II (if $r > 6$), I
2	R_{123}, R_{345}	L	L (if $r > 5$), I, O
3	$R_{123}, R_{456}, R_{678}$	IL	L, II, I
3	$R_{123}, R_{145}, R_{167}$	Y	Y (if $r > 7$), L, O
3	$R_{123}, R_{145}, R_{267}$	F	F (if $r > 7$), II, L, I
3	$R_{123}, R_{345}, R_{561}$	Δ	Δ (if $r > 6$), L, I
4	$R_{123}, R_{345}, R_{561}, R_{789}$	IΔ	Δ, IL, II
4	$R_{123}, R_{145}, R_{167}, R_{189}$	X	Y, O
4	$R_{123}, R_{145}, R_{267}, R_{389}$	E	F, II
4	$R_{123}, R_{124}, R_{134}, R_{234}$	$\binom{4}{3}$	$\binom{4}{3}$ (if $r > 4$), I
10	$R_{ijk},\ 1 \le i < j < k \le 5$	$\binom{5}{3}$	$\binom{4}{3}$ ($r = 5$)

For convenience we also record the pattern in abbreviated form, referred to as the regulus *type,* see the third column of the table. Hopefully the symbol which we adopt for the type immediately conveys the pattern. Thus in the case of type X, the four arms of the symbol X represent the presence of four reguli, and the fact that the centre of the X lies on all four arms indicates that one of the lines, λ_1 for the pattern in the table, belongs to all four reguli. A partial spread $\mathcal{S}_9$ of type X will be referred to as an $\mathcal{S}_9(\mathsf{X})$; a similar interpretation applies to $\mathcal{S}_5(\mathsf{O})$, $\mathcal{S}_6(\Delta)$, $\mathcal{S}_7(\mathsf{II})$, ... For $\lambda \in \mathcal{S}_r$ the possible reguli types for $\mathcal{S}_{r-1} = \mathcal{S}_r\diagdown\{\lambda\}$ are listed in the final column of the table. Since $r \le 9$, see lemma 1.2, and since types IΔ, X, E require $r = 9$, these three types can not occur in the final column.

Remark 1.4 *Certain other feasible types, for example type W, with pattern* $R_{123}, R_{345}, R_{567}, R_{789}$, *and type LV, with pattern* $R_{123}, R_{345}, R_{567}, R_{589}$, *do not appear in the table because, see corollary 5.2, they are not possible for a partial spread in* $\mathrm{PG}(4,2)$. *Incidentally it will later emerge that there are just two classes of partial spread* $\mathcal{S}_9$ *of type* IΔ, *and it then proves useful to refer to them as* $\mathsf{I}^\rho\Delta$ *and* $\mathsf{I}^\kappa\Delta$, *see theorem 4.5. These two classes of* $\mathcal{S}_9$ *give rise to two classes of* $\mathcal{S}_8$ *of type* IL, *and these will be referred to as* $\mathsf{I}^\rho\mathsf{L}$ *and* $\mathsf{I}^\kappa\mathsf{L}$, *see section 5.2.*

1.3 Maximal partial spreads and reguli types

1.3.1 Types $\binom{4}{3}$ and $\binom{5}{3}$

In $\mathrm{PG}(3,2)$ there exists a projectively unique spread Σ_5 of five lines, see [11]. Since any $\mathcal{S}_3$ in $\mathrm{PG}(3,2)$ is a regulus, the spread Σ_5 is of type $\binom{5}{3}$. Moreover any $\mathcal{S}_4$ in $\mathrm{PG}(3,2)$ is also projectively unique, of the form $\Sigma_5\setminus\{\lambda\}$, and so necessarily of type $\binom{4}{3}$. In $\mathrm{PG}(4,2)$ observe that

$$R_{ijk} \;\&\; R_{ijl} \Longrightarrow R_{ikl} \;\&\; R_{jkl}, \tag{1.5}$$

since if $\{\lambda_i,\lambda_j,\lambda_k\}$ and $\{\lambda_i,\lambda_j,\lambda_l\}$ are both reguli then the four lines λ_i, λ_j, λ_k, λ_l lie in a common $\mathrm{PG}(3,2)$ and constitute an $\mathcal{S}_4$ in the $\mathrm{PG}(3,2)$.

Lemma 1.5 *(i) A partial spread $\mathcal{S}_5$ in* $\mathrm{PG}(4,2)$ *which is a spread in a* $\mathrm{PG}(3,2)$ *subspace σ is a maximal partial spread in* $\mathrm{PG}(4,2)$.

(ii) If $\mathcal{S}_4$ in $\mathrm{PG}(4,2)$ *is of type $\binom{4}{3}$ then $\mathcal{S}_4$ lies in some hyperplane σ, and so is of the form $\Sigma_5\setminus\{\lambda\}$ where Σ_5 is a spread in σ.*

(iii) Suppose that $\mathcal{S}_r$ is a partial spread in $\mathrm{PG}(4,2)$ *which is an extension of an $\mathcal{S}_4$ of type $\binom{4}{3}$. Then either*
(a) $\mathcal{S}_r$ is of type $\binom{5}{3}$ and is a spread Σ_5 in a hyperplane, or
(b) $r=5,\ 6$ or 7 and $\mathcal{S}_r$ is itself of type $\binom{4}{3}$.

(iv) Any $\mathcal{S}_7$ of type $\binom{4}{3}$ is maximal.

Proof. (i) Any line in $\mathrm{PG}(4,2)$ intersects σ, and hence meets a line of $\mathcal{S}_5$; thus $\mathcal{S}_5$ can not be extended to an $\mathcal{S}_6$.

(ii) This follows from (1.5) and surrounding remarks.

(iii) By (i) $\mathcal{S}_4=\Sigma_5\setminus\{\lambda\}$ where $\Sigma_5=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda\}$ is a spread in a hyperplane σ. Extending $\mathcal{S}_4$ using λ yields the possibility (a) in part (iii). If instead, possibility (b), lines $\lambda_5,\lambda_6,\lambda_7$, ... other than λ are used to extend $\mathcal{S}_4$ these must meet σ in distinct points a_5,a_6,a_7, ... of λ. But λ only has 3 points, whence $r\le 7$. We still need to show that for possibility (b) no further reguli are present. Suppose, to the contrary, first of all that $\{\lambda_i,\lambda_5,\lambda_6\}$ is a regulus ρ for some $i\in\{1,2,3,4\}$. Now for some $x\in\lambda_i$ the line $\mu=\mathrm{join}(a_5,x)$ will belong to ρ^{opp} and so meet λ_6 in some point b. But $\mu\subset\sigma$, and so $b=a_6$. Hence $\mu=\mathrm{join}(a_5,a_6)=\lambda$, which is impossible since x can not lie on both lines of the skew pair λ_i, λ. Suppose secondly (for $r=7$) that $\{\lambda_5,\lambda_6,\lambda_7\}$ is a regulus ρ, and so $\lambda=\{a_5,a_6,a_7\}\in\rho^{\mathrm{opp}}$. Choosing $\nu\in\rho^{\mathrm{opp}},\nu\ne\lambda$, the point $p=\nu\cap\sigma$ lies on λ_j for some $j\in\{5,6,7\}$; but $p\in\sigma\setminus\{\lambda\}$ and so p also lies on λ_i for some $i\in\{1,2,3,4\}$, contradicting $\mathcal{S}_7$ being a partial spread. (Alternatively, $\{\lambda_5,\lambda_6,\lambda_7\}$ being a regulus would yield $N_7\ge 5$, contradicting the result $N_7\le 4$ in (1.6d) below.)

(iv) Such an $\mathcal{S}_7$ is an extension of an $\mathcal{S}_4$ of type $\binom{4}{3}$, and so, see the proof of $r\le 7$ in (iii), can not extend to an $\mathcal{S}_8$. ■

1.3.2 The possibilities for $(L,\ L',\ N_r)$

Theorem 1.6 *For a partial spread $\mathcal{S}_r$ in* PG(4,2), *the only possibilities for L, L', N_r and regulus type are as given in table 2. In particular:*

$$(a)\ N_4 \notin \{2,3\};\quad (b)\ N_5 \notin \{3,5,6,7,8,9\};\quad (c)\ N_6 \le 4;$$
$$(d)\ N_7 \le 4;\quad (e)\ N_8 \le 3,\ \text{but } N_8 \ne 1;\quad (f)\ N_9 = 4. \tag{1.6}$$

Table 2 Possibilities for $L,\ L',\ N_r$

r	L	L'	N_r	Types
3	6	50	1	I
	4	52	0	O
4	16	25	4	$\binom{4}{3}$
	10	31	1	I
	8	33	0	O
5	35	0	10	$\binom{5}{3}$; maximal
	23	12	4	$\binom{4}{3}$
	19	16	2	L
	17	18	1	I
	15	20	0	O
6	34	4	4	$\binom{4}{3}$
	32	6	3	Δ

Table 2 (Cont.)

r	L	L'	N_r	Types
6	30	8	2	L, II
	28	10	1	I
	26	12	0	O
7	50	0	4	$\binom{4}{3}$; max.
	48	2	3	F, Y, Δ
	46	4	2	L, II
	44	6	1	I
	42	8	0	O
8	70	1	3	F, Y, IL, Δ
	68	3	2	II, (not L)
	64	7	0	O
9	101	0	4	E, IΔ, X; max.

Proof. Most of the entries in table 2 follow immediately from eqs. (1.3) and (1.4). For example if $r = 7$ then (1.3) and (1.4) assert $L + L' = 50$ and $42 + 2N_7 = L$; hence the only possibilities for (L, L', N_7) are the five listed in the table. In particular, since $L' \ge 0$, we have $L \le 50$ and so $N_7 \le 4$, as stated in (1.6d). Similarly $N_8 \le 3$, as stated in (1.6e). If $r = 9$ then (1.3) and (1.4) assert $L + L' = 101$ and $93 + 2N_9 = L$; but $L' = 0$, by lemma 1.2, and so $N_9 = 4$, as stated in (1.6f).

Concerning (1.6a,b), these follow from the results in section 1.3.1. In particular from eq. (1.5) we see that $N_4 > 1$ implies $N_4 = 4$; similarly $N_5 > 2$ implies $N_5 \ge 4$ and $N_5 > 4$ implies $N_5 = 10$.

Concerning (1.6c), the possibility $(L, L', N_6) = (36, 2, 5)$ for an $\mathcal{S}_6$ is ruled out, since an extension to an $\mathcal{S}_7$ would then exist (because $L' > 0$) and have $N_7 \ge 5$, contradicting (1.6d) $N_7 \le 4$. The only other possibility $(L, L', N_6) = (38, 0, 6)$ for an $\mathcal{S}_6$ is also ruled out: an $\mathcal{S}_6$ with $N_6 = 6$ would, by (1.5), contain an $\mathcal{S}_4$ of type $\binom{4}{3}$ and hence by lemma 1.5(iii) be also of type $\binom{4}{3}$, having $N_6 = 4$ contradicting $N_6 = 6$.

We have still to prove $N_8 \ne 1$. Suppose to the contrary that an $\mathcal{S}_8$ had $(L, L', N_8) = (66, 5, 1)$. Then $\psi(\mathcal{S}_8)^c$ would be a 7-set having five internal lines, which is not possible. (A 7-set in PG(4,2) with ≥ 5 internal lines is necessarily a plane, having 7 internal lines.)

Concerning the final column of the table, we delay until corollary 5.2 the proof that no other reguli types occur. ■

Theorem 1.7 *If $\mathcal{S}_r$ is a maximal partial spread of r lines in* PG(4, 2) *then one of the following holds:*

$$(a)\ r = 5,\ N_5 = 10; \quad (b)\ r = 7,\ N_7 = 4; \quad (c)\ r = 9,\ N_9 = 4. \tag{1.7}$$

Moreover each of the possibilities (a)-(c) is realized.

Proof. From table 2 the possibilities (1.7a,b,c) are the only ones for which $L' = 0$. Possibilities (a) and (b) are realized, see lemma 1.5i, iv (and also section 7.1 for more details). Concerning possibility (c), see section 5.1 for its realizations. ∎

1.3.3 The possible regulus types for an $\mathcal{S}_9$

Given a partial spread $\mathcal{S}_r = \{\lambda_1, \ldots, \lambda_r\}$ which contains N_r reguli, suppose that λ_i belongs to precisely n_i of these reguli. We refer to n_i as the *valency* of the line λ_i. Observe that these valencies satisfy $\Sigma_{i=1}^{r} n_i = 3N_r$.

Lemma 1.8 *For an $\mathcal{S}_9$ the valencies $n_1, \ldots, n_9$ satisfy*

$$(i)\ \Sigma_{i=1}^{9} n_i = 12, \qquad (ii)\ n_i > 0. \tag{1.8}$$

Proof. Since $N_9 = 4$, result (i) follows. Concerning (ii), if a line $\lambda_i \in \mathcal{S}_9$ were to have $n_i = 0$ then $\mathcal{S}_8 = \mathcal{S}_9 \setminus \{\lambda_i\}$ would have 4 reguli, contradicting the fact, see eq. (1.6e), that $N_8 \leq 3$. ∎

Let types LV and W be as described in remark 1.4.

Lemma 1.9 *A $\mathcal{S}_9$ has regulus type X, LV, E, IΔ or W.*

Proof. After a suitable re-labelling, only three possibilities are allowed by lemma 1.8, namely

$$\begin{aligned} &(a)\ n_1 = 4,\ n_2 = \ldots = n_9 = 1; \\ &(b)\ n_1 = 3,\ n_2 = 2,\ n_3 = \ldots = n_9 = 1; \\ &(c)\ n_1 = n_2 = n_3 = 2,\ n_4 = \ldots = n_9 = 1. \end{aligned} \tag{1.9}$$

Bearing in mind the result (1.5), the only possible types for the four reguli of an $\mathcal{S}_9$ are correspondingly

$$(a)\ \mathsf{X}; \quad (b) \mathsf{LV}; \quad (c)\ \mathsf{E},\ \mathsf{I\Delta},\ \mathsf{W}. \tag{1.10}$$

(In fact types LV and W do not occur, see corollary 5.2.) ∎

Corollary 1.10 *Every $\mathcal{S}_9$ contains an $\mathcal{S}_5$ of type L.* ∎

1.4 Further considerations

1.4.1 Dual partial spreads

Nonzero elements of the vector space V_n^* dual to V_n are identified with the points of the dual projective space $\mathrm{PG}(n-1,2)^* = \mathbb{P}V_n^*$. Recall that the annihilator $U^{\mathrm{O}} := \{f \in V_n^* \mid f(u) = 0$, for all $u \in U\}$ of any subset $U \subset V_n$ is always a subspace of V_n^*; moreover if $\dim \prec U \succ = k$ then $\dim U^{\mathrm{O}} = n - k$. *We use the same notation also for projective subspaces.* Thus if α is a plane in $\mathrm{PG}(4,2)$ then α^{O} is a line in $\mathrm{PG}(4,2)^*$.

Quite generally the notion of a partial spread dualizes to yield the notion of a *dual partial spread.* In the present case of $\mathrm{PG}(4,2)$ a dual partial spread $\mathcal{P}_r$ of size $r(>1)$ is a set $\{\pi_1, \dots, \pi_r\}$ of r planes in $\mathrm{PG}(4.2)$ such that $\langle \pi_i, \pi_j \rangle = \mathrm{PG}(4,2)$ for each $i \neq j$. By setting $\mu_i^* = (\pi_i)^{\mathrm{O}}$ note that dual partial spreads $\mathcal{P}_r = \{\pi_1, \dots, \pi_r\}$ in $\mathrm{PG}(4,2)$ are in bijective correspondence with partial spreads $\mathcal{S}_r^* = \{\mu_1^*, \dots, \mu_r^*\}$ in the dual space $\mathrm{PG}(4,2)^*$. (The condition $\langle \pi_i, \pi_j \rangle = \mathrm{PG}(4,2)$ is equivalent to the condition $\mu_i \cap \mu_j = \emptyset$.) Consequently the classification of partial spreads in $\mathrm{PG}(4,2)$ obtained in this paper amounts also to a classification of dual partial spreads in $\mathrm{PG}(4,2)$.

See section 2.3.2 below for an instance where duality considerations bear fruit.

1.4.2 The Grassmannian $G_{1,4,2}$ in $\mathrm{PG}(9,2)$

Along with the vector space $V_5 := V(5,2)$, and its projective space $\mathrm{PG}(4,2) = \mathbb{P}V_5$, we have the associated space $V_{10} := \wedge^2 V_5$ of bivectors, and its projective space $\mathrm{PG}(9,2) = \mathbb{P}(\wedge^2 V_5)$. Each $A \in \mathrm{GL}(5,2)$ gives rise to a corresponding element $T_A = \wedge^2 A$ of $\mathrm{GL}(V_{10})$ whose effect on the decomposable bivectors $u \wedge v \in V_{10}$ is $T_A(u \wedge v) = Au \wedge Av$, $A \in \mathrm{GL}(5,2)$. Under the action T of $\mathrm{GL}(5,2)$ the projective space $\mathrm{PG}(9,2)$ is the union $\mathrm{Rk}_2 \cup \mathrm{Rk}_4$ of two $\mathrm{GL}(5,2)$-orbits, consisting of those bivectors having rank 2 and rank 4, respectively. The Grassmann map $\prec u, v \succ \mapsto \prec u \wedge v \succ$ sends the 2-spaces of V_5 to those 1-spaces of $\wedge^2 V_5$ which are spanned by decomposable bivectors. Projectively, the lines of $\mathrm{PG}(4,2)$ are mapped onto the points of the orbit Rk_2, the latter, being the Grassmannian $G_{1,4,2} \subset \mathrm{PG}(9,2)$ of lines of $\mathrm{PG}(4,2)$, having length 155. Consequently $|\mathrm{Rk}_4| = 1023 - 155 = 868$. (Similarly the projective space $\mathrm{PG}(9,2)^* := \mathbb{P}(\wedge^3 V_5)$ is the union of two orbits of lengths 155 and 868; using the Grassmann map $\prec u, v, w \succ \mapsto \prec u \wedge v \wedge w \succ$, the 155 planes of $\mathrm{PG}(4,2)$ are mapped onto the 155 points of the Grassmannian $G_{2,4,2} \subset \mathrm{PG}(9,2)^*$.)

Let $\mathcal{C}_r = \{m_1, \dots, m_r\} \subset \mathrm{PG}(9,2)$ denote the Grassmann image of a partial spread $\mathcal{S}_r = \{\mu_1, \dots, \mu_r\}$ in $\mathrm{PG}(4,2)$. Observe that the study of partial spreads $\mathcal{S}_r$ in $\mathrm{PG}(4,2)$ is equivalent to the study of r-sets $\mathcal{C}_r = \{m_1, \dots, m_r\}$ in $\mathrm{PG}(9,2)$ which satisfy the two conditions

$$\text{(i) } \mathcal{C}_r \subset \mathrm{Rk}_2 (= G_{1,4,2}); \qquad \text{(ii) } m_i + m_j \in \mathrm{Rk}_4, \text{ for each } i \neq j. \tag{1.11}$$

2 Outline and outlook

2.1 Overview

In this paper we will obtain a classification, summarized in tables B.2a and B.2b (see appendix B) of all the partial spreads $\mathcal{S}_r$ of lines in PG(4, 2). The various GL(5, 2)-orbits of the underlying point-sets $\psi(\mathcal{S}_r)$ which occur are classified in table B.1. In these tables roman numerals are used to indicate the value of r. Thus the six orbits for a $\psi(\mathcal{S}_7)$ are labelled VIIa, VIIb, ... , VIIf. Also the five equivalence classes of partial spreads $\mathcal{S}_7$ whose point-sets lie on the orbit VIIe are labelled VIIe.1, VIIe.2, ... ,VIIe.5.

Up to the action of GL(5, 2) we will show that there are exactly 64 distinct classes of partial spreads, the number s_r of equivalence classes of partial spreads $\mathcal{S}_r$ of size r being

$$s_1 = 1,\ s_2 = 1,\ s_3 = 2,\ s_4 = 4,\ s_5 = 10,\ s_6 = 14,\ s_7 = 19,\ s_8 = 9,\ s_9 = 4.$$

The chief interest no doubt lies with the maximal partial spreads. We will show that *there are precisely eight equivalence classes of maximal partial spreads, one of size 5, three of size 7 and four of size 9:*

$$\text{Vj.1;}\quad \text{VIIf.1, VIIf.2, VIIf.3;}\quad \text{IXa.1, IXa.2, IXa.3, IXa.4.} \tag{2.1}$$

But certain of the non-maximal partial spreads are also worthy of attention, for example the cyclic partial spreads and the regulus-free (type O) partial spreads. For $r \geq 3$ there are nine classes of cyclic partial spreads:

$$\text{IIIa.1, b.1;}\quad \text{IVa.1, d.1;}\quad \text{Va.1, e.1, j.1;}\quad \text{VIa.1;}\quad \text{VIIa.1.}$$

If $\mathcal{S}_r$ is cyclic, recall that the profile of $\mathcal{S}_r$ has to be of the form $(m_1, m_2, m_3)^r$, with the integers $m_i > 0$ satisfying $(m_1+m_2+m_3)r = 3L$. For example, the profile of a cyclic $\mathcal{S}_5$ is seen to be $(3, 3, 3)^5$, $(2, 3, 4)^5$ or $(7, 7, 7)^5$, according as $\mathcal{S}_5$ belongs to the class Va.1, Ve.1 or Vj.1.

For $r \geq 3$ there are twelve classes of regulus-free partial spreads:

$$\text{IIIa.1;}\quad \text{IVa.1, b.1;}\quad \text{Va.1, ... , e.1;}\quad \text{VIa.1, b.1;}\quad \text{VIIa.1;}\quad \text{VIIIa.1.}$$

Partial spreads belonging to three of these classes, namely Va.1, VIb.1 and VIIIa.1, have the maximality property of admitting no extension to a larger regulus-free partial spread. Class Va.1 consists of spreads on parabolic quadrics $\mathcal{P}_4$, and its influence also spills over to the classes Vb.1, Vc.1, Vd.1, see sections 3.4.1, 3.4.2; it has stabilizer group $\mathcal{G}(\mathcal{S}_5) \cong$ Sym(5). Class VIIIa.1 also has a large stabilizer group, with $\mathcal{G}(\mathcal{S}_8) \cong 2^3{:}\mathrm{F}_{21}$, of order 168, which acts transitively on $\mathcal{S}_8$. (The stabilizer groups for the other eight classes of $\mathcal{S}_8$ are all of order ≤ 6.) This high symmetry is perhaps best seen using the construction in section 6.1. For an interesting aspect of partial spreads of class VIIIa.1, see section 2.3.2 below.

The group Sym(6), isomorphic to both Sp(4, 2) and to O(5, 2), makes a (somewhat unexpected!) appearance in our description of roughly half of the classes of partial spreads.

So in Appendix A we summarize the relevant mathematics surrounding the group isomorphisms

$$\text{(i)}\ \ \mathrm{Sp}(4,2) \cong \mathrm{Sym}(6), \qquad \text{(ii)}\ \ \mathrm{O}(5,2) \cong \mathrm{Sym}(6). \tag{2.2}$$

A partial spread $\mathcal{S}_r$ comes along with various *invariants,* and by computing one or more of these we are in many cases able to assign $\mathcal{S}_r$ to its class. First of all there is the orbit $\Theta(\psi)$ of the underlying point-set $\psi(\mathcal{S}_r)$. Indeed the orbit $\Theta(\psi)$ alone suffices to distinguish the classes of $\mathcal{S}_r$ for the cases $r \leq 5$. The orbit $\Theta(\psi)$ may in turn be determined, by reference to table B.1, by computing signature(ψ) and signature(ψ^c). Usually one needs only to compute the easiest one of these, namely the signature of the smaller of the two sets $\psi,\ \psi^c$. However, see section B.3.1(i), the signature of ψ^c is needed to distinguish between the orbits Va and Vc. Other invariants of a partial spread $\mathcal{S}_r$ include its regulus type, the structure of its stabilizer group $\mathcal{G}(\mathcal{S}_r)$ and its profile. All of these invariants (including a refined version of the profile, see section B.3.2) are given in the tables in appendix B.

However, for $r > 5$, a number of cases arise where two partial spreads $\mathcal{S}_r$ and $\mathcal{S}'_r$ share the same profile, and also the other invariants of the last paragraph, but which are inequivalent. In such cases it may at first (or even second!) glance be hard to distinguish between $\mathcal{S}_r$ and $\mathcal{S}'_r$. In the majority of such cases the distinction can be made by use of symplectic considerations arising from the isomorphism (2.2i). See for example the discussion of the classes VIc.1, VIc.2 and VId.1, VId.2 in section 7.3.1, which follows on from the $\mathrm{Sp}(4,2)$ treatment in section 7.1 of the three maximal partial spreads of size 7. In a few cases, see classes VIe.1, e.2 in section 7.3.1 and classes VIIc.1, c.2 in section 7.2.2, the distinction proves more intractable. But in all cases we provide at least one way of making the distinction.

2.2 Plan of the proof

In outline, the main steps involved in the proof of our classification are as follows. First of all we deal with partial spreads $\mathcal{S}_r$ for $r \leq 5$. See section 3. Here the two classes IIIa.1 and Vh.1 should be particularly noted. If $\mathcal{S}_3(\mathsf{O}) \in$ class IIIa.1 then, see section 3.2, it determines a privileged *even hyperplane* equipped with a distinguished *null polarity*, and it turns out that the resulting $\mathrm{Sp}(4,2)$ considerations help in the understanding of many of the classes considered later. The class Vh.1, consisting of partial spreads $\mathcal{S}_5$ of type L and considered further in section 4.1, is of especial importance, since by corollary 1.10 every partial spread $\mathcal{S}_9$ contains an $\mathcal{S}_5(\mathsf{L}) \in$ class Vh.1.

Secondly, using this last fact, in sections 4 and 5 we determine all partial spreads $\mathcal{S}_r$ for $r > 5$ which are not of type O, I, II or $\binom{4}{3}$. Then, in section 6, we consider those partial spreads $\mathcal{S}_6$, $\mathcal{S}_7$ and $\mathcal{S}_8$ which are of type O. Finally, in section 7, we classify the remaining partial spreads, namely those $\mathcal{S}_r,\ r > 5$, which are of type I, II or $\binom{4}{3}$, the latter including the three classes of maximal partial spreads of size 7.

2.2.1 Use of the computer

The proof that there are just the eight classes (2.1) of maximal partial spreads in PG$(4,2)$ is computer-free. Our determination of all equivalence classes of non-maximal partial spreads is also essentially computer-free. Nevertheless use of the computer has been of great help to the project, especially in respect of the detailed information provided in tables B.1, B.2a and B.2b, as we now describe.

First of all consider the entries in table B.1 under the headings signature(ψ) and signature(ψ^c). On the one hand it is straightforward to compute these by hand, especially for the smaller of the two sets $\psi(\mathcal{S}_r),\ \psi(\mathcal{S}_r)^c$. See for example the computation immediately before theorem 6.4 of the signature of a 13-set $\psi(\mathcal{S}_6)^c$. However such computations are rather tedious, and especially so for the larger of the sets $\psi,\ \psi^c$. So, for many of the signatures, we have relied on computer-generated results using Magma. (For details of Magma, see [2] and [3].)

Secondly, a similar remark applies to the entries in tables B.2a and B.2b under the heading profile($\mathcal{S}_r$): many entries were computed by hand, but all were computed using Magma. The latter was also used to analyze output from GRAPE, see below, especially in connection with the regulus pattern of partial spreads.

Thirdly, in assigning the underlying point-sets of distinct classes of partial spreads, such as VIIe.1, VIIe.2, ... ,VIIe.5, to the same GL$(5,2)$-orbit, in this case VIIe, we make use of the following fact: given $\psi = \psi(\mathcal{S}_r)$ and $\psi' = \psi(\mathcal{S}'_r)$ then

$$\left.\begin{array}{r}\text{signature}(\psi) = \text{signature}(\psi')\\ \text{signature}(\psi^c) = \text{signature}((\psi')^c)\end{array}\right\} \Longrightarrow \Theta(\psi) = \Theta(\psi').$$

For this fact we appeal to a certain more general result which may be read off from [7, Table 8.4.6], the derivation of which used the computer, see section B.3.1(ii). We also borrowed from [7] information concerning the orders of the stabilizer groups of point-sets, see column 4 of table B.1, although many of these were also obtained by hand, see for example section 3.1.

Finally we mention the extremely useful contribution from GRAPE, see below, at a half-way stage of the project. At this stage the determination of the eight classes (2.1) of maximal partial spreads was almost complete, but we were somewhat daunted by the task of describing all the non-maximal ones. The results from GRAPE pointed the way to what was still to be done by hand, and provided invaluable checks on our assertions concerning the stabilizer groups $\mathcal{G}(\mathcal{S})$, and that there are indeed precisely 56 classes of non-maximal partial spreads.

GRAPE [25] is a GAP [6] package for computing with finite graphs endowed with group actions. The GRAPE package is designed primarily for constructing and analyzing graphs related to groups, designs and finite geometries. The GRAPE philosophy is that a graph Γ always comes together with a known subgroup G of the automorphism group of Γ (the group G usually comes from the construction of Γ). Then G is used to store Γ efficiently and to speed up computations with Γ.

In the present context, the graph Γ constructed by GRAPE is the graph whose vertices are the lines of PG$(4,2)$, with two lines (vertices) joined by an edge if and only if they are skew. The associated group G of automorphisms is $\mathrm{GL}(5,2)$ in its action on the lines of $\mathrm{PG}(4,2)$. Now the partial spreads in PG(4,2) are precisely the vertex-sets of the complete subgraphs of Γ. We determined these complete subgraphs up to G-equivalence using GRAPE (version 4.0), first determining the maximal complete subgraphs up to G-equivalence. GAP (version 4.1) was then used to determine the stabilizer $\mathcal{G}(\mathcal{S})$ for each partial spread $\mathcal{S}$ classified. The complete classification of partial spreads and the determination of their stabilizers took about 10 minutes of CPU-time on a 350 MHz Pentium II PC running Linux. This computation is detailed explicitly within the web page [26]. (With the latest version of GRAPE (version 4.2), this computation can be done more quickly.)

2.3 Outlook

We claim that the classification in this paper is currently the only complete classification of partial spreads in a projective space PG(4,q). Moreover we are of the strong belief that any future complete classification in $\mathrm{PG}(4,q)$ for any value of $q > 2$ will only be achieved with much use of the computer and with much less theoretical understanding than in the present $q = 2$ case.

Naturally we hope that our classification will prove to be of interest and use to other researchers in the field. In support of this, we sketch, in the rest of this section, three instances where the classification has already proved to be of considerable use.

2.3.1 Flats in $\mathrm{PG}(9,2)$ external to the Grassmannian $G_{1,4,2}$

In this section the terms *internal* and *external* refer to the Grassmannian $\mathrm{Rk}_2 = G_{1,4,2} \subset \mathrm{PG}(9,2)$, see section 1.4.2. The internal flats of $G_{1,4,2}$ are well understood, the maximal ones being of two kinds, *Greek planes* and *Latin solids,* see [10, Section 24.2]. However it is much more difficult to determine all $\mathrm{GL}(5,2)$-orbits of external flats. But in fact this formidable classification problem has recently been completely solved, see [15], [16], [17] — *aided in no small way by the results of the classification in the present work.*

For $k = 0, 1, 2, 3, 4$, the external k-flats in $\mathrm{PG}(9,2)$ are proved in [17] to fall respectively into $1, 2, 3, 2, 2$ distinct $\mathrm{GL}(5,2)$-orbits:

$$\begin{array}{llll} k=0: & \mathrm{Rk}_4; & k=1: & \mathrm{orb}(1\alpha),\ \mathrm{orb}(1\beta); \\ k=2: & \mathrm{orb}(2\alpha),\ \mathrm{orb}(2\beta),\ \mathrm{orb}(2\gamma); & & \\ k=3: & \mathrm{orb}(3\alpha),\ \mathrm{orb}(3\beta); & k=4: & \mathrm{orb}(4+),\ \mathrm{orb}(4-). \end{array} \tag{2.3}$$

Furthermore no external k-flats exist for $k > 4$. One way to see this last is to use the fact, see [13], [14], that $G_{1,4,2}$ is a hypersurface in $\mathrm{PG}(9,2)$ of degree 5, and then to apply the theorem in [19]. It was demonstrated in [17] that of the ten orbits (2.3) no less than seven can be simply constructed out of partial spreads in $\mathrm{PG}(4,2)$, by means of the following *even hyperplane construction.*

The even hyperplane construction of external flats. As in section 1.4.2, let $\mathcal{C}_r = \{m_1, \dots, m_r\} \subset \mathrm{PG}(9,2)$ denote the Grassmann image of a partial spread $\mathcal{S}_r = \{\mu_1, \dots, \mu_r\}$ in $\mathrm{PG}(4,2)$. In searching for external flats the property (1.11)(ii) suggests that we consider the projective space $\mathcal{E}(\mathcal{C}_r)$ defined to be that generated by the $\binom{r}{2}$ points $m_i + m_j$. In cases where the m_i are linearly independent then $\mathcal{E}(\mathcal{C}_r)$ is the unique hyperplane inside the $(r-1)$-flat $\langle\mathcal{C}_r\rangle$ which is disjoint from the r points $m_i \in \mathcal{C}_r$; since the latter points are internal, take note that no hyperplane of $\langle\mathcal{C}_r\rangle$ other than $\mathcal{E}(\mathcal{C}_r)$ is a candidate to be an external flat.

Theorem 2.1 *(See [17, Theorems 2.5, 2.10, 2.11].) If $\mathcal{S}_r, r \geq 2$, is a partial spread in* $\mathrm{PG}(4,2)$, *then $\mathcal{E}(\mathcal{C}_r)$ is an external flat if and only if one of the following holds:*

(o) $r = 2$; $\mathcal{E}(\mathcal{C}_2)$ is an external point;
(i) $r = 3$ and one of the following holds:
  (a) $\mathcal{S}_3 \in$ class IIIa.1, in which case $\mathcal{E}(\mathcal{C}_3) \in \mathrm{orb}(1\alpha)$;
  (b) $\mathcal{S}_3 \in$ class IIIb.1, in which case $\mathcal{E}(\mathcal{C}_3) \in \mathrm{orb}(1\beta)$;
(ii) $r = 4$ and one of the following holds:
  (a) $\mathcal{S}_4 \in$ class IVb.1, in which case $\mathcal{E}(\mathcal{C}_4) \in \mathrm{orb}(2\alpha)$;
  (b) $\mathcal{S}_4 \in$ class IVc.1, in which case $\mathcal{E}(\mathcal{C}_4) \in \mathrm{orb}(2\beta)$;
(iii) $r = 5$ and one of the following holds:
  (a) $\mathcal{S}_5 \in$ class Ve.1, in which case $\mathcal{E}(\mathcal{C}_5) \in \mathrm{orb}(3\alpha)$;
  (b) $\mathcal{S}_5 \in$ class Vg.1, in which case $\mathcal{E}(\mathcal{C}_5) \in \mathrm{orb}(3\beta)$.

2.3.2 Conclaves of planes in $\mathrm{PG}(4,2)$ and certain planes in $\mathrm{PG}(9,2)$ external to $\mathcal{G}_{1,4,2}$

By theorem 2.1, of the ten orbits (2.3) of flats, external to $G_{1,4,2}$, only the three orbits $\mathrm{orb}(2\gamma)$, $\mathrm{orb}(4+)$ and $\mathrm{orb}(4-)$ can *not* be obtained using the foregoing even hyperplane construction. Concerning the external planes belonging to $\mathrm{orb}(2\gamma)$ their stabilizer group is known, [17, Theorem 4.1], to have the structure $2^3{:}\,\mathrm{F}_{21}$. Now we also have $\mathcal{G}(\mathcal{S}_8) \cong 2^3{:}\,\mathrm{F}_{21}$ in the case of a partial spread $\mathcal{S}_8$ of class VIIIa.1. So we naturally enquire: *can we construct external planes of* $\mathrm{orb}(2\gamma)$ *out of partial spreads of class VIIIa.1?* The answer is in the affirmative, as we now sketch.

First we need the notion of a *conclave of planes* in $\mathrm{PG}(4,2)$.

Definition 2.2 *An 8-set $\mathcal{P}_8 = \{\pi_1, \dots, \pi_8\}$ of planes in* $\mathrm{PG}(4,2)$ *is termed a* conclave *if the $\binom{8}{2} = 28$ intersections $\pi_i \cap \pi_j$, $i \neq j$, are distinct points. (See [23] for a more general definition, of a* conclave $\{\alpha_1, \dots, \alpha_N\}$ *of* $\mathrm{PG}(m,q)$*'s.)*

Lemma 2.3 *Let $\mathcal{P}_8 = \{\pi_1, \dots, \pi_8\}$ be a dual partial spread in* $\mathrm{PG}(4,2)$ *which is in bijective correspondence — via $\mu_i^* = (\pi_i)^O$, see section 1.4.1 — with a partial spread $\mathcal{S}_8^* = \{\mu_1^*, \dots, \mu_8^*\}$ in the dual space* $\mathrm{PG}(4,2)^*$. *Suppose $\mathcal{S}_8^*$ is of class VIIIa.1; then $\mathcal{P}_8$ is a conclave.*

Proof. To each of the solids $\sigma^*_{ij} = \langle \mu^*_i, \mu^*_j \rangle$ of $\mathrm{PG}(4,2)^*$ corresponds a point $v_{ij} := (\sigma^*_{ij})^{\mathrm{O}} = \pi_i \cap \pi_j$ of $\mathrm{PG}(4,2)$. But because $\mathcal{S}^*_8$ is regulus-free, the $\binom{8}{2} = 28$ solids σ^*_{ij} are distinct. Hence so are the 28 points v_{ij}, that is $\mathcal{P}_8$ is a conclave. ■

Theorem 2.4 *(See [18, Theorem 3.6].) Let* $\mathcal{P}_8 = \{\pi_1, \dots, \pi_8\}$ *be a conclave of planes in* $\mathrm{PG}(4,2)$*, and let its Grassmann image be the 8-set* $\mathcal{K}_8 = \{p_1, \dots, p_8\}$ *of points on the Grassmannian* $G_{2,4,2} \subset PG(9,2)^*$*. Then the annihilator* $P := (\mathcal{K}_8)^{\mathrm{O}}$ *is an external plane in* $\mathrm{PG}(9,2)$ *which belongs to* $\mathrm{orb}(2\gamma)$*.*

Consult [18] for more details of the bijective correspondences $\mathcal{S}^*_8 \leftrightarrow \mathcal{P}_8$, and $\mathcal{P}_8 \leftrightarrow P$ between the three kinds of geometric objects just considered, and of their shared symmetry group $\mathcal{G}_{\mathcal{S}^*_8} = \mathcal{G}_{\mathcal{P}_8} = \mathcal{G}_P$.

Remark 2.5 *In the terminology of [5] a regulus-free partial spread* $\mathcal{S}_r$ *in* $\mathrm{PG}(4,2)$ *is a* $(n,q) = (4,2)$ *instance of a* generalized r-arc. *In the terminology of [12] a conclave of planes* $\mathcal{P}_8$ *in* $\mathrm{PG}(4,2)$ *is a* $(n,q) = (4,2)$ *instance of a* 2-dimensional dual hyperoval. *Concerning this last, we prefer to keep to the term* conclave of planes*; at least it met with the approval of J.H. Conway during the talk of one of us (R.S., see [22]) at the First Pythagorean Conference, Spetses, Greece 1996.*

2.3.3 An invariant for a spread of lines in $\mathrm{PG}(5,2)$

The problem of finding all spreads of lines in $\mathrm{PG}(5,2)$ will probably only ever be solved with the aid of computers. In particular, given two spreads $\mathcal{S}$, $\mathcal{S}'$ in $\mathrm{PG}(5,2)$, it is usually a highly nontrivial task to decide whether or not they are equivalent (belonging to the same $\mathrm{GL}(6,2)$-orbit). It is therefore of some interest that the present classification of partial spreads in $\mathrm{PG}(4,2)$ provides us with an invariant for spreads of lines in $\mathrm{PG}(5,2)$, as we now describe.

Lemma 2.6 *Let* $\mathcal{S}$ *be a spread of lines in* $\mathrm{PG}(5,2)$*, and let* π *be any hyperplane of* $\mathrm{PG}(5,2)$*. Then precisely 5 of the 21 lines of* $\mathcal{S}$ *lie inside* π*.*

Proof. The 32 points of π^c account for 16 lines of $\mathcal{S}$ which meet π in a point. The remaining $31 - 16 = 15$ points of π must therefore support the remaining $21 - 16 = 5$ lines of $\mathcal{S}$. ■

Each line-spread $\mathcal{S}$ in $\mathrm{PG}(5,2)$ thus determines a partial spread $\mathcal{S}_5$ in each $\mathrm{PG}(4,2)$ of $\mathrm{PG}(5,2)$. Suppose that of these 63 partial spreads $\mathcal{S}_5$, one for each hyperplane of $\mathrm{PG}(5,2)$, precisely N_x belong to class Vx.1, x = a, b, ... , j, in table B.1.

Definition 2.7 *The sequence* $(N_a, N_b, \dots, N_j)$ *is the* invariant sequence *of the spread* $\mathcal{S}$*.*

Clearly spreads $\mathcal{S}$, $\mathcal{S}'$ in $\mathrm{PG}(5,2)$ which have different invariant sequences will be inequivalent.

As an example of an invariant sequence, consider the spread $\mathcal{S} = \mathcal{S}_7 \cup \mathcal{S}'_7 \cup \mathcal{S}''_7$ in $\mathrm{PG}(5,2)$ defined as follows. *In the shorthand notation described at the start of the next section,* let $\{1,2,3,4,5,6,7\}$ be a *hyperbasis* (= basis + unit point) for $V(6,2)$. So $123456 = 7$, etc. The 63 points of $\mathrm{PG}(5,2)$ are then the 7 points i, the 21 points ij, and the 35 points ijk, where $i < j < k$. Let A in $\mathrm{GL}(6,2)$ be the element of order 7 which effects the cyclic permutation (1234567), and let $\mathcal{S}_7, \mathcal{S}'_7, \mathcal{S}''_7$ be the A-cyclic partial spreads which contain the respective lines $\{12, 14, 24\}$, $\{1, 235, 467\}$, $\{127, 136, 145\}$. Then $\mathcal{S}_7$ accounts for the 21 points ij, $\mathcal{S}'_7$ accounts for the 7 points i and for 14 of the points ijk, and $\mathcal{S}''_7$ accounts for the remaining 21 points ijk.

Claim 2.8 *The spread* $\mathcal{S} = \mathcal{S}_7 \cup \mathcal{S}'_7 \cup \mathcal{S}''_7$ *in* $\mathrm{PG}(5,2)$ *has* $(7,0,0,28,28,0,0,0,0,0)$ *for its invariant sequence.*

The computation of the invariant sequence was fairly painless, but only because one could take advantage of the Z_7 (in fact $Z_7 \rtimes Z_3$) symmetry.

3 Small partial spreads: $r \leq 5$

Shorthand notation. Having made a choice $\mathcal{B} = \{e_1, \ldots, e_5\}$ of basis for V_5 it will often prove convenient to use $i,\ ij,\ \ldots$ as shorthand for $e_i,\ e_i + e_j,\ \ldots$. We also set $u = e_1 + \ldots + e_5 = 12345$, and use $4u$ as shorthand for $e_4 + u = 1235$, etc. From time to time we make use of the *hyperbasis* $\mathcal{B}^* = \mathcal{B} \cup \{u\}$, see section A.2.1 of the Appendix. Concerning generators for the stabilizer $\mathcal{G}(\mathcal{S}_r)$ of a partial spread $\mathcal{S}_r$, that element $A \in \mathrm{GL}(5,2)$ whose action on the basis $\mathcal{B}$ is, for example $A : 1 \mapsto 134,\ 2 \mapsto 23,\ 3 \mapsto 3,\ 4 \mapsto 34,\ 5 \mapsto 125$, is written $A : (1,2,3,4,5) \mapsto (134, 23, 3, 34, 125)$, or, more often, simply $A = (134, 23, 3, 34, 125)$.

3.1 Introduction

We have considered earlier those (projectively unique) partial spreads which lie in some hyperplane $\sigma \subset \mathrm{PG}(4,2)$, namely partial spreads $\mathcal{S}_3$ of type I (reguli), $\mathcal{S}_4$ of type $\binom{4}{3}$ and $\mathcal{S}_5$ of type $\binom{5}{3}$. These three orbits are listed as IIIb.1, IVd.1 and Vj.1 in table B.2a. Now associated with the hyperplane σ is a group $\mathcal{N} \cong (Z_2)^4$ of involutions, where $\mathcal{N} \setminus \{I\}$ consists of those 15 transvections (projectively, elations) $\{J(a) : a \in \sigma\}$ which fix σ pointwise. Here $J(a)x = x$ if $x \in \sigma$, and $J(a)x = x + a$ if $x \in \sigma^c$. In each of the three cases, $\mathcal{S} = \mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_5$, under consideration clearly the stabilizer groups $\mathcal{G}(\psi)$ and $\mathcal{G}(\mathcal{S})$ will contain $\mathcal{N}$ as a normal subgroup.

In the case of $\psi_5 = \psi(\mathcal{S}_5) = \sigma$, we have $\mathcal{G}(\psi_5) = \mathcal{N} \rtimes \mathrm{GL}(4,2)$ and $\mathcal{G}(\mathcal{S}_5) = \mathcal{N} \rtimes \mathcal{K}$, where $\mathcal{K} \cong -\mathrm{L}(2,4)$, of order $|\,\mathrm{GL}(4,2)| \div 56 = 360$, is the stabilizer within $\mathrm{GL}(4,2)$ of a spread in $\mathrm{PG}(3,2)$. So $|\mathcal{G}(\mathcal{S}_5)| = 16 \times 360 = 5760$. In the case of $\psi_4 = \psi(\mathcal{S}_4) = \sigma \setminus \lambda$ the stabilizer within $\mathrm{GL}(4,2)$ of the line $\lambda \subset \mathrm{PG}(3,2)$ is $\cong (Z_2)^4 \rtimes (\mathrm{GL}(2,2) \times \mathrm{GL}(2,2))$, of order $16 \times 6 \times 6 = 576$. Since ψ_4 contains eight $\mathcal{S}_4$s the stabilizer within $\mathrm{GL}(4,2)$ of a particular $\mathcal{S}_4$ has order $576 \div 8 = 72$. Hence $|\mathcal{G}(\psi_4)| = |\mathcal{N}| \times 576 = 9216$ and $|\mathcal{G}(\mathcal{S}_4)| = |\mathcal{N}| \times 72 = 1152$, as recorded in tables B.1 and B.2a. In the case of $\psi_3 = \psi(\mathcal{S}_3) = \mathcal{H}$, the stabilizer within $\mathrm{GL}(4,2)$ of the quadric $\mathcal{H} \subset \mathrm{PG}(3,2)$ is $\mathrm{O}^+(4,2) \cong$

$(\mathrm{GL}(2,2) \times \mathrm{GL}(2,2)) \rtimes Z_2$, of order 72. Hence $|\mathcal{G}(\psi_3)| = |\mathcal{N}| \times 72 = 1152$ and, since $\mathcal{H}$ contains two $\mathcal{S}_3$ s, $|\mathcal{G}(\mathcal{S}_3)| = 1152 \div 2 = 576$, again as recorded in tables B.1 and B.2a.

At this point we may as well consider a partial spread $\mathcal{S}_5$ of the form $\mathcal{S}_4 \cup \{\mu\}$ where $\mathcal{S}_4$ is of type $\binom{4}{3}$, with $\psi_4 = \psi(\mathcal{S}_4) = \sigma \diagdown \lambda$, and where the line μ meets λ in a point p. So, see lemma 1.5, $\mathcal{S}_5$ is necessarily of type $\binom{4}{3}$. *It is projectively unique.* For if $\mathcal{S}'_5 = \mathcal{S}_4 \cup \{\mu'\}$ then there exists $T \in \mathcal{G}(\mathcal{S}_4)$ sending $p' = \lambda \cap \mu'$ to p, and there exists $J(a) \in \mathcal{N}$ sending $T\mu'$ to μ, whence $J(a)T$ sends $\mathcal{S}'_5$ to $\mathcal{S}_5$. Moreover, for a given $\mathcal{S}_4$ of type $\binom{4}{3}$ there are $3 \times 8 = 24$ choices for μ. So the orders of $\mathcal{G}(\psi_5)$ and $\mathcal{G}(\mathcal{S}_5)$ are $|\mathcal{G}(\psi_4)| \div 24 = 384$ and $|\mathcal{G}(\mathcal{S}_4)| \div 24 = 48$, as recorded in the entries Vi in tables B.1 and B.2a. The structure of $\mathcal{G}(\mathcal{S}_5)$ is seen to be

$$\mathcal{G}(\mathcal{S}_5) = \mathcal{G}_0 \times \langle J(p) \rangle \cong \mathrm{Sym}(4) \times Z_2, \tag{3.1}$$

with $\mathcal{G}_0$ effecting all 4! permutations of the lines of $\mathcal{S}_4$.

Consider also a partial spread $\mathcal{S}_4$ of type I, and so of the form $\mathcal{S}_3 \cup \{\mu\}$ where $\mathcal{S}_3$ is of type I, with $\psi(\mathcal{S}_3) = \mathcal{H} \subset \sigma$, and where the line μ meets σ in a point p. Since $\mathcal{G}(\mathcal{S}_3)$ is transitive on the points $p \in \sigma \diagdown \mathcal{H}$, a corresponding argument to the previous one shows that *a partial spread S_4 of type I is projectively unique.* Moreover, for a given regulus $\mathcal{S}_3$ there are $6 \times 8 = 48$ choices for μ. So the orders of $\mathcal{G}(\psi_4)$ and $\mathcal{G}(\mathcal{S}_4)$ are $|\mathcal{G}(\psi_3)| \div 48 = 24$ and $|\mathcal{G}(\mathcal{S}_3)| \div 48 = 12$, as recorded in the entries IVc in tables B.1 and B.2a. See eq. (3.28) for the structure $\mathrm{Sym}(3) \times Z_2$ of $\mathcal{G}(\mathcal{S}_4)$.

We now proceed to consider all the remaining partial spreads of sizes $3, 4$ or 5. One of these is a projectively unique partial spread $\mathcal{S}_5$ of type L, and because it turns out to play a particularly important role in our determination of all partial spreads, we will also consider it later, in greater detail, in section 4.1.

3.2 The projectively unique $\mathcal{S}_3(\mathsf{O})$ and its extensions to an $\mathcal{S}_4$

Without loss of generality any partial spread $\mathcal{S}_3$ which is not a regulus can be taken to be $\mathcal{S}_3 = \mathcal{S}_3(\mathsf{O}) = \{\lambda_1, \lambda_2, \lambda_3\}$ where, in shorthand notation with respect to the basis $\mathcal{B} = \{e_1, \dots, e_5\}$,

$$\lambda_1 = \{1, 2, 12\}, \ \lambda_2 = \{3, 4, 34\}, \ \lambda_3 = \{5, u, 5u\}. \tag{3.2}$$

Selecting a particular choice of $\mathcal{S}_3 = \mathcal{S}_3(\mathsf{O})$ equips $\mathrm{PG}(4,2)$ with a surprising amount of structure. First of all there is the unique transversal $\tau = \{12, 34, 5u\}$ of the lines $\lambda_1, \lambda_2, \lambda_3$. Secondly $\mathcal{S}_3$ determines the hyperbasis $\mathcal{B}^* = \psi(\mathcal{S}_3) \diagdown \tau = \mathcal{B} \cup \{u\}$ for V_5, and so, see section A.2.1, it follows that $\mathcal{S}_3$ also determines a privileged hyperplane, the *even hyperplane* σ of $\mathcal{B}^*$, which comes along with a distinguished null polarity. The $10 + 5 = 15$ points of σ are those, $\{ij\}$ and $\{iu\}$, which are of even weight in a(ny) basis $\mathcal{B} = \mathcal{B}^* \diagdown \{p\}, p \in \mathcal{B}^*$, and the scalar product $x.y$ is such that $ij.ik = ij.iu = iu.ju = 1$, and $ij.kl = ij.ku = 0$, whenever i, j, k, l are distinct, see (A.7). Observe that $\mathcal{S}_3$ also selects one line, namely $\tau = \lambda(12\,34\,5u)$ $(= \lambda_{s_{23}})$, see after eq. (A.8)), out of the fifteen self-polar lines (A.8i) of σ.

Next $\mathcal{S}_3$ determines that involution $J = J(\mathcal{S}_3) \in \mathrm{GL}(5,2)$ which, for $i = 1, 2, 3$, interchanges the two points of λ_i not on τ. Thus, for $\mathcal{S}_3$ as in (3.2), we have, in shorthand notation, $J = (12)(34)(5u)$ — meaning that J is that element of $\mathrm{GL}(5,2)$ with effect $J: e_1 \rightleftarrows e_2,\ e_3 \rightleftarrows e_4,\ e_5 \rightleftarrows u$. Consequently $\mathcal{S}_3$ also determines that plane α which is the fixed-point-set $\mathrm{fix}(J)$ of J, and which is seen to be

$$\mathrm{fix}(J) = \alpha = \tau \cup \xi, \qquad \text{where} \quad \xi = \{135, 145, 235, 245\}. \tag{3.3}$$

So $\mathcal{S}_3$ also determines the 4-set $\xi = \mathrm{fix}(J) \setminus \tau$.

(We might also mention the three solids (hyperplanes) $\sigma_{12}, \sigma_{13}, \sigma_{23}$ which are determined by $\mathcal{S}_3$, where $\sigma_{ij} = \langle \lambda_i, \lambda_j \rangle$, and the associated three 3-dimensional hyperbolic quadrics $\mathcal{H}_{12}, \mathcal{H}_{13}, \mathcal{H}_{23}$, where $\mathcal{H}_{ij} = \sigma_{ij} \setminus (\lambda_i \cup \lambda_j)$, and note the alternative definition $\xi = (\sigma_{12} \cup \sigma_{13} \cup \sigma_{23})^c$ of the 4-set ξ. Moreover if σ_{ij} has equation $f_{ij} = 0$, where $f_{ij} \in V_5^*$ (= the dual of V_5), then we find that σ is that solid whose equation is $f = 0$, where $f = f_{12} + f_{13} + f_{23}$.)

In lemma 3.1 below we describe the stabilizer group $\mathcal{G}(\mathcal{S}_3)$ of $\mathcal{S}_3 = \mathcal{S}_3(\mathsf{O})$. Since any $A \in \mathcal{G}(\mathcal{S}_3)$ must preserve the various structures $\tau, \mathcal{B}^*, \sigma, \xi, \ldots$ described above, it is not difficult to arrive at these details of $\mathcal{G}(\mathcal{S}_3)$ by elementary means. Nevertheless it is of some interest to see how to obtain $\mathcal{G}(\mathcal{S}_3)$ from the material in section A.1.2 of the Appendix. For note that $A \in \mathcal{G}(\mathcal{S}_3)$ if and only if A preserves both $\mathcal{B}^*$ and the particular self-polar line $\tau = \lambda(s_{23}) \subset \sigma$ which is labelled by the syntheme $s_{23} = \{12, 34, 5u\}$ in the array (A.2). Let us identify $\mathrm{Sym}(6)$ with $\mathrm{Sym}(\mathcal{B}^*) = \mathrm{Sym}(\{1,2,3,4,5,u\})$. Using lemma A.2(ii) it follows that

$$\begin{aligned} \mathcal{G}(\mathcal{S}_3) = \mathrm{stab}(s_{23}) = \theta(\mathrm{stab}\{2,3\}), \qquad \text{where} \\ \mathrm{stab}\{2,3\} = \mathrm{Sym}(\{1,4,5,u\}) \times \langle (23) \rangle \cong \mathrm{Sym}(4) \times Z_2, \end{aligned} \tag{3.4}$$

and where $\theta = \theta^{-1}$ is that outer involutory automorphism of $\mathrm{Sym}(6)$ which is described in section A.1.2. The elements of $\mathcal{G}(\mathcal{S}_3)$ can now be obtained from elements of $\mathrm{Sym}(\{1,4,5,u\}) \times \langle (23) \rangle$ upon using the property (A.6) of θ: if $s_{ab} = \{ij,\ kl,\ mn\}$, then $\theta(ab) = (ij)(kl)(mn)$. Thus, for example, the central involution $(23) \in \mathrm{stab}\{2,3\}$ yields the central involution $\theta(23) = (12)(34)(5u) = J \in \mathcal{G}(\mathcal{S}_3)$. As another example, since $s_{14} = \{13, 24, 5u\}$ and $s_{15} = \{12, 45, 3u\}$, see (A.2), we have

$$\theta(154) = \theta(14)\theta(15) = (13)(24)(5u)\,(12)(45)(3u) = (14u)(235),$$

and so arrive at the element $A_3 \in \mathcal{G}(\mathcal{S}_3)$ in the lemma.

Lemma 3.1 *If $\mathcal{S}_3$ is of type O, with involution $J = J(\mathcal{S}_3)$, then*

$$\mathcal{G}(\mathcal{S}_3) = \mathcal{G}_0(\mathcal{S}_3) \times \langle J \rangle, \quad \textit{where } \mathcal{G}_0(\mathcal{S}_3) \cong \mathrm{Sym}(4), \tag{3.5}$$

with $\mathcal{G}_0$ effecting all $4!$ permutations of the four points of ξ. Explicitly, for $\mathcal{S}_3$ as in (3.2), the subgroup $\mathcal{G}_{00} \lhd \mathcal{G}_0$, isomorphic to $\mathrm{Alt}(4)$, contains the four Z_3 subgroups $\langle A_i \rangle$, $i = 1,2,3,4$, where, in shorthand notation, the A_i effect the following permutations of the six elements $1 = e_1, \ldots, 5 = e_5, u$:

$$A_1 : (135)(24u),\ A_2 : (145)(23u),\ A_3 : (235)(14u),\ A_4 : (245)(13u). \tag{3.6}$$

Also the four-group inside $\mathcal{G}_{00}$ *is* $\{I, (12)(34), (12)(5u), (34)(5u)\}$. *Each* A_i *lies in class 3B of* $\mathrm{GL}(5,2)$ *and is of cycle type* $3^{10}1^1$ *in its action upon* $\mathrm{PG}(4,2)$, *(see [4, p.70] and [9, Table 4]), and the unique fixed points of the four* A_i *are precisely the four points* $135(=24u)$, $145(=23u)$, $235(=14u)$ *and* $245(=13u)$ *of the set* ξ. *The six involutions* $\in \mathcal{G}_0 \diagdown \mathcal{G}_{00}$ *may be represented by* $K = (13)(24)$, *and the six elements of order 4 by* (1324). ■

Remark 3.2 *The stabilizer group* $\mathcal{G}(\mathcal{S}_3)$ *contains* two *subgroups isomorphic to* $\mathrm{Sym}(4)$, *namely* $\mathcal{G}_0(\mathcal{S}_3)$ *and* $\mathcal{G}_0(\mathcal{S}_3)'$, *where* $\mathcal{G}_0(\mathcal{S}_3) = \mathcal{G}_{00} \cup K\mathcal{G}_{00}$ *is as in the theorem and* $\mathcal{G}_0(\mathcal{S}_3)' = \mathcal{G}_{00} \cup JK\mathcal{G}_{00}$. *The subgroup* $\mathcal{G}_0(\mathcal{S}_3)'$ *will occur later, see the lead-in to theorem 7.1.*

Consider now an extension $\mathcal{S}_4 = \mathcal{S}_3(\mathsf{O}) \cup \{\lambda_4\}$, and note from table 2 that there are $L' = 52$ choices for λ_4. Of these 52 lines, 16 lie in the even hyperplane σ and 36 intersect σ in a point, each of the 12 points of $\sigma \diagdown \tau$ contributing 3 lines of the 36.

Under the action of $\mathcal{G}(\mathcal{S}_3)$ the 16 lines of σ which are skew to τ form two orbits Ω and Ω', each of length 8, with representatives

$$\lambda_4 = \{13, 25, 4u\}, \quad \lambda_4' = \{13, 15, 35\}, \tag{3.7}$$

the lines of Ω being self-polar and those of Ω' being nonpolar, see eq. (A.8). Each of the partial spreads $\mathcal{S}_4 = \mathcal{S}_3 \cup \{\lambda_4\}$ and $\mathcal{S}_4' = \mathcal{S}_3 \cup \{\lambda_4'\}$ is of type O. However they are inequivalent because they have distinct *profiles* (see section 1.1):

$$\text{profile}(\mathcal{S}_4) = (2,2,2)^4, \qquad \text{profile}(\mathcal{S}_4') = (1,2,3)^3(2,2,2), \tag{3.8}$$

each line of $\mathcal{S}_4$ being balanced, while in the case of $\mathcal{S}_4'$ only λ_4' is balanced. The resulting two classes of partial spreads $\mathcal{S}_4$ of type O are listed, respectively, as IVa.1 and IVb.1 in table B.2a. They will be considered further below in section 3.3.

Of the remaining 36 lines the three which meet σ in the point 13 are:

$$\lambda_4'' = \{13, 124, 234\}, \quad \lambda_4''' = \{13, 125, 235\}, \quad \lambda_4^* = \{13, 145, 345\}. \tag{3.9}$$

Now $\mathcal{S}_4'' = \mathcal{S}_3 \cup \{\lambda_4''\}$ is of type I, since $\{\lambda_1, \lambda_2, \lambda_4''\}$ is a regulus, while $\mathcal{S}_4''' = \mathcal{S}_3 \cup \{\lambda_4'''\}$ is seen to be of type O. There is no need to consider $\mathcal{S}_3 \cup \{\lambda_4^*\}$ because λ_4^* lies on the same $\mathcal{G}(\mathcal{S}_3)$-orbit Ω''' as λ_4''', the element $K = (5u)$ of $\mathcal{G}(\mathcal{S}_3)$ effecting the interchange $\lambda_4^* \rightleftarrows \lambda_4'''$. Since $\mathcal{G}(\mathcal{S}_3)$ is transitive on the 12-set $\sigma \diagdown \tau$ it follows that the 36 choices for λ_4 which meet σ in a point form two orbits Ω'' and Ω''', of lengths 12 and 24, with representatives λ_4'' and λ_4'''.

Lemma 3.3 *Up to equivalence, there exist precisely four partial spreads* $\mathcal{S}_4$ *of size 4. Three of these arise as extensions of an* $\mathcal{S}_3(\mathsf{O})$, *represented by the partial spreads* $\mathcal{S}_3 \cup \{\lambda_4\}$, $\mathcal{S}_3 \cup \{\lambda_4'\}$ *and* $\mathcal{S}_3 \cup \{\lambda_4''\}$ *just considered; they are of the respective types* O, O *and* I, *listed as IVa.1, b.1 and c.1 in table B.2a. (For more details concerning classes IVa.1 and IVb.1, see section 3.3; for class IVc.1, see end of section 3.1 and the beginning of section 3.5.) The fourth is of type* $\binom{4}{3}$, *as considered in section 1.3.1, and listed as IVd.1.*

Proof. By our lead-in to the lemma the only other possibility for an extension of an $\mathcal{S}_3(\mathsf{O})$ would have $\mathcal{S}_4''' = \mathcal{S}_3 \cup \{\lambda_4'''\}$ as representative. But in fact $\mathcal{S}_4'''$ is equivalent to $\mathcal{S}_4'$, with λ_3 being the sole balanced line of $\mathcal{S}_4'''$. To see this, start out from $\mathcal{S}_3^\dagger = \{\lambda_1, \lambda_2, \lambda_4'''\}$ and note that λ_3 is a nonpolar line in the even hyperplane $\sigma^\dagger$of $\mathcal{S}_3^\dagger$.

Any other kind of $\mathcal{S}_4$ must be an extension of a regulus, since the latter, of type I, is the only class of $\mathcal{S}_3$ other than one of type O already considered. Recalling from theorem 1.6(a) that $N_4 \notin \{2, 3\}$, an extension $\mathcal{S}_4$ of a regulus is either a partial spread in a hyperplane, of type $\binom{4}{3}$, or else it is of type I. But, see end of section 3.1, there is only the one class, IVc.1, of $\mathcal{S}_4(\mathsf{I})$ s. ■

3.3 The two classes of partial spread $\mathcal{S}_4(\mathbf{O})$

3.3.1 The cyclic partial spread $\mathcal{S}_4(\mathbf{O})$

In section 3.2 we proved that there exists a unique class IVa.1 of partial spreads $\mathcal{S}_4$ of type O and profile $(2, 2, 2)^4$, represented by $\mathcal{S}_4 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$:

$$\lambda_1 = \{1, 2, 12\},\ \lambda_2 = \{3, 4, 34\},\ \lambda_3 = \{5, u, 5u\},\ \lambda_4 = \{13, 25, 4u\}. \tag{3.10}$$

That each line of $\mathcal{S}_4$ is balanced, contributing $(2, 2, 2)$ to profile($\mathcal{S}_4$), comes about because each of the twelve points of $\psi(\mathcal{S}_4)$ lies on just one of the four transversals of $\mathcal{S}_4$. *It follows that these four transversals themselves constitute a* S_4 *of type* O *and profile* $(2, 2, 2)^4$. The 12-set ψ thus has the structure of a *double-four* of lines (the terminology here being that used in [23]), and may conveniently be represented in a 4×4 array

$$\psi = \begin{pmatrix} - & 2 & 1 & 12 \\ 4 & - & 3 & 34 \\ u & 5 & - & 5u \\ 4u & 25 & 13 & - \end{pmatrix}. \tag{3.11}$$

Here the lines of the original partial spread $\mathcal{S}_4$ are given by the four rows, and the transversals $\tau_{234}, \tau_{134}, \tau_{124}, \tau_{123}$ of $\mathcal{S}_4$, forming, let us say, the *opposite* partial spread $\mathcal{S}_4^{\text{opp}}$, are given by the four columns.

Given *any* double-four $\psi \subset \mathrm{PG}(4, 2)$,

$$\psi = \begin{pmatrix} - & x_{12} & x_{13} & x_{14} \\ x_{21} & - & x_{23} & x_{24} \\ x_{31} & x_{32} & - & x_{34} \\ x_{41} & x_{42} & x_{43} & - \end{pmatrix}, \tag{3.12}$$

then its twelve points $x_{ij}, i \neq j$, satisfy various relations. First of all we have the defining properties $\Sigma_{j \neq i} x_{ij} = 0$, and $\Sigma_{i \neq j} x_{ij} = 0$. From these it quickly follows that $x_{ij} + x_{ji} = x_{kl} + x_{lk}$ holds for any permutation $ijkl$ of 1234, and also that $x_{ij} + x_{jk} + x_{ki}$ does not depend on the choice of 3-set $\{i, j, k\} \subset \{1, 2, 3, 4\}$. If $x_{ij} + x_{jk} + x_{ki} = 0$ then ψ lies in a hyperplane and the rows, also columns, of ψ form an $\mathcal{S}_4$ of type $\binom{4}{3}$. (In this case of a *degenerate* double-four, ψ can be expressed as a double-four in more than one way.) However we are interested in a *non-degenerate* double-four, such as (3.11), where the x_{ij} generate PG$(4,2)$; in which case the rows of ψ are an $\mathcal{S}_4(\mathsf{O})$, as are the columns. So a non-degenerate double-four ψ comes along with a privileged point given by $n = x_{ij} + x_{jk} + x_{ki}$ for any 3-set $\{i,j,k\} \subset \{1,2,3,4\}$. The significance of the point n is explained in the next lemma.

Lemma 3.4 *If ψ is a non-degenerate double-four in* PG$(4,2)$ *then it extends to a unique parabolic quadric $\mathcal{P}_4$. In detail, if ψ is as in (3.12), and if the points y_1, y_2, y_3 are defined by $y_i = x_{i4} + x_{4i} = x_{jk} + x_{kj}$, for any permutation ijk of* 123, *then $\mathcal{P}_4 = \psi \cup \lambda$, where $\lambda = \{y_1, y_2, y_3\}$. Moreover, for any 3-set $\{i,j,k\} \subset \{1,2,3,4\}$, the nucleus n of $\mathcal{P}_4$ is given by $n = x_{ij} + x_{jk} + x_{ki}$.*

Proof. Recall from section A.3.2 that if $\mathcal{S}_5$ is a spread on a parabolic quadric $\mathcal{P}_4$ then $\mathcal{S}_4 = \mathcal{S}_5 \diagdown \{\lambda\}$ has profile $(2,2,2)^4$ and, by lemma A.9, type O, and so is of class IVa.1. So, by projective uniqueness, any $\mathcal{S}_4 \in$ class IVa.1 can be viewed in this way, as an $\mathcal{S}_4$ on a $\mathcal{P}_4$, with double-four $\psi_4 = \psi(\mathcal{S}_4)$ of the form $\mathcal{P}_4 \diagdown \lambda$. It is easy to see that distinct parabolic quadrics can not share the same 12-set ψ_4, that is the extension $\mathcal{P}_4$ of ψ_4 is unique. Taking ψ_4 to be as in (A.13), a straightforward check confirms the validity of the recipe in the lemma for finding the nucleus n and the missing line λ. ■

Example 3.5 *If ψ is the non-degenerate double-four given in (3.14) below, then $\mathcal{P}_4 = \psi \cup \lambda$, where $\lambda = \{245, 5u, 135\}$ and $n = u$.*

If we view $\mathcal{S}_4 \in$ class IVa.1 as an $\mathcal{S}_4$ on a $\mathcal{P}_4$, with double-four $\psi_4 = \psi(\mathcal{S}_4)$ as in (A.13), then from lemma A.8 we deduce that

$$\mathcal{G}(\psi_4) \cong \mathrm{Sym}(4) \times Z_2, \quad \mathcal{G}(\mathcal{S}_4) \cong \mathrm{Sym}(4), \tag{3.13}$$

and that $\mathcal{S}_4$ is cyclic. (Here $Z_2 = \langle J \rangle$ where in terms of (3.12) J is $x_{ij} \rightleftarrows x_{ji}$.)

The cyclic symmetry of a partial spread $\mathcal{S}_4$ belonging to the class IVa.1 is made manifest if we employ the canonical form

$$\psi = \begin{pmatrix} - & 1 & 125 & 25 \\ 35 & - & 2 & 235 \\ 345 & 45 & - & 3 \\ 4 & 415 & 15 & - \end{pmatrix} \tag{3.14}$$

for the double-four $\psi(\mathcal{S}_4)$ as an alternative to that in (3.11). Let $\mathcal{S}_4 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be given by the rows and $\mathcal{S}_4^{\mathrm{opp}} = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ by the columns of the array (3.14). Observe that $A = (2,3,4,1,5)$ achieves the 4-cycle $(\lambda_1\lambda_2\lambda_3\lambda_4)$ and so $A \in \mathcal{G}(\mathcal{S}_4)$. It is also easy to use (3.14) to fill in further details of the isomorphisms (3.13). For note that $B = (2, 345, 25, 145, 24)$ keeps λ_4 fixed and achieves the 3-cycle $(\lambda_1\lambda_2\lambda_3)$; so $B \in \mathcal{G}(\mathcal{S}_4)$. Hence $\mathcal{G}(\mathcal{S}_4)$ contains the subgroup $\langle A, B \rangle \cong \mathrm{Sym}(4)$ which effects all permutations of the four lines $\lambda_i \in \mathcal{S}_4$. (Similarly $\langle A, B \rangle \subseteq \mathcal{G}(\mathcal{S}_4^{\mathrm{opp}})$, with A, B effecting the permutations

$(\mu_1\mu_2\mu_3\mu_4)$, $(\mu_1\mu_2\mu_3)$.) But the subgroup $\mathcal{G}_0 < \mathcal{G}(\mathcal{S}_4)$ which stabilizes each $\lambda_i \in \mathcal{S}_4$ is trivial, since any $T \in \mathcal{G}_0$ will also fix each $\mu_j \in \mathcal{S}_4^{\text{opp}}$, hence fixes each point $p_{ij} = \lambda_i \cap \mu_j$, whence $T = I$. Hence $\mathcal{G}(\mathcal{S}_4) = \langle A, B\rangle \cong \text{Sym}(4)$. Since the class of partial spreads $\mathcal{S}_4$ of type O and profile $(2,2,2)^4$ is unique, it *must* be the case that $\mathcal{S}_4^{\text{opp}}$ is equivalent to $\mathcal{S}_4$. But the 12-set $\psi_4 = \psi(\mathcal{S}_4)$ supports just the two partial spreads $\mathcal{S}_4$ and $\mathcal{S}_4^{\text{opp}}$, and so $|\mathcal{G}(\psi_4)| = 2|\mathcal{G}(\mathcal{S}_4)|$. In fact the equivalence of $\mathcal{S}_4^{\text{opp}}$ with $\mathcal{S}_4$ may be realized by the involution $J = (35, 45, 15, 25, 5)$, which in terms of (3.12) is $x_{ij} \rightleftarrows x_{ji}$. Now J not only achieves the interchanges $\lambda_i \rightleftarrows \mu_i$ but also commutes with A and B. Hence $\mathcal{G}(\psi_4) = \mathcal{G}(\mathcal{S}_4) \times \langle J\rangle$. Summarizing:

$$\mathcal{G}(\psi_4) = \langle A, B, J\rangle \cong \text{Sym}(4) \times Z_2, \quad \mathcal{G}(\mathcal{S}_4) = \langle A, B\rangle \cong \text{Sym}(4). \tag{3.15}$$

3.3.2 The non-cyclic partial spread $\mathcal{S}_4(\mathsf{O})$

Recalling eqs. (3.7) and (3.8), the partial spread $\mathcal{S}_4 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$:

$$\lambda_1 = \{1, 2, 12\},\ \lambda_2 = \{3, 4, 34\},\ \lambda_3 = \{5, u, 5u\},\ \lambda_4 = \{13, 15, 35\}, \tag{3.16}$$

representing the class IVb.1, is of type O and profile $(1,2,3)^3(2,2,2)$. Any $T \in \mathcal{G}(\mathcal{S}_4)$ must permute amongst themselves the three unbalanced lines $\lambda_1, \lambda_2, \lambda_3$ and must keep fixed (set-wise) the sole balanced line λ_4. (So $\mathcal{S}_4$ is a *non-cyclic* partial spread.) More can be said: T must permute amongst themselves the three points $1, 3, 5$, since only these points lie on two transversals, and similarly for the three points $2, 4, u$ which lie on no transversals. It quickly follows that

$$\mathcal{G}(\mathcal{S}_4) = \mathcal{G}(\psi_4) = \langle A, B\rangle \cong \text{Sym}(3), \tag{3.17}$$

where A and B in $\text{GL}(5,2)$ effect the permutations $(135)(24u)$ and $(13)(24)(5)(u)$ of the hyperbasis $\mathcal{B}^*$.

3.4 The five classes of $\mathcal{S}_5$ of type O

Any regulus-free partial spread $\mathcal{S}_5$ must arise as an extension of a regulus-free $\mathcal{S}_4$. Recall that an $\mathcal{S}_4(\mathsf{O})$ is either cyclic, of class IVa.1, or non-cyclic, of class IVb.1. First of all let us consider a partial spread $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_5\}$ where $\mathcal{S}_4$ is of class IVa.1. Then $\mathcal{S}_5$ is regulus-free for eighteen choices of the line λ_5, and under the action of $\mathcal{G}(\mathcal{S}_4) \cong \text{Sym}(4)$ these 18 form four orbits $\Omega_a, \Omega_b, \Omega_c$ and Ω_d, of lengths $1, 3, 6$ and 8. For let $\mathcal{S}_4 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be as given by the rows of (3.14):

$$\lambda_1 = \{1, 125, 25\},\ \lambda_2 = \{2, 235, 35\},\ \lambda_3 = \{3, 345, 45\},\ \lambda_4 = \{4, 415, 15\}. \tag{3.18}$$

Then, see (3.15), $\mathcal{G}(\mathcal{S}_4) = \langle A, B\rangle \cong \text{Sym}(4)$, where $A = (2, 3, 4, 1, 5)$ and $B = (2,\ 345, 25, 145, 24)$, and representatives of the four orbits $\Omega_a, \Omega_b, \Omega_c, \Omega_d$ are seen to be

$$\lambda_a = \{5u, 245, 135\}, \quad \lambda_b = \{5u, 5, u\}, \quad \lambda_c = \{5u, 12, 34\}, \quad \lambda_d = \{12, 1u, 2u\}. \tag{3.19}$$

Below we describe some distinguishing features of the four $\mathcal{S}_5(\mathsf{O})$ which arise from the four choices (3.19), and in particular see that they are inequivalent. In table B.2a they are assigned classes Va.1, Vb.1, Vc.1 and Vd.1, since the four supporting point-sets $\psi_r = \psi(\mathcal{S}_r)$ belong to distinct orbits, listed Va, Vb, Vc and Vd in table B.1.

3.4.1 The parabolic $\mathcal{S}_5(\mathsf{O})$

The singleton orbit Ω_a is occupied by that unique line λ_a such that $\psi(\mathcal{S}_4) \cup \lambda_a$ is the unique parabolic quadric $\mathcal{P}_4$ which extends the double-four $\psi(\mathcal{S}_4)$, see lemma 3.4 and example 3.5. The partial spread $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_a\}$ is thus a spread $\mathcal{S}_5$ on the quadric $\mathcal{P}_4$, see appendix §A.3.2. It has profile $(3,3,3)^5$, see lemma A.7. So, in the present case of an $\mathcal{S}_5$ of class Va.1, we have

$$\text{classVa.1:} \quad \mathcal{G}(\psi_5) \cong \mathrm{O}(5,2) \cong \mathrm{Sym}(6), \quad \mathcal{G}(\mathcal{S}_5) \cong \mathrm{Sym}(5). \tag{3.20}$$

(See lemma A.8(i) for the isomorphism of $\mathcal{G}(\mathcal{S}_5)$ with Sym(5), and lemma A.9 for the fact that $\mathcal{S}_5$ is regulus-free.) Here $\mathcal{G}(\mathcal{S}_5)$ effects all 5! permutations of the five lines; in particular $\mathcal{S}_5$ is a cyclic partial spread.

It should be noted that if $\mathcal{S}_5$ is spread on a parabolic quadric $\mathcal{P}_4$ then $\mathcal{S}_5$ has the *maximality property of admitting no extension to a larger regulus-free partial spread in* PG(4, 2). Indeed, see lemma A.10(i), all 20 extensions $\mathcal{S}_5 \cup \{\lambda\}$ are of type I, and moreover are all of class VIc.2.

3.4.2 The classes Vb.1, Vc.1, Vd.1, Ve.1 of partial spreads $\mathcal{S}_5(\mathsf{O})$

If now $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_b\}$ then we see that $\mathcal{S}_5$ has profile $(3,3,3)^4(1,3,5)$, with $\lambda_5 = \lambda_b$ contributing the $(1,3,5)$. This distinguishing feature of λ_5 goes along with the fact that $\mathcal{S}_5 \setminus \{\lambda_i\}$ is non-cyclic, of class IVb.1, for $i = 1,2,3,4$ and is cyclic, of class IVa.1, only for $i = 5$. It follows that $\mathcal{G}(\mathcal{S}_5)$ is a subgroup of $\mathcal{G}(\mathcal{S}_4)$, and has order $|\mathcal{G}(\mathcal{S}_4)|/|\Omega_b| = 8$. But if $A = (2,3,4,1,5)$, see after (3.14), then A is not only in $\mathcal{G}(\mathcal{S}_4)$ but also fixes λ_b. Similarly $K = (25,15,45,35,5)$ is in $\mathcal{G}(\mathcal{S}_4)$, since it effects the permutation $(\lambda_1)(\lambda_3)(\lambda_2\lambda_4)$, and it also fixes λ_b. Hence

$$\text{class Vb.1:} \qquad \mathcal{G}(\mathcal{S}_5) = \langle A, K\rangle \cong D_8. \tag{3.21}$$

Now the 15-set $\psi_5 = \psi(\mathcal{S}_5)$ supports just two partial spreads $\mathcal{S}_5$, namely $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_b\}$ and $\mathcal{S}_5^* = \mathcal{S}_4^{\mathrm{opp}} \cup \{\lambda_b\}$. Moreover the involution $J = (35,45,15,25,5)$, see before (3.15), fixes λ_b and so effects the interchange $\mathcal{S}_5 \rightleftarrows \mathcal{S}_5^*$. Consequently $\mathcal{G}(\psi_5) = \mathcal{G}(\mathcal{S}_5) \times \langle J\rangle \cong D_8 \times Z_2$.

We next consider $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_5\}$ with $\lambda_5 = \lambda_c$, and we find that $\mathcal{S}_5$ has the same profile $(3,3,3)^5$ as an $\mathcal{S}_5$ of class Va.1. Nevertheless the classes Va.1 and Vc.1 are distinct, since $\mathcal{S}_5 \setminus \{\lambda_i\}$ is non-cyclic for $i \in \{1,3\}$ and cyclic for $i \in \{2,4,5\}$. Thus $\mathcal{G}(S_5)$ is *not* transitive on $\mathcal{S}_5$, since any $T \in \mathcal{G}(S_5)$ must necessarily respect the $3+2$ partition $\{\lambda_2,\lambda_4,\lambda_5\} \cup \{\lambda_1,\lambda_3\}$ of $\mathcal{S}_5$. By looking at $\mathcal{S}_5 \setminus \{\lambda_1\}$, which has λ_3 as sole balanced line, our attention is drawn to $C = (125,15,45,34,5)$, which effects the permutation $(\lambda_1)(\lambda_3)(\lambda_2\lambda_4\lambda_5)$, and to $K = (25,15,45,35,5)$, which effects the permutation $(\lambda_1)(\lambda_3)(\lambda_5)(\lambda_2\lambda_4)$. Thus $\mathcal{G}(S_5)$ contains a subgroup $\langle C, K\rangle \cong \mathrm{Sym}(3)$ which achieves all permutations of $\{\lambda_2,\lambda_4,\lambda_5\}$

and which separately stabilizes both λ_1 and λ_3. But $\mathcal{G}(S_5)$ also contains the involution $L = (45, 35, 25, 15, 5\}$ which effects the permutation $(\lambda_1\lambda_3)(\lambda_2)(\lambda_4)(\lambda_5)$, leading to:

$$\text{class Vc.1:}\quad \mathcal{G}(S_5) = \langle C, K\rangle \times \langle L\rangle \cong \mathrm{Sym}(3) \times Z_2. \tag{3.22}$$

In the present case the 15-set ψ_5 supports precisely four partial spreads $\mathcal{S}_5$, all equivalent, the further three arising by replacing $\mathcal{S}_5\diagdown\{\lambda_i\}$, $i \in \{2, 4, 5\}$, by $(\mathcal{S}_5\diagdown\{\lambda_i\})^{\mathrm{opp}}$. Consequently $|\mathcal{G}(\psi_5)| = 4 \times |\mathcal{G}(S_5)| = 48$.

For $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_5\}$ with $\lambda_5 = \lambda_d$ we find that $\mathcal{S}_5$ has profile $(2, 3, 4)^3(3, 3, 3)^2$, with λ_4 and λ_5 being the two balanced lines. Nevertheless λ_4 and λ_5 do not enter $\mathcal{S}_5$ on an equal footing, since $\mathcal{S}_5\diagdown\{\lambda_i\}$ is cyclic only for $i = 5$. It follows, cf. the discussion of class Vb.1, that $|\mathcal{G}(\mathcal{S}_5)| = |\mathcal{G}(\mathcal{S}_4)|/|\Omega_d| = 3$, whence $\mathcal{G}(\mathcal{S}_5) \cong Z_3$. But recall that $B = (2, 345, 25, 145, 24)$ effects the permutation $(\lambda_1\lambda_2\lambda_3)(\lambda_4)$, and note that B also stabilizes $\lambda_5 = \lambda_d$. Hence

$$\text{class Vd.1:}\quad \mathcal{G}(S_5) = \langle B\rangle \cong Z_3. \tag{3.23}$$

In the present case the 15-set ψ_5 supports just two partial spreads $\mathcal{S}_5$, namely $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_d\}$ and $\mathcal{S}_5^* = \mathcal{S}_4^{\mathrm{opp}} \cup \{\lambda_d\}$. Moreover the involution $K' = (345, 2, 145, 25, 24)$ not only fixes λ_d but also effects the interchanges $\lambda_1 \rightleftarrows \mu_1$, $\lambda_2 \rightleftarrows \mu_3$, $\lambda_3 \rightleftarrows \mu_2$ and $\lambda_4 \rightleftarrows \mu_4$, and hence $\mathcal{S}_5 \rightleftarrows \mathcal{S}_5^*$. Consequently $\mathcal{G}(\psi_5) = \langle B, K'\rangle \cong \mathrm{Sym}(3)$.

Remark 3.6 *From the spread $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_a\}$ on the parabolic quadric $\mathcal{P}_4$, partial spreads belonging to the four classes Va.1, Vb.1, Vc.1, and Vd.1 have arisen above as $(\mathcal{S}_5\diagdown\{\lambda_a\}) \cup \{\lambda\}$, with λ belonging respectively to the $\mathcal{G}(\mathcal{S}_4)$-orbits $\Omega_a, \Omega_b, \Omega_c$ and Ω_d. Observe that these last orbits are subsets of the* $\mathrm{O}(5, 2)$*-orbits $\Lambda_1, \Lambda_2, \Lambda_3$ and Λ_4, respectively, in lemma A.6. Of the 33 external lines λ of $\psi(\mathcal{S}_4)$ we have used the* $(1+3+6+8 =)18$ *choices which give rise to a regulus-free $\mathcal{S}_5$. The remaining* $33 - 18 = 15$ *choices comprise: 3 further lines of Λ_3 (non-nuclear tangents, one at each point of λ_a), which give rise to an $\mathcal{S}_5$ of type* L, *and 12 further lines of Λ_4 (the remaining lines external to $\mathcal{P}_4$), which give rise to an $\mathcal{S}_5$ of type* I, *of class Vf.1.*

Any regulus-free partial spread $\mathcal{S}_5 = \{\lambda_1, \ldots, \lambda_5\}$ which is not of the classes Va.1, Vb.1, Vc.1, Vd.1 so far considered must be such that $\mathcal{S}_5\diagdown\{\lambda_i\}$ is of class IVb.1 for each $\lambda_i \in \mathcal{S}_5$. At least one such class exists, for consider the partial spread $\mathcal{S}_5$ where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are

$$\{1u, 2, 345\},\ \{2u, 3, 451\},\ \{3u, 4, 512\},\ \{4u, 5, 123\},\ \{5u, 1, 234\}. \tag{3.24}$$

Then $\mathcal{S}_5$ is cyclic, since if $A \in \mathrm{GL}(5, 2)$ is defined by $A : 1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 5 \mapsto 1$, then A effects the permutation $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)$. Hence, see after (3.20), $\mathcal{S}_5$ is regulus-free. Under the action of $\langle A\rangle \cong Z_5$ the ten transversals τ_{ijk} fall into two orbits Ω and Ω', each of length 5, with Ω represented by $\tau_{123} = \{2, 451, 3u\}$ and Ω' represented by $\tau_{124} = \{1u, 451, 123\}$. Also the fifteen points of ψ_5 fall into three orbits, each of length 5, with representatives $1u$, 2 and 345. Since the points $1u$, 2, 345 lie on 2, 1, 3 transversals, respectively, it follows that $\mathcal{S}_5$ has profile $(3, 2, 4)^5$; moreover we see that ψ_5 has the distinctive signature $(15, 15, 0)(5^2 5^3 5^4)(15^0)$. *Hence S_5 belongs to a different class, Ve.1, from the four already discovered.*

Now any $T \in \mathcal{G}(\mathcal{S}_5)$ must permute amongst themselves those points which lie on only one transversal, namely the points $1, 2, 3, 4, 5$. It then quickly follows from (3.24) that $\mathcal{G}(\mathcal{S}_5)$ is precisely $\langle A \rangle$:

$$\text{classVe.1:} \quad \mathcal{G}(\mathcal{S}_5) = \langle A \rangle \cong Z_5. \tag{3.25}$$

Note that the 15-set ψ_5 supports just one other partial spread, namely the partial spread

$$\mathcal{S}'_5 = \{\tau_{345}, \tau_{451}, \tau_{512}, \tau_{123}, \tau_{234}\} = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\} \tag{3.26}$$

which consists of the five transversals belonging to the orbit Ω. Since the involution $K : 1 \mapsto 1, 2 \rightleftarrows 5, 3 \rightleftarrows 4$ achieves the interchanges $\lambda_1 \rightleftarrows \mu_2$, $\lambda_2 \rightleftarrows \mu_1$, $\lambda_3 \rightleftarrows \mu_5$, $\lambda_4 \rightleftarrows \mu_4$, $\lambda_5 \rightleftarrows \mu_3$ we see that $\mathcal{G}(\mathcal{S}'_5) = \mathcal{G}(\mathcal{S}_5)$ and $\mathcal{G}(\psi_5) \cong \langle A, K \rangle \cong D_{10}$.

Finally, to show that there are no further classes of $\mathcal{S}_5(\mathsf{O})$, consider a partial spread $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_5\}$ where $\mathcal{S}_4$ is of class IVb.1. Of the eighteen choices of the line λ_5 for which $\mathcal{S}_5$ is regulus-free, we find that twelve yield an $\mathcal{S}_5$ which contains a cyclic $\mathcal{S}_4(\mathsf{O})$, in which case $\mathcal{S}_5$ is of class Vb.1, Vc.1 or Vd.1. The remaining six choices are seen to form a *single* orbit under the action of $\mathcal{G}(\mathcal{S}_4) \cong \mathrm{Sym}(3)$. So there is just one class of $\mathcal{S}_5(\mathsf{O})$ with the property that $\mathcal{S}_5 \setminus \{\lambda_i\}$ is of class IVb.1 for each $\lambda_i \in \mathcal{S}_5$, namely the class Ve.1 described above.

Lemma 3.7 *There are precisely five equivalence classes of regulus-free partial spreads* $\mathcal{S}_5$, *namely Va.1, Vb.1, Vc.1, Vd.1 and Ve.1, with stabilizer groups isomorphic to* $\mathrm{Sym}(5)$, D_8, $\mathrm{Sym}(3) \times Z_2$, Z_3 *and* Z_5, *respectively.* ■

3.5 The remaining partial spreads of size 5

Any $\mathcal{S}_5$ which is not of type O will either be of type $\binom{5}{3}$ or will arise as an extension of an $\mathcal{S}_4(\mathsf{I})$. Recall from section 3.1 that there is just one equivalence class IVc.1 of partial spreads $\mathcal{S}_4$ of type I. Let us take as representative of this class the partial spread $\mathcal{S}_4 = \mathcal{S}_4(\mathsf{I}) = \mathcal{S}_3(\mathsf{I}) \cup \{\lambda_4\}$ given by

$$\lambda_1 = \{5, 4, 45\},\ \lambda_2 = \{3, 2, 23\},\ \lambda_3 = \{35, 24, 1u\},\ \lambda_4 = \{34, 1, 134\}. \tag{3.27}$$

For this choice, the ambient hyperplane of the regulus $\mathcal{S}_3(\mathsf{I})$ is $\sigma = \langle 2, 3, 4, 5 \rangle$ and λ_4 meets σ in the point $p = 34$. We now wish to show that the stabilizer of this partial spread of type I is

$$\mathcal{G}_4 = \mathcal{G}(\mathcal{S}_4) = \langle C, D \rangle \times \langle J \rangle \cong \mathrm{Sym}(3) \times Z_2, \tag{3.28}$$

with $J = (134, 2, 3, 4, 5)$, $C = (1, 1u, 24, 23, 2)$ and $D = (1, 5, 4, 3, 2)$,

To this end let $\beta_1 = \tau_{423}$, $\beta_2 = \tau_{413}$ and $\beta_3 = \tau_{412}$ denote the three transversals of $\mathcal{S}_4$ which meet λ_4. In terms of the hyperbolic quadric $\mathcal{H} = \psi(\mathcal{S}_3)$, $\beta_1, \beta_2, \beta_3$ are the three bisecants from p to $\mathcal{H}$. If for $j \neq i$ the bisecant β_j meets λ_i in the point p_{ij} then $\lambda_i = \{p_{i1}, p_{i2}, p_{i3}\}$ where $\langle p, p_{ii} \rangle$ is one of the three tangents from p to $\mathcal{H}$. (For $\mathcal{S}_4(\mathsf{I})$ as in (3.27) we have $\beta_1 = \{34, 23, 24\}$, $\beta_2 = \{34, 45, 35\}$ and $\beta_3 = \{34, 4, 3\}$, and the three tangent points are $p_{11} = 5$, $p_{22} = 2$ and $p_{33} = 1u$.) Now any $T \in \mathcal{G}(\mathcal{S}_4)$ must of course permute amongst themselves the three lines $\lambda_1, \lambda_2, \lambda_3$ of $\mathcal{S}_3(\mathsf{I})$, preserve the line λ_4 of valency 0 and fix the point $p = 34$ where λ_4 meets σ. But T must also permute amongst themselves the three bisecants β_1, β_2 and β_3, and also the three tangent points p_{11}, p_{22} and p_{33}. It follows that there exists a homomorphism $\pi : T \mapsto \pi(T)$ from $\mathcal{G}(\mathcal{S}_4)$ to Sym(3) such that $Tp_{ij} = p_{\pi(T)i\,\pi(T)j}$. Now $\langle C, D\rangle$ is a subgroup of $\mathcal{G}(\mathcal{S}_4)$ which is isomorphic to Sym(3), since C and D, of orders 3 and 2, fix λ_4 and effect the permutations $(\lambda_1\lambda_2\lambda_3)$ and $(\lambda_1\lambda_2)(\lambda_3)$ of the lines of $\mathcal{S}_3$. So π is surjective. If K is in the kernel $\ker\pi$ of this homomorphism then K fixes each point of $\mathcal{H}$ and hence of σ, whence $K \in \mathcal{N}$, where $\mathcal{N} = \{J(a) : a \in \sigma\} \cup \{I\}$ is as in section 3.1. But since K must also preserve λ_4, it follows that $\ker\pi = \langle J(p)\rangle \cong Z_2$. Since $J(p) = J(34) = (134, 2, 3, 4, 5)$ commutes with C and D, we have completed the proof of (3.28).

Incidentally we also see that $\psi_4 = \psi(\mathcal{S}_4)$ has stabilizer $\mathcal{G}(\psi_4) = \mathcal{G}_4 \times \langle L\rangle \cong \mathrm{Sym}(3) \times (Z_2)^2$, where $L = (1, 2, 4, 3, 5)$ fixes λ_4 and maps $\mathcal{S}_3(\mathsf{I})$ to $\mathcal{S}_3(\mathsf{I})^{\mathrm{opp}}$.

External to ψ_4 are 31 lines, which fall into five $\mathcal{G}_4$-orbits Ω, Ω', Ω'', Ω''' and Ω^{iv}, of lengths 12, 6, 6, 6 and 1, with respective representatives

$$\begin{aligned}&\lambda_5 = \{245, 12, 145\}, \quad \lambda_5' = \{25, 15, 12\}, \quad \lambda_5'' = \{25, 13, 4u\},\\ &\lambda_5''' = \{25, 124, 145\}, \quad \lambda_5^{\mathrm{iv}} = \{25, 234, 345\}.\end{aligned} \tag{3.29}$$

Note that λ_5^{iv} lies in the hyperplane σ_{123}, and so the extension $\mathcal{S}_4(\mathsf{I}) \cup \{\lambda_5^{\mathrm{iv}}\}$ is of type $\binom{4}{3}$. Next, the extension $\mathcal{S}_4(\mathsf{I}) \cup \{\lambda_5'''\}$ is seen to be of type L, with $\{\lambda_3, \lambda_4, \lambda_5'''\}$ as the other regulus. The remaining extensions

$$\mathcal{S}_5 = \mathcal{S}_4(\mathsf{I}) \cup \{\lambda_5\}, \qquad \mathcal{S}_5' = \mathcal{S}_4(\mathsf{I}) \cup \{\lambda_5'\}, \qquad \mathcal{S}_5'' = \mathcal{S}_4(\mathsf{I}) \cup \{\lambda_5''\} \tag{3.30}$$

are all seen to be of type I. Consequently, since $\mathcal{S}_4(\mathsf{I})$ is projectively unique, we deduce
(i) *there is a projectively unique* $S_5(\binom{4}{3})$;
(ii) *there is a projectively unique* $S_5(\mathsf{L})$.

Concerning (i), it is of the form $\mathcal{S}_5 = \mathcal{S}_4(\binom{4}{3}) \cup \{\lambda_4\}$ and was encountered earlier, in section 3.1; the structure $\mathcal{G}(\mathcal{S}_5) \cong \mathrm{Sym}(4) \times Z_2$ is easily obtained from our present concerns, with J in (3.28) being $J(p)$ in (3.1). Concerning (ii), $S_5(\mathsf{L})$, it is listed as Vh.1 in table B.2a. It will be considered in more detail in section 4.1, see lemma 4.1.

Now $\sigma_{123} = \mathcal{S}_3(\mathsf{I}) \cup \{\mu, \mu'\}$, where $\mu = \{25, 234, 345\} = \lambda_5^{\mathrm{iv}}$ and $\mu' = \{34, 245, 235\}$, and the points where $\lambda_4, \lambda_5, \lambda_5'$ and λ_5'' meet σ_{123} are $a_4 = 34, a_5 = 245, a_5' = 25$ and $a_5'' = 25$, respectively. Note that the two points a_4, a_5 both lie on μ' while for a_4, a_5' and for a_4, a_5'' one point a_4 lies on μ' and the other point $a_5' = a_5''$ lies on μ. Hence $\mathcal{S}_5$ is not equivalent to $\mathcal{S}_5'$ nor to $\mathcal{S}_5''$. On the other hand $\mathcal{S}_5'$ and $\mathcal{S}_5''$ *are* equivalent. For if $T \in \mathrm{GL}(5, 2)$, of order 4, is defined by $T : (1, 2, 3, 4, 5) \mapsto (13, 3, 2, 5, 4)$, then T fixes each of $\lambda_1, \lambda_2, \lambda_3$ and sends λ_4 to λ_5'' and λ_5' to λ_4.

That $\mathcal{S}_5$ and $\mathcal{S}_5'$ are inequivalent follows also from the fact that the profile $(2, 3, 4)^2(3, 3, 5)$ $(3, 4, 4)^2$ of $\mathcal{S}_5$ is distinct from that $(1, 3, 5)$ $(2, 2, 5)(3, 4, 4)^3$ of $\mathcal{S}_5'$. In fact even the signatures of $\psi(\mathcal{S}_5)$ and $\psi(\mathcal{S}_5')$ are distinct, and in table B.2a the partial spreads $\mathcal{S}_5$ and $\mathcal{S}_5'$ are listed under the entries Vf.1 and Vg.1, respectively, with the stabilizer group being of order 2 in each case. To see this last, note that the stabilizer group $\mathcal{G}(\mathcal{S}_5)$ must preserve the pair of lines $\{\lambda_4, \lambda_5\}$ of valency 0, and $\mathcal{G}(\mathcal{S}_5')$ must preserve the pair of lines

$\{\lambda_4, \lambda_5'\}$. Now λ_5 is on a $\mathcal{G}_4$-orbit of length $12 = |\mathcal{G}_4|$, and so the subgroup of $\mathcal{G}(\mathcal{S}_5)$ which preserves λ_5 is trivial. But $K : (1, 2, 3, 4, 5) \mapsto (145, 35, 24, 5, 4)$ effects the permutation $(\lambda_1)(\lambda_2\lambda_3)(\lambda_4\lambda_5)$, whence $\mathcal{G}(\mathcal{S}_5) = \langle K\rangle \cong Z_2$. On the other hand λ_5' is on a $\mathcal{G}_4$-orbit of length 6, and so the subgroup of $\mathcal{G}(\mathcal{S}_5')$ which preserves λ_5 has order $|\mathcal{G}_4|/6 = 2$; indeed this subgroup is $\langle D\rangle$. But in this case no element of $\mathcal{G}(\mathcal{S}_5')$ interchanges λ_4 and λ_5'; so $\mathcal{G}(\mathcal{S}_5') = \langle D\rangle \cong Z_2$.

Lemma 3.8 *There are just two classes of partial spreads $\mathcal{S}_5$ which are of type* I*. One class, listed as Vf.1 in table B.2a, is represented by $\mathcal{S}_4(\mathsf{I}) \cup \{\lambda_5\}$ in (3.30) and the other class, listed as Vg.1, is represented by $\mathcal{S}_4(\mathsf{I}) \cup \{\lambda_5'\}$ in (3.30). In each case $\mathcal{G}(\mathcal{S}_5) \cong Z_2$.* ■

3.6 Summary

Theorem 3.9 *There are precisely ten equivalence classes of partial spreads in* PG(4, 2) *of size 5. Five of these, Va.1, Vb.1, Vc.1, Vd.1, and Ve.1, are regulus-free. Two, Vf.1 and Vg.1, are of type* I*, and there is just one class in the case of types* L*, $\binom{4}{3}$ and $\binom{5}{3}$.* ■

4 Further preliminary results

Prior to our discussion of maximal partial spreads of orders nine (in section 5.1) and seven (in section 7.1), it will be helpful to list some facts surrounding non-maximal partial spreads of the kinds $\mathcal{S}_5(\mathsf{L})$ and $\mathcal{S}_6(\Delta)$. First we show afresh that there is a projectively unique partial spread $\mathcal{S}_5$ of type L, listed as Vh.1 in table B.2a, but this time determine its stabilizer group. We then proceed, in sections 4.2 - 5.2, to classify, up to equivalence, all those partial spreads which properly contain an $\mathcal{S}_5$ of type L, that is all those partial spreads $\mathcal{S}_r$, $r > 5$, which are not of type O, I, II or $\binom{4}{3}$. In particular, on account of lemma 1.9, we classify all $\mathcal{S}_9$ s.

4.1 The projectively unique $\mathcal{S}_5(\mathsf{L})$

Given any $\mathcal{S}_5 = \{\lambda_1, \dots, \lambda_5\}$ of type L, we may, after a suitable re-labelling, suppose that the two reguli are $\rho = \rho_{123} = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\rho' = \rho_{145} = \{\lambda_1, \lambda_4, \lambda_5\}$. Let the plane of intersection of the two solids $\sigma = \sigma_{123}$ and $\sigma' = \sigma_{145}$ be denoted by α, and let $\mathcal{H} = \mathcal{H}_{123} \subset \sigma$ and $\mathcal{H}' = \mathcal{H}_{145} \subset \sigma'$ denote the 3-dimensional hyperbolic quadrics associated with the reguli ρ and ρ'. Now in a PG(3, 2) a plane intersects a hyperbolic quadric either in a conic (= 3-arc) or in two intersecting lines. Since the intersections $\alpha \cap \mathcal{H}$ and $\alpha \cap \mathcal{H}'$ contain λ_1 they are thus necessarily of the form $\lambda_1 \cup \mu$ and $\lambda_1 \cup \mu'$ where the line $\mu \in \rho^{\text{opp}}$ and the line $\mu' \in (\rho')^{\text{opp}}$ meet in a privileged point p of λ_1. (The four lines of α other than λ_1, μ and μ' belong to none of the reguli $\rho, \rho^{\text{opp}}, \rho', (\rho')^{\text{opp}}$.) Let us choose a basis such that $\alpha = \langle 3, 4, 5\rangle$ and $\lambda_1 = \langle 4, 5\rangle = \{5, 4, 45\}$, with privileged point $p = 5$. So, without loss of generality, we may display $\mathcal{S}_5 = \mathcal{S}_5(\mathsf{L})$ in the form

$$\begin{aligned} &\lambda_1 = \{5, 4, 45\}, \quad \lambda_2 = \{3, x, x3\}, \quad \lambda_3 = \{35, x4, x345\}, \\ &\lambda_4 = \{34, y, y34\}, \quad \lambda_5 = \{345, y4, y35\}, \end{aligned} \tag{4.1}$$

for some $y \in \sigma \setminus \alpha$ and $x \in \sigma' \setminus \alpha$, the two hyperbolic quadrics $\mathcal{H} = \mathcal{H}_{123}$ and $\mathcal{H}' = \mathcal{H}_{145}$ being given by the arrays

$$\mathcal{H} = \begin{pmatrix} 5 & 4 & 45 \\ 3 & x & x3 \\ 35 & x4 & x345 \end{pmatrix}, \quad \mathcal{H}' = \begin{pmatrix} 5 & 4 & 45 \\ 34 & y & y34 \\ 345 & y4 & y35 \end{pmatrix}. \tag{4.2}$$

Here the two reguli ρ, ρ' are given by the rows, and the two opposite reguli ρ^{opp}, $(\rho')^{\text{opp}}$ by the columns, of these two arrays.

Lemma 4.1 *(i) With respect to a suitable choice of basis any $\mathcal{S}_5 = \mathcal{S}_5(\mathsf{L}) = \{\lambda_1, \ldots, \lambda_5\}$, with regulus pattern R_{123}, R_{145}, may be cast in the form*

$$\begin{aligned} &\lambda_1 = \{5, 4, 45\}, \quad \lambda_2 = \{3, 2, 23\}, \quad \lambda_3 = \{35, 24, 1u\}, \\ &\lambda_4 = \{34, 1, 134\}, \quad \lambda_5 = \{345, 14, 135\}. \end{aligned} \tag{4.3}$$

(ii) The stabilizer $\mathcal{G}_5 = \mathcal{G}(\mathcal{S}_5)$ of this partial spread of type L is

$$\mathcal{G}_5 = \langle A, K \rangle \cong D_8, \tag{4.4}$$

with $A = (1u, 134, 34, 45, 5)$ of order 4 and $K = (2, 1, 34, 4, 5)$ of order 2.

Proof. (i) In (4.1) x, y, e_3, e_4, e_5 are necessarily independent and so (4.3) follows from the choice $e_1 = y,\ e_2 = x$.

(ii) Observe that $KAK = A^{-1} = A^3$, and so $\langle A, K \rangle \cong D_8$. Also note that A and K effect permutations π_A and π_K of the five lines of $\mathcal{S}_5$ given by

$$\pi_A = (\lambda_1)(\lambda_2\lambda_4\lambda_3\lambda_5), \qquad \pi_K = (\lambda_1)(\lambda_2\lambda_4)(\lambda_3\lambda_5), \tag{4.5}$$

whence $\langle A, K \rangle$ lies inside $\mathcal{G}_5$. Now if $T \in \mathcal{G}_5$ then T necessarily stabilizes the line λ_1 (the only line of $\mathcal{S}_5$ of valency 2), fixes the privileged point $5 \in \lambda_1$ and either effects permutations of $\{\lambda_2, \lambda_3\}$ and of $\{\lambda_4, \lambda_5\}$ separately, or else swaps $\{\lambda_2, \lambda_3\}$ with $\{\lambda_4, \lambda_5\}$. On noting that $\langle \pi_A, \pi_K \rangle \cong D_8$ it follows that a suitable $S \in \langle A, K \rangle$ exists such that $T^* = ST$ stabilizes each of the five lines and also fixes 5. It quickly follows that $T^* = I$, whence $T \in \langle A, K \rangle$, and so $\mathcal{G}_5 = \langle A, K \rangle$. ■

If from $\mathcal{S}_5 = \mathcal{S}_5(\mathsf{L})$ we remove the line λ_1 of valency 2 the resulting $\mathcal{S}_4(\mathsf{O}) = \mathcal{S}_5 \setminus \{\lambda_1\}$ is of class IVa.1. So $\psi_4 = \psi(\mathcal{S}_4)$ is a double-four and is of the form $\psi_4 = \mathcal{P}_4 \setminus \{\lambda\}$, where the parabolic quadric $\mathcal{P}_4$, the line $\lambda \subset \mathcal{P}_4$ and the nucleus n of $\mathcal{P}_4$ are uniquely determined, see lemma 3.4. Now, cf. remark 3.6, λ_1 is a non-nuclear tangent to $\mathcal{P}_4$ which meets λ in a point p (the privileged point of $\mathcal{S}_5(\mathsf{L})$). Since n, λ and λ_1 are coplanar it follows that every line through n meets $\psi_5 = \psi(\mathcal{S}_5)$ in a point. This distinguished point $n \in (\psi_5)^c$ accounts for the 1^0 in the signature $(16, 16, 0)(1^0 6^2 9^4)(16^0)$ of $(\psi_5)^c$, see the orbit Vh in table B.1. (For $\mathcal{S}_5(\mathsf{L})$ as in (4.3), $\lambda_1 = \{5, 4, 45\},\ p = 5$ and, using lemma 3.4, $n = 3u$, $\lambda = \{5, 12, 125\}$.) Along with p and n the point $n' = p + n$ is also fixed by $\mathcal{G}_5$, and the lines through n' which lie in $(\psi_5)^c$ form a single $\mathcal{G}_5$-orbit of length 4. (For $\mathcal{S}_5(\mathsf{L})$ as in (4.3), $n' = 124$ and the $\mathcal{G}_5$-orbit is the orbit Ω'' at the start of the next section.)

4.2 The three extensions of an $\mathcal{S}_5(\mathsf{L})$ to an $\mathcal{S}_6$

Let $\mathcal{S}_5 = \mathcal{S}_5(\mathsf{L})$ be as in lemma 4.1, and, from table 2, note that $\psi(\mathcal{S}_5)$ has $L' = 16$ external lines. Under the action of $\mathcal{G}_5$ these 16 lines are seen to fall into three orbits Ω, Ω' and Ω'', of lengths 8, 4 and 4, with respective representatives

$$\lambda_6 = \{25, 13, 4u\}, \quad \lambda_6' = \{25, 15, 12\}, \quad \lambda_6'' = \{235, 2u, 124\}. \tag{4.6}$$

Consider the resulting three extensions of $\mathcal{S}_5 = \mathcal{S}_5(\mathsf{L})$ to an $\mathcal{S}_6$:

$$\mathcal{S}_6 = \mathcal{S}_5 \cup \{\lambda_6\}, \quad \mathcal{S}_6' = \mathcal{S}_5 \cup \{\lambda_6'\}, \quad \mathcal{S}_6'' = \mathcal{S}_5 \cup \{\lambda_6''\},$$

and denote by $\mathcal{G}_6$, $\mathcal{G}_6'$ and $\mathcal{G}_6''$ their stabilizers. We see that $\mathcal{S}_6''$ is of type Δ, with regulus pattern R_{123}, R_{145}, R_{365}, and so is inequivalent to both $\mathcal{S}_6$ and $\mathcal{S}_6'$ which are of type L. But $\mathcal{S}_6$ and $\mathcal{S}_6'$ are also inequivalent. For if $T \in \mathrm{GL}(5,2)$ were to map $\mathcal{S}_6$ on to $\mathcal{S}_6'$ then T would lie in $\mathcal{G}(\mathcal{S}_5)$, since T must map lines of nonzero valency to lines of nonzero valency, whence $T\lambda_6 = \lambda_6'$, in contradiction to λ_6, λ_6' belonging to different $\mathcal{G}(\mathcal{S}_5)$-orbits. *Hence, up to equivalence, $S_5 = S_5(L)$ has precisely three extensions to a partial spread of size 6.*

In the case of $\mathcal{S}_6$ any $T \in \mathcal{G}_6$ must stabilize separately both $\mathcal{S}_5$ and λ_6, and so $\mathcal{G}_6$ is a subgroup of $\mathcal{G}_5$. By the orbit-stabilizer theorem $|\mathcal{G}_6| = |\mathcal{G}_5|/|\Omega| = 1$. Similarly in the case of $\mathcal{S}_6'$ we have $|\mathcal{G}_6'| = |\mathcal{G}_5|/|\Omega'| = 2$; in fact, since λ_6' is fixed by the involution $K = (2, 1, 34, 4, 5)$ in lemma 4.1, $\mathcal{G}_6' = \langle K \rangle \cong Z_2$. Hence we have proved the following lemma.

Lemma 4.2 *Up to equivalence, there exist precisely two partial spreads of size 6 of regulus type L, with respective stabilizers of order 1 and 2, listed as VIh.1 and VIh.2 in table B.2a.*

Incidentally, besides having different orders for their stabilizers, VIh.1 and VIh.2 may be distinguished by their having different profiles:

$$\mathrm{profile}(\mathcal{S}_6) = (355)(447)(456)^2(555)(566), \quad \mathrm{profile}(\mathcal{S}_6') = (355)(456)^4(557).$$

In particular the line λ_1 of valency 2 contributes (566) in the case of $\mathcal{S}_6$, and (557) in the case of $\mathcal{S}_6'$.

In the case of $\mathcal{S}_6''$ we see that the subgroup of $\mathcal{G}_5$ which fixes λ_6'' is $\langle K \rangle \cong Z_2$. But there also exist elements of $\mathcal{G}_6''$ which send λ_6'' to another of the lines λ_2, λ_4 of valency 1, and so $|\mathcal{G}_6''| = 3\times\ |\langle K \rangle| = 6$. Indeed we find that $\mathcal{G}_6'' = \langle B, K \rangle \cong \mathrm{Sym}(3)$, where $B = (3, 2u, 235, 35, 24)$ effects the permutation $\pi_B = (\lambda_1\lambda_3\lambda_5)(\lambda_2\lambda_6''\lambda_4)$ of the six lines of $\mathcal{S}_6''$. Hence we have proved the following lemma. (See [24] for another proof.)

Lemma 4.3 *(i) Up to equivalence, there exists a unique partial spread of size 6 of type Δ. With respect to a suitable choice of basis any $\mathcal{S}_6 = \mathcal{S}_6(\Delta) = \{\lambda_1, \ldots, \lambda_6\}$ of regulus pattern R_{123}, R_{145}, R_{365} may be cast in the form*

$$\lambda_1 = \{5, 4, 45\}, \quad \lambda_2 = \{3, 2, 23\}, \quad \lambda_3 = \{35, 24, 1u\},$$
$$\lambda_4 = \{34, 1, 134\}, \quad \lambda_5 = \{345, 14, 135\}, \quad \lambda_6 = \{235, 2u, 124\}. \tag{4.7}$$

(ii) The stabilizer of this partial spread of type Δ is

$$\mathcal{G}(\mathcal{S}_6) = \langle B, K\rangle \cong \mathrm{Sym}(3), \tag{4.8}$$

where $B = (3, 2u, 235, 35, 24)$ has order 3, and where $K = (2, 1, 34, 4, 5)$. ∎

Remark 4.4 *Under the action of its stabilizer $\mathcal{G}(\mathcal{S}_6)$, a partial spread $\mathcal{S}_6(\Delta)$ decomposes as $\mathcal{S}_6(\Delta) = \mathcal{S}_3(\mathsf{O}) \cup \mathcal{S}_3(\mathsf{O})'$ where the two orbits $\mathcal{S}_3(\mathsf{O})$, $\mathcal{S}_3(\mathsf{O})'$ consist of those lines whose valency is 2, 1, respectively. For $\mathcal{S}_6(\Delta)$ as in (4.7), $\mathcal{S}_3(\mathsf{O}) = \{\lambda_1, \lambda_3, \lambda_5\}$ and $\mathcal{S}_3(\mathsf{O})' = \{\lambda_6, \lambda_4, \lambda_2\}$. Note the appropriateness here of the equilateral triangle symbol Δ, acted upon by its symmetry group $\cong \mathrm{Sym}(3)$, upon viewing $\lambda_1, \lambda_3, \lambda_5$ as the vertices and $\lambda_6, \lambda_4, \lambda_2$ (in that order!) as the midpoints of the opposite sides. The order here is important, since the regulus pattern R_{123}, R_{145}, R_{365} sets up a pairing $\lambda_1 \leftrightarrow \lambda_6$, $\lambda_3 \leftrightarrow \lambda_4$, $\lambda_5 \leftrightarrow \lambda_2$ which any $T \in \mathcal{G}(\mathcal{S}_6)$ must respect. For example K, see (4.5), effects the symmetry of Δ which keeps fixed the "vertex" λ_1 and the opposite "midpoint" λ_6.*

4.3 The extensions of an $\mathcal{S}_6(\Delta)$

If $\mathcal{S}_6 = \mathcal{S}_6(\Delta) = \mathcal{S}_3(\mathsf{O}) \cup \mathcal{S}_3(\mathsf{O})'$ is as in lemma 4.3 and remark 4.4 then the 13-set complementary to $\psi(\mathcal{S}_6)$ is of the form

$$\psi(\mathcal{S}_6)^c = \mathcal{H} \cup \xi, \qquad \xi = \{123, 125, 3u, 5u\}, \tag{4.9}$$

where the nine points of the hyperbolic quadric $\mathcal{H}$ are as in the array

$$\mathcal{H} = \begin{pmatrix} 25 & 15 & 12 \\ 13 & u & 245 \\ 4u & 234 & 145 \end{pmatrix}. \tag{4.10}$$

Observe that the hyperplane $\sigma = \langle \mathcal{H} \rangle$ is of the form

$$\sigma = \mathcal{H} \cup \tau \cup \tau', \qquad \tau = \{4, 35, 345\},\ \tau' = \{124, 134, 23\}, \tag{4.11}$$

where τ and τ' are the transversals τ_{135} and τ_{642} of the partial spreads $\mathcal{S}_3(\mathsf{O}) = \{\lambda_1, \lambda_3, \lambda_5\}$ and $\mathcal{S}_3(\mathsf{O})' = \{\lambda_6, \lambda_4, \lambda_2\}$. (Also note that the four points of ξ lie in a plane α which intersects σ in τ.)

Let ρ and $\kappa = \rho^{\mathrm{opp}}$ denote the reguli consisting of the rows and columns, respectively, of the hyperbolic quadric $\mathcal{H}$ in (4.10). Thus $\rho = \{\lambda_7, \lambda_8, \lambda_9\}$ and $\kappa = \{\lambda_7^{\mathrm{o}}, \lambda_8^{\mathrm{o}}, \lambda_9^{\mathrm{o}}\}$ where

$$\lambda_7 = \{25, 15, 12\}, \quad \lambda_8 = \{13, u, 245\}, \quad \lambda_9 = \{4u, 234, 145\}, \tag{4.12}$$

$$\lambda_7^{\mathrm{o}} = \{25, 13, 4u\}, \quad \lambda_8^{\mathrm{o}} = \{15, u, 234\}, \quad \lambda_9^{\mathrm{o}} = \{12, 245, 145\}. \tag{4.13}$$

Now these six generators of the quadric H are precisely the $L' = 6$ lines, see table 2, external to ψ. Under the action of the group $\mathcal{G}(\mathcal{S}_6) = \langle B, K\rangle \cong \mathrm{Sym}(3)$ these six lines fall into three orbits, of lengths 1, 2 and 3, namely $\kappa_1 = \{\lambda_9^{\mathrm{o}}\}$, $\kappa_2 = \{\lambda_7^{\mathrm{o}}, \lambda_8^{\mathrm{o}}\}$ and $\rho = \{\lambda_7, \lambda_8, \lambda_9\}$. The next theorem now follows.

Theorem 4.5 *(i) There are precisely three inequivalent partial spreads $\mathcal{S}_7$ of type Δ, with representatives $\mathcal{S}_6(\Delta)\cup\{\lambda_9^o\}$, $\mathcal{S}_6(\Delta)\cup\{\lambda_7^o\}$, and $\mathcal{S}_6(\Delta)\cup\{\lambda_7\}$, the stabilizers being $\langle B,K\rangle\cong \mathrm{Sym}(3)$, $\langle B\rangle\cong Z_3$ and $\langle K\rangle\cong Z_2$, respectively. (These are listed as VIIe.1, VIIe.2 and VIIe.3 in table B.2b.)*

(ii) There are precisely three inequivalent partial spreads $\mathcal{S}_8$ of type Δ, with representatives $\mathcal{S}_6(\Delta)\cup\{\lambda_7^o,\lambda_8^o\}$, $\mathcal{S}_6(\Delta)\cup\{\lambda_7^o,\lambda_9^o\}$, and $\mathcal{S}_6(\Delta)\cup\{\lambda_7,\lambda_8\}$, the stabilizers being $\langle B,K\rangle\cong \mathrm{Sym}(3)$, $\langle B\rangle\cong Z_3$ and $\langle K\rangle\cong Z_2$, respectively. (These are listed as VIIIc.1, VIIIc.2 and VIIIc.3 in table B.2b.)

(iii) There are precisely two inequivalent partial spreads of type $\mathsf{I}\Delta$, with representatives

$$\mathcal{S}_9^\rho=\mathcal{S}_6(\Delta)\cup\rho \qquad \text{and} \qquad \mathcal{S}_9^\kappa=\mathcal{S}_6(\Delta)\cup\kappa, \tag{4.14}$$

listed as IXa.2, type $\mathsf{I}^\rho\Delta$, and IXa.3, type $\mathsf{I}^\kappa\Delta$, in table B.2b. Both $\mathcal{S}_9^\rho$ and $\mathcal{S}_9^\kappa$ have the same stabilizer group, which they share with $\mathcal{S}_6(\Delta)$:

$$\mathcal{G}(\mathcal{S}_9^\rho)=\mathcal{G}(\mathcal{S}_9^\kappa)=\mathcal{G}(\mathcal{S}_6(\Delta))=\langle B,K\rangle\cong \mathrm{Sym}(3). \tag{4.15}$$

Given any partial spread $\mathcal{S}_9$ of type $\mathsf{I}\Delta$, let $\eta=\mathcal{S}_9\setminus\mathcal{S}_6(\Delta)$ be its distinguished regulus *(consisting of those lines which contribute the I to the $\mathsf{I}\Delta$). Then $\mathcal{S}_9$ is equivalent to $\mathcal{S}_9^\rho$ if $\mathcal{G}(\mathcal{S}_9)$ is transitive on η, and to $\mathcal{S}_9^\kappa$ if η contains a line λ_{fxd} which is fixed by $\mathcal{G}(\mathcal{S}_9)$.* ■

(Concerning $\mathcal{G}(\mathcal{S}_9^\rho)$, it is no larger than that given in (4.15) since any symmetry of $\mathcal{S}_9^\rho=\mathcal{S}_6(\Delta)\cup\rho$ must preserve the distinguished regulus ρ, and hence also preserves $\mathcal{S}_6(\Delta)$; similarly for $\mathcal{G}(\mathcal{S}_9^\kappa)$.)

Given an $\mathcal{S}_9(\mathsf{I}\Delta)$ *we can determine whether it is of the kind $\mathsf{I}^\rho\Delta$ or $\mathsf{I}^\kappa\Delta$ without recourse to its stabilizer group.* One recipe for so doing uses the fact that the partial spread $\mathcal{S}_3(\mathsf{O})$ in remark 4.4 has σ in (4.11) for its even hyperplane, and so $\mathcal{S}_3(\mathsf{O})$ induces a null polarity in σ, see section 3.2. For type $\mathsf{I}^\rho\Delta$ the distinguished regulus of $\mathcal{S}_9(\mathsf{I}\Delta)$ is of polar type $(3,0)$, see section A.2.2, while for type $\mathsf{I}^\kappa\Delta$, it is of polar type $(1,2)$. Equally well, note that σ is also the even hyperplane of the partial spread $\mathcal{S}_3(\mathsf{O})'$ in remark 4.4, and is thereby equipped with another null polarity. Using this second null polarity, $\mathcal{S}_9(\mathsf{I}\Delta)$ is of type $\mathsf{I}^\rho\Delta$ or $\mathsf{I}^\kappa\Delta$ according as its distinguished regulus is of polar type $(0,3)$ or $(2,1)$. (The three inequivalent partial spreads $\mathcal{S}_7(\Delta)$, and three inequivalent partial spreads $\mathcal{S}_8(\Delta)$, may similarly be distinguished.)

A second recipe is as follows. Let the lines of $\mathcal{S}_9=\mathcal{S}_9(\mathsf{I}\Delta)$ be numbered such that the regulus pattern is R_{123}, R_{145}, R_{365}, R_{789}, and let σ be the ambient hyperplane of the distinguished regulus $\eta=\{\lambda_7,\lambda_8,\lambda_9\}$ of $\mathcal{S}_9$. For $i=1,2,\ldots,6$, let λ_i meet σ in the point a_i. Then, recall the pairing in remark 4.4, $\lambda_{\mathrm{fxd}}=\{a_1+a_6,\ a_3+a_4,\ a_5+a_2\}$ is the line of $\mathcal{H}=\psi(\eta)$ which is fixed by $\mathcal{G}(\mathcal{S}_9)$, and $\mathcal{S}_9(\mathsf{I}\Delta)$ is of type $\mathsf{I}^\rho\Delta$ or $\mathsf{I}^\kappa\Delta$ according as $\lambda_{\mathrm{fxd}}\in\eta^{\mathrm{opp}}$ or $\lambda_{\mathrm{fxd}}\in\eta$.

A third recipe is simplest of all: using the null polarity induced in σ by $\mathcal{S}_3(\mathsf{O})$, the polar $(\tau')^\perp=\{12,245,145\}$ of the transversal $\tau'=\{124,134,23\}$ of $\mathcal{S}_3(\mathsf{O})'$ lies in the distinguished regulus $\mathcal{S}_9\setminus\mathcal{S}_6(\Delta)$ of $\mathcal{S}_9(\mathsf{I}^\kappa\Delta)$ but not in that of $\mathcal{S}_9(\mathsf{I}^\rho\Delta)$.

4.4 The nine extensions of an $\mathcal{S}_5(\mathsf{L})$ to an $\mathcal{S}_7$

For $\mathcal{S}_5 = \mathcal{S}_5(\mathsf{L})$ as in lemma 4.1 and section 4.2, any extension to an $\mathcal{S}_r,\ r = 7, 8, 9$, which makes use of one of the lines from the orbit Ω'' will be of type Δ, and so equivalent to one already considered in theorem 4.5. So to obtain anything different we look at extensions of an $\mathcal{S}_5(\mathsf{L})$ to an $\mathcal{S}_7$ which use lines solely from the orbits Ω and Ω' (see beginning of section 4.2). Of these extensions we will find four which are of type L, these being inequivalent from the manner of their construction in view of the following remark.

Remark 4.6 *If $\mathcal{S}_7 = \mathcal{S}_5(\mathsf{L}) \cup \{\lambda_6, \lambda_7\}$ is of type L then any $T \in \mathcal{G}(S_7)$ must separately stabilize both $\mathcal{S}_5(\mathsf{L})$ and $\{\lambda_6, \lambda_7\}$, since λ_6, λ_7 are the only lines $\in \mathcal{S}_7$ of zero valency. Thus $\mathcal{G}(S_7)$ is a subgroup of $\mathcal{G}(S_5(\mathsf{L})) \cong D_8$.*

First of all consider adding two lines λ_6, λ_7 both belonging to Ω. We may as well choose $\lambda_6 = \{25, 13, 4u\}$, as in eq. (4.6). Since λ_7 has to be skew to λ_6, we find that there are four possible choices (a)-(d) for $\lambda_7 \in \Omega$, namely

$$\begin{array}{ll} \text{(a) } \lambda_7 = K\lambda_6, \text{ yielding an } \mathcal{S}_7(\mathsf{L}); & \text{(b) } \lambda_7 = KA^2\lambda_6, \text{ yielding an } \mathcal{S}_7(\mathsf{L}); \\ \text{(c) } \lambda_7 = A^2\lambda_6, \text{ yielding an } \mathcal{S}_7(\mathsf{Y}); & \text{(d) } \lambda_7 = KA\lambda_6, \text{ yielding an } \mathcal{S}_7(\mathsf{F}). \end{array} \tag{4.16}$$

(Here A and K are as in lemma 4.1.) Recalling that the stabilizer of $\mathcal{S}_6(\mathsf{L}) = \mathcal{S}_5(\mathsf{L}) \cup \{\lambda_6\}$ is trivial, it follows that

$$\text{choice (a) } \mathcal{G}(S_7) = \langle K \rangle; \qquad \text{choice (b) } \mathcal{G}(S_7) = \langle KA^2 \rangle, \tag{4.17}$$

so that $|\mathcal{G}(S_7)| = 2$ for both of the choices (a) and (b). In each of these two cases we see that $\psi(S_7)^c$ has signature $(10, 4, 0)(2^0 4^1 4^2)(10^0)$, and so ψ belongs to the orbit VIIc of table B.1. The two partial spreads $\mathcal{S}_7$ are inequivalent, see before remark 4.6; they are listed as VIIc.3, VIIc.4 in table B.2b.

For choice (c) we see that $N_7 = 3$, with $\rho_{167} = \{\lambda_1, \lambda_6, \lambda_7\}$ being a regulus, so that $\mathcal{S}_7$ is of type Y. Defining $Y : (1, 2, 3, 4, 5) \longmapsto (4u, 345, 135, 5, 45)$, note that $Y \in \mathcal{G}(S_7)$ since it effects the permutation $\pi_Y = (\lambda_1)(\lambda_2\lambda_5\lambda_7\lambda_3\lambda_4\lambda_6)$ of the seven lines of $\mathcal{S}_7$, and cyclically permutes the three reguli $\rho_{123}, \rho_{145}, \rho_{167}$. Consider the subgroup $\mathcal{G}_0$ of $\mathcal{G}(S_7)$ which fixes the regulus ρ_{167}, hence stabilizes $\{\rho_{123}, \rho_{145}\}$, and hence stabilizes $\mathcal{S}_5(\mathsf{L})$. So $\mathcal{G}_0$ is that subgroup of $\mathcal{G}(\mathcal{S}_5(\mathsf{L}))$ which stabilizes $\{\lambda_6, \lambda_7\}$. From lemma 4.1(ii) we see that $\mathcal{G}_0 = \langle A^2 \rangle$. But $A^2 = Y^3$, and so $\mathcal{G}(S_7)$ is precisely $\langle Y \rangle \cong Z_6$.

For choice (d) we see that $N_7 = 3$, with $\{\lambda_2, \lambda_6, \lambda_7\}$ being a regulus, so that $\mathcal{S}_7$ is of type F, and we find that $\mathcal{G}(S_7) = \langle KA \rangle \cong Z_2$. One way to see this last result $\mathcal{G}(S_7) = \langle KA \rangle$ is to apply our results in section 7.2 to $\mathcal{S}_6(\mathsf{II}) = \rho_{145} \cup \rho_{267}$. From (7.4) we see that $\mathcal{S}_6(\mathsf{II})$ has stabilizer $\langle C \rangle$ where $C : (1, 2, 3, 4, 5) \longmapsto (13, 45, 5, 2, 3)$. Noting that $\lambda_3 = \{35, 24, 1u\}$ is stabilized by C^2 but not by C it follows that $\mathcal{G}(S_7) = \langle C^2 \rangle = \langle KA \rangle$.

For both of the choices (c) and (d) we see that $\psi(S_7)^c$ has signature $(10, 2, 0)(4^0 6^2)(10^0)$, and so ψ belongs to the orbit VIIe of table B.1; the partial spreads are listed in table B.2b as (c) VIIe.4 and (d) VIIe.5.

Next we consider adding two lines $\lambda_6 \in \Omega,\ \lambda'_7 \in \Omega'$, and again we may as well choose $\lambda_6 = \{25, 13, 4u\}$, as in eq. (4.6). Since λ'_7 has to be skew to λ_6, we find that there are two possible choices for $\lambda'_7 \in \Omega'$, namely

$$\text{choice (e) } \lambda'_7 = A\lambda'_6, \quad \text{yielding an } \mathcal{S}_7 \text{ of type } \mathsf{L};$$
$$\text{choice (f) } \lambda'_7 = A^2\lambda'_6, \quad \text{yielding an } \mathcal{S}_7 \text{ of type } \mathsf{L}.$$

Here $\lambda'_6 = \{25, 15, 12\}$, as in eq. (4.6). Since λ_6 and λ'_7 now belong to distinct orbits of $\mathcal{G}(S_5(\mathsf{L}))$ it follows, cf. remark 4.6, that any $T \in \mathcal{G}(S_7)$ lies in $\mathcal{G}(\mathcal{S}_6) \cap \mathcal{G}(\mathcal{S}'_6)$, where $\mathcal{S}_6 = S_5(\mathsf{L}) \cup \{\lambda_6\}$ and $\mathcal{S}'_6 = S_5(\mathsf{L}) \cup \{\lambda'_7\}$. But, lemma 4.2, $\mathcal{G}(\mathcal{S}_6)$ is trivial, and hence so is $\mathcal{G}(S_7)$ for either of the choices (e), (f). Again for each of these two choices we see that $\psi(S_7)^c$ has signature $(10, 4, 0)(2^0 4^1 4^2)(10^0)$, and so ψ belongs to the orbit VIIc of table B.1. The two partial spreads $\mathcal{S}_7$ are inequivalent, see before remark 4.6; they are listed as VIIc.5, VIIc.6 in table B.2b.

Finally we consider adding two lines λ'_6, λ'_7 both belonging to Ω'. We may as well choose $\lambda'_6 = \{25, 15, 12\}$, as in eq. (4.6). Since λ'_7 has to be skew to λ'_6, we find that there are two possible choices for $\lambda'_7 \in \Omega'$, namely (g) $A\lambda'_6$ and (h) $KA\lambda'_6$. But K stabilizes λ'_6, and hence the two choices lead to equivalent partial spreads, so we may as well adopt choice (g):

$$\begin{aligned} &\mathcal{S}'_7 = S_5(\mathsf{L}) \cup \{\lambda'_6, \lambda'_7\}, \qquad \text{where} \\ &\lambda'_6 = \{25, 15, 12\}, \qquad \lambda'_7 = A\lambda'_6 = \{2u, 234, 125\}. \end{aligned} \tag{4.18}$$

We see that $N_7 = 3$, with $\{\lambda_4, \lambda'_6, \lambda'_7\}$ being a regulus, so that $\mathcal{S}'_7$ is of type F. But recall that we have already found an extension of $S_5(\mathsf{L})$ of type F, namely, see choice (d) above,

$$\begin{aligned} &\mathcal{S}_7 = S_5(\mathsf{L}) \cup \{\lambda_6, \lambda_7\}, \qquad \text{where} \\ &\lambda_6 = \{25, 13, 4u\}, \qquad \lambda_7 = KA\lambda_6 = \{235, 15, 123\}. \end{aligned} \tag{4.19}$$

In fact, as in the proof of the next theorem, these two partial spreads $\mathcal{S}_7$ and $\mathcal{S}'_7$ are equivalent.

Theorem 4.7 *Up to equivalence there are precisely nine extensions of a $S_5(\mathsf{L})$ to a S_7. Of these four are of type L, listed as VIIc.3 - VIIc.6 in table B.2b, and three are of type Δ, listed as VIIe.1 - VIIe.3. One is of type Y, listed as VIIe.4, and one is of type F, listed as VIIe.5.*

Proof. Because of our preceding work it only remains to show that the two partial spreads $\mathcal{S}_7$ and $\mathcal{S}'_7$ of type F given in (4.19), (4.18) are equivalent. Defining $T \in \mathrm{GL}(5,2)$ by $T : (1, 2, 3, 4, 5) \mapsto (25, 4, 5, 1, 34)$ note that T maps $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$ to $\lambda_4, \lambda_1, \lambda_5, \lambda'_6, \lambda'_7, \lambda_2, \lambda_3$, respectively, and so $T(\mathcal{S}_7) = \mathcal{S}'_7$. (Note incidentally that T in this proof is a Singer element of $\mathrm{GL}(5,2) : \langle T \rangle \cong Z_{31}$.) ■

Corollary 4.8 *A partial spread $\mathcal{S}_7$ is maximal if and only if it is of type $\binom{4}{3}$, that is if and only if it is of the form*

$$\mathcal{S}_7 = \mathcal{S}_4(\tbinom{4}{3}) \cup \mathcal{S}_3(\mathsf{O}). \tag{4.20}$$

Proof. If $\mathcal{S}_7$ is of the form (4.20) then, lemma 1.5, it is maximal. To prove the reverse implication, we will derive a contradiction from supposing that $\mathcal{S}_7$ is maximal and yet does not contain an $\mathcal{S}_4$ of type $\binom{4}{3}$. For if this last is the case then $\mathcal{S}_7$, which by theorem 1.7 has $N_7 = 4$, must contain an $\mathcal{S}_5(\mathsf{L})$. But, see theorem 4.7, an extension $\mathcal{S}_7$ of an $\mathcal{S}_5(\mathsf{L})$ has $N_7 \leq 3$, and so is not maximal. ■

We conclude this section by providing a way to distinguish between the classes VIIc.3, VIIc.4, VIIc.5 and VIIc.6, all four consisting of partial spreads of type L and profile $(5,6,7)^2(6,6,8)^2(6,7,7)^2(7,7,8)$. It is true that classes VIIc.3 and VIIc.4 have already been distinguished from classes VIIc.5 and VIIc.6, since for the former two $\mathcal{G}(S_7) \cong Z_2$ while for the latter two $\mathcal{G}(S_7)$ is trivial. The following recipe has the virtue of providing distinctions between all four classes without the need to computer the stabilizers. Given that $\mathcal{S}_7 = \mathcal{S}_5(\mathsf{L}) \cup \{\lambda_6, \lambda_7\}$ is of type L, suppose that λ_1 is the line of valency 2 and put $\mathcal{S}_4 = \mathcal{S}_5(\mathsf{L}) \diagdown \{\lambda_1\}$. Consider the subset $\mathcal{L} \subset \mathcal{S}_4$ consisting of those lines of $\mathcal{S}_4$ which lie inside the even hyperplane σ of $\mathcal{S}_3(\mathsf{O}) = \{\lambda_1, \lambda_6, \lambda_7\}$. Then one finds that $|\mathcal{L}| = 2$ if $\mathcal{S}_7$ is of class VIIc.3 or VIIc.4, and $|\mathcal{L}| = 1$ if $\mathcal{S}_7$ is of class VIIc.5 or VIIc.6. Moreover if $|\mathcal{L}| = 2$, then for one class, say VIIc.3, the two lines of $\mathcal{L}$ are self-polar, and for the other class, say VIIc.4, the two lines of $\mathcal{L}$ are nonpolar. Also if $|\mathcal{L}| = 1$, then for one class, say VIIc.5, the line $\lambda \in \mathcal{L}$ is self-polar, and for the other class, say VIIc.6, the line $\lambda \in \mathcal{L}$ is nonpolar.

5 Partial spreads of sizes 8 and 9

5.1 The four maximal partial spreads of size 9

By corollary 1.10, every $\mathcal{S}_9$ must be of the form $\mathcal{S}_9 = \mathcal{S}_5(\mathsf{L}) \cup \mathcal{S}_4$ for suitable $\mathcal{S}_4$. Consider $\mathcal{S}_5(\mathsf{L})$ as in lemma 4.1, and recall our opening remarks in section 4.4 concerning the three $\mathcal{G}(\mathcal{S}_5(\mathsf{L}))$-orbits $\Omega, \Omega', \Omega''$ of lines external to $\psi(\mathcal{S}_5)$, with representatives $\lambda_6, \lambda'_6, \lambda''_6$ as in eq. (4.6). If $\mathcal{S}_4$ contains a line $\lambda \in \Omega''$ then $\mathcal{S}_9$ will be equivalent to one of the two classes of $\mathcal{S}_9$ found in theorem 4.5(iii), and so be of type $\mathsf{I}^\rho\Delta$ or type $\mathsf{I}^\kappa\Delta$. So to obtain an $\mathcal{S}_9$ of a different class, $\mathcal{S}_4$ must consist solely of lines from orbits Ω and Ω'. The four members of Ω' do not form an $\mathcal{S}_4$, and so, without loss of generality, we may suppose that $\lambda_6 \in \mathcal{S}_4$. Now skew to λ_6 are the four lines $K\lambda_6, KA^2\lambda_6, A^2\lambda_6, KA\lambda_6 \in \Omega$ and the two lines $A\lambda'_6, A^2\lambda'_6 \in \Omega'$, see choices (a)-(f) in section 4.4. From these six lines we see that we may form an $\mathcal{S}_3$ in just two ways, $\{A^2\lambda_6, K\lambda_6,\ KA^2\lambda_6\}$ and $\{KA\lambda_6, A\lambda'_6,\ A^2\lambda'_6\}$, leading to the existence of precisely two choices of $\mathcal{S}_4 = \{\lambda_6, \lambda_7, \lambda_8, \lambda_9\} \subset \Omega \cup \Omega'$ which contain λ_6, namely

$$\mathcal{S}_4^{\mathrm{i}} = \{\lambda_6, A^2\lambda_6, K\lambda_6, KA^2\lambda_6\} \qquad \text{and} \qquad \mathcal{S}_4^{\mathrm{ii}} = \{\lambda_6, KA\lambda_6, A\lambda'_6, A^2\lambda'_6\}. \tag{5.1}$$

(Here A and K are as in lemma 4.1.)

Theorem 5.1 *(i) Up to equivalence there are precisely four partial spreads of size 9. One is of type* X*, represented by* $\mathcal{S}_5(\mathsf{L}) \cup \mathcal{S}_4^i$*, one is of type* E*, represented by* $\mathcal{S}_5(\mathsf{L}) \cup \mathcal{S}_4^{ii}$*, and the* *other two are of types $\mathsf{I}^\rho\Delta$ and $\mathsf{I}^\kappa\Delta$ as described in theorem 4.5(iii). These four classes are listed as IXa.1, IXa.4, IXa.2 and IXa.3, respectively, in table B.2b.*

(ii) The stabilizer group of an $\mathcal{S}_9(\mathsf{X})$ is of order 24, and each of the other three classes has stabilizer group of order 6. In more detail

$$\mathcal{G}(\mathcal{S}_9(\mathsf{X})) \cong \mathrm{Alt}(4) \times Z_2, \qquad \mathcal{G}(\mathcal{S}_9(\mathsf{E})) \cong Z_6,$$
$$\mathcal{G}(\mathcal{S}_9(\mathsf{I}^\rho\Delta)) \cong \mathrm{Sym}(3), \qquad \mathcal{G}(\mathcal{S}_9(\mathsf{I}^\kappa\Delta)) \cong \mathrm{Sym}(3). \tag{5.2}$$

Proof. (i) Up to equivalence there is a unique partial spread $\mathcal{S}_5$ of type L, see 4.1, so part (i) follows from the above run-in to the theorem, after checking that $\mathcal{S}_5(\mathsf{L}) \cup \mathcal{S}_4^{\mathrm{i}}$ is of type X (having regulus pattern $R_{123}, R_{145}, R_{167}, R_{189}$), and that $\mathcal{S}_5(\mathsf{L}) \cup \mathcal{S}_4^{\mathrm{ii}}$ is of type E (having regulus pattern $R_{123}, R_{145}, R_{267}, R_{389}$).

(ii) Let the four reguli $\rho_{123}, \rho_{145}, \rho_{167}, \rho_{189}$ in $\mathcal{S}_9(\mathsf{X}) = \mathcal{S}_5(\mathsf{L}) \cup \mathcal{S}_4^{\mathrm{i}}$ be denoted $\rho_a, \rho_b, \rho_c, \rho_d$, and note that $\mathcal{S}_9(\mathsf{X})$ contains the four partial spreads $\mathcal{S}_7^a$, $\mathcal{S}_7^b$, $\mathcal{S}_7^c$, $\mathcal{S}_7^d$ of type Y, where the three reguli in $\mathcal{S}_7^x$ are $\{\rho_a, \rho_b, \rho_c, \rho_d\} \smallsetminus \rho_x$. Now, table 2, each $\mathcal{S}_7^x$ has just two external lines, which when added to $\mathcal{S}_7^x$ yield the original $\mathcal{S}_9(\mathsf{X})$; hence $\mathcal{G}(\mathcal{S}_7^x) < \mathcal{G}(S_9)$ for each x. Consequently $\mathcal{G}(S_9)$ contains four subgroups $Z_3^x \cong Z_3$, where Z_3^x cyclically permutes the three reguli in $\mathcal{S}_7^x$. For example, recall that in considering $\mathcal{S}_7^d = \mathcal{S}_5(\mathsf{L}) \cup \{\lambda_6, A^2\lambda_6\}$, in connection with choice (c) in eq. (4.16), we found that $\mathcal{G}(S_7^d) = \langle Y \rangle \cong Z_6$, with $Y : (1,2,3,4,5) \longmapsto (4u, 345, 135, 5, 45)$. Here $Y = Y^d$ effects the permutation $\pi_Y = (\lambda_1)(\lambda_2\lambda_5\lambda_7\lambda_3\lambda_4\lambda_6)(\lambda_8\lambda_9)$ of the nine lines of $\mathcal{S}_9(\mathsf{X})$ and hence the permutation $(\rho_a\rho_b\rho_c)(\rho_d)$ of the four reguli; so $(Y^d)^2 = Y^2$, of order 3, effects the permutation $(\rho_a\rho_c\rho_b)(\rho_d)$. Since $\mathcal{G}(S_9)$ is thus transitive upon the four $\mathcal{S}_7^x$, it has order $4 \times \mathcal{G}(\mathcal{S}_7^x) = 24$. Concerning its structure, by the foregoing $\mathcal{G}(S_9)$ achieves all even permutations of the four reguli and so contains a subgroup $\cong \mathrm{Alt}(4)$. But note that it also contains the central involution $Y^3 (= (Y^x)^3$, $x = a, b, c, d)$ which fixes each of the four reguli (but which swaps the two lines $\neq \lambda_1$ of each regulus). Consequently $\mathcal{G}(\mathcal{S}_9(\mathsf{X})) \cong \mathrm{Alt}(4) \times Z_2$. Alternatively, by similar reasoning, we may arrive at $\mathcal{G}(\mathcal{S}_9)$ in the form

$$\mathcal{G}(\mathcal{S}_9(\mathsf{X})) = \langle Y, K \rangle = \langle Y^2, K \rangle \times \langle Y^3 \rangle \cong \mathrm{Alt}(4) \times Z_2, \tag{5.3}$$

where $K : (1,2,3,4,5) \longmapsto (2,1,34,4,5)$ is as in lemma 4.1 and effects the permutation $(\rho_a\rho_b)(\rho_c\rho_d)$ of the four reguli. (See also (6.13) below for another derivation of the $\mathrm{Alt}(4) \times Z_2$ structure of $\mathcal{G}(\mathcal{S}_9(\mathsf{X}))$.)

Concerning $\mathcal{S}_9(\mathsf{E}) = \mathcal{S}_5(\mathsf{L}) \cup \mathcal{S}_4^{\mathrm{ii}}$ observe that it contains three different partial spreads $\mathcal{S}_7$ of type F. Now the stabilizer of each $\mathcal{S}_7(\mathsf{F})$ is seen to be $\langle KA \rangle \cong Z_2$, see the discussion of choice (d) in eq. (4.16). But there exists an element of the stabilizer which cyclically permutes $\lambda_1, \lambda_2, \lambda_3$, and hence the three $\mathcal{S}_7(\mathsf{F})$s, whence $|\mathcal{G}(\mathcal{S}_9(\mathsf{X}))| = 3 \times 2 = 6$. Indeed we have

$$\mathcal{G}(\mathcal{S}_9(\mathsf{E})) = \langle E \rangle \cong Z_6,$$

where $E : (1,2,3,4,5) \mapsto (2u, 45, 5, 1u, 35)$ effects the permutation $(\lambda_1\lambda_3\lambda_2)\,(\lambda_4\lambda_8\lambda_7\lambda_5\lambda_9\lambda_6)$ and where $E^3 = KA (= A^3K)$.

See eq. (4.15) for the other two stabilizers in (5.2). ■

Corollary 5.2 *The regulus type of a partial spread $\mathcal{S}_r$ in* PG(4, 2) *is necessarily one of those listed in table 1.*

Proof. Any $\mathcal{S}_r$ lies inside a maximal partial spread; but by our results in theorem 1.7, corollary 4.8 and theorem 5.1 the type of a maximal partial spread is $\binom{5}{3}$, $\binom{4}{3}$, X, E or $\mathsf{I}\Delta$. ∎

5.2 The nine partial spreads of size 8

By removing a line from each of the above four kinds of $\mathcal{S}_9$ we will obtain all possible kinds of $\mathcal{S}_8$. For an $\mathcal{S}_9$ denote by $\mathcal{L}_i$ the subset of lines of $\mathcal{S}_9$ of valency i.

First of all consider a partial spread $\mathcal{S}_8 = \mathcal{S}_9 \setminus \{\lambda\}$ in the cases where λ has valency 1. Then $\mathcal{S}_8$ has $N_8 = 3$ reguli; thus, see table 2, $\psi(\mathcal{S}_8)$ has $L' = 1$ external lines, and so $\mathcal{S}_8$ has a unique extension back to the original $\mathcal{S}_9$. According as $\mathcal{S}_9$ is of type (i) X (ii) E (iii) $\mathsf{I}^\rho\Delta$ (iv) $\mathsf{I}^\kappa\Delta$ then $\mathcal{S}_8$ is of type (i) Y (ii) F (iii) either Δ or $\mathsf{I}^\rho\mathsf{L}$ (iv) either Δ or $\mathsf{I}^\kappa\mathsf{L}$. (Here, and in table B.2b, an $\mathcal{S}_8(\mathsf{IL})$ is said to be of type $\mathsf{I}^\rho\mathsf{L}$ or $\mathsf{I}^\kappa\mathsf{L}$ according as its unique extension to an $\mathcal{S}_9$ is of type $\mathsf{I}^\rho\Delta$ or $\mathsf{I}^\kappa\Delta$.) In cases (i) and (ii) the stabilizer group $\mathcal{G}(\mathcal{S}_9)$ is transitive on $\mathcal{L}_1$, and so, using $|\mathcal{G}(\mathcal{S}_8)| = |\mathcal{G}(\mathcal{S}_9)|/|\mathcal{L}_1|$, the order of $\mathcal{G}(\mathcal{S}_8(\mathsf{Y}))$ is $24/8 = 3$ and the order of $\mathcal{G}(\mathcal{S}_8(\mathsf{F}))$ is $6/6 = 1$.

In case (iii) $\mathcal{L}_1$ consists of two $\mathcal{G}(\mathcal{S}_9)$-orbits ρ and δ, each of length 3, and $\mathcal{S}_8$ is of type Δ or $\mathsf{I}^\rho\mathsf{L}$ according as $\lambda \in \rho$ or $\lambda \in \delta$; in either case $|\mathcal{G}(\mathcal{S}_8)| = 6/3 = 2$. In case (iv) $\mathcal{L}_1$ consists, see theorem 4.5 (with $\kappa_1 = \{\lambda_9^\circ\}$, $\kappa_2 = \{\lambda_7^\circ, \lambda_8^\circ\}$ and $\delta = \{\lambda_2, \lambda_4, \lambda_6\}$), of three $\mathcal{G}(\mathcal{S}_9)$-orbits, κ_1, κ_2 and δ, of lengths 1, 2, 3, and $\mathcal{S}_8$ is of type Δ, Δ or $\mathsf{I}^\kappa\mathsf{L}$ according as $\lambda \in \kappa_1$, $\lambda \in \kappa_2$ or $\lambda \in \delta$, with $\mathcal{G}(\mathcal{S}_8)$ of corresponding orders $6/1 = 6$, $6/2 = 3$ and $6/3 = 2$.

It follows from the foregoing that there exist projectively unique partial spreads S_8 of types Y, F, $\mathsf{I}^\rho\mathsf{L}$ *and* $\mathsf{I}^\kappa\mathsf{L}$, *and precisely three classes of S_8 of type* Δ. (These last three classes, and their stabilizer groups, were considered previously in theorem 4.5(ii).)

Secondly consider a partial spread $\mathcal{S}_8 = \mathcal{S}_9 \setminus \{\lambda\}$ in those of the above cases, namely $\mathcal{S}_9$ of type (ii) E (iii) $\mathsf{I}^\rho\Delta$ (iv) $\mathsf{I}^\kappa\Delta$, where λ has valency 2. In each of these three cases $\mathcal{G}(\mathcal{S}_9)$ is transitive on the three lines of $\mathcal{L}_2$, and $\mathcal{S}_8$ is of type II with stabilizer of order $6/3 = 2$. So there exist at most three classes of $\mathcal{S}_8(\mathsf{II})$. But now $N_8 = 2$ and so there are $L' = 3$ external lines. Thus any $\mathcal{S}_8(\mathsf{II})$ has precisely three extensions to an $\mathcal{S}_9$; moreover one finds that these extensions are one each of types E, $\mathsf{I}^\rho\Delta$ and $\mathsf{I}^\kappa\Delta$. *Consequently there exists a projectively unique partial spread S_8 of type* II.

Finally from an $\mathcal{S}_9$ of type X we may remove the unique line λ of valency 4 and thereby deduce that *there exists a projectively unique partial spread S_8 of type* O. In this case $N_8 = 0$ and so, see table 2, the 7-set $\psi(\mathcal{S}_8)^c$ has $L' = 7$ internal lines, *which entails that $\psi(S_8)^c$ is a plane, say α.* Hence an $\mathcal{S}_8(\mathsf{O})$ has precisely seven extensions $\mathcal{S}_8(\mathsf{O}) \cup \{\nu\}$, $\nu \subset \alpha$, to an $\mathcal{S}_9$, each extension being an $\mathcal{S}_9(\mathsf{X})$ since for no other type of $\mathcal{S}_9$ can $\mathcal{S}_9 \setminus \{\lambda\}$ be of type O. Since an $\mathcal{S}_9(\mathsf{X})$ is projectively unique, note that $\mathcal{G}(\mathcal{S}_8)$ is transitive on the seven lines $\nu \subset \alpha$, with "point"-stabilizer $\mathcal{G}(\mathcal{S}_9(\mathsf{X}))$ of order 24. Hence $|\mathcal{G}(\mathcal{S}_8)| = 24 \times 7 = 168$. (For the structure of $\mathcal{G}(\mathcal{S}_8)$, and also for a different construction of an $\mathcal{S}_8(\mathsf{O})$, see section 6.1 below.) Summarizing:

Theorem 5.3 *Up to equivalence there are precisely nine partial spreads of size 8 in* $\mathrm{PG}(4,2)$, *three of type* Δ, *listed as VIIIc.1, VIIIc.2 and VIIIc.3 in table B.2b, and one each of types* O, II, Y, $\mathsf{I}^\rho\mathsf{L}$, $\mathsf{I}^\kappa\mathsf{L}$, F, *listed as VIIIa, VIIIb, VIIIc.4 - VIIIc.7.* ■

6 Partial spreads $\mathcal{S}_r(\mathsf{O})$, $r > 5$

6.1 The unique partial spreads $\mathcal{S}_8(\mathsf{O})$ and $\mathcal{S}_7(\mathsf{O})$

In section 5.2 we saw that a partial spread S_8 of type O is projectively unique and may be constructed from an $\mathcal{S}_9(\mathsf{X})$ by removing the line of valency 4. In this section we give a different construction of an $\mathcal{S}_8(\mathsf{O})$, and provide full details of its high symmetry ($|\mathcal{G}(\mathcal{S}_8)| = 168$). (For further interesting aspects of an $\mathcal{S}_8(\mathsf{O})$ see [18].) The less symmetrical $\mathcal{S}_9(\mathsf{X})$ is then constructed from the more symmetrical $\mathcal{S}_8(\mathsf{O})$ by adding a line. We also show that there is a projectively unique $\mathcal{S}_7(\mathsf{O})$, obtained from an $\mathcal{S}_8(\mathsf{O})$ by removing any line, with stabilizer $\mathcal{G}(\mathcal{S}_7)$ of order 21.

Recalling that $\alpha = \psi(\mathcal{S}_8(\mathsf{O}))^c$ is a plane, let us start out from $\mathrm{PG}(4,2)$ as the join of a line $\lambda_0 \in \mathcal{S}_8$ and the plane α. So in vector space terms our construction starts out from a preferred direct sum decomposition $V_5 = V_2 \oplus V_3$, where $\mathbf{P}V_2 = \lambda_0$ and $\mathbf{P}V_3 = \alpha$. (It will later emerge that the seven lines of $\mathcal{S}_8(\mathsf{O})$ other than λ_0 enter on the same democratic footing as λ_0.) We choose a Z_7-subgroup $\langle A\rangle$ of $\mathrm{GL}(V_3)$, and suppose (for the sake of definiteness) that the generator A of this Singer cyclic subgroup satisfies $A^3 = A + I$. The commutant $[A]$ of A is a field $\mathbb{F} = \{0\} \cup \{I, A, \ldots, A^6\}$ isomorphic to $\mathrm{GF}(8)$. Let F_{21} denote the normalizer of $\langle A\rangle$ in $\mathrm{GL}(V_3)$ and let $\langle C\rangle$ be one out of the seven Z_3-subgroups of F_{21}. Then $F_{21} = \langle A\rangle \rtimes \langle C\rangle$ and (after replacing C by C^{-1} if necessary) C satisfies $CAC^{-1} = A^2$, and indeed $CFC^{-1} = F^2$ for all $F \in \mathbb{F}$.

Now there is a unique line $\nu_0 \subset \alpha$ which is stabilized by C, and we can choose $u \in \nu_0$ so that

$$\nu_0 = \{u, Au, A^3u\} \qquad \text{and} \qquad Cu = Au. \tag{6.1}$$

Observe that from $Cu = Au$ and $CAC^{-1} = A^2$ it follows that $CA^ru = A^{2r+1}u$. Let us also make a choice of linear isomorphism M mapping V_2 onto the 2-dimensional vector space $\{0\} \cup \nu_0$. For the sake of definiteness, put $\lambda_0 = \{a, b, c\}$ and set $Ma = u$, $Mb = Au$ and $Mc = A^3u$. If we define $C_2 \in \mathrm{GL}(V_2)$ by $C_2 : a \mapsto b \mapsto c \mapsto a$, so as to mirror $C : u \mapsto Au \mapsto A^3u \mapsto u$, we thereby arrange for M to intertwine C_2 with (the restriction to $\{0\} \cup \nu_0$ of) C:

$$MC_2 = CM. \tag{6.2}$$

A general vector $v \in V_5 = V_2 \oplus V_3$ will be written $v = x + y$, with $x \in V_2$ and $y \in V_3$. For $F \in \mathbb{F}$ we define the 2-dimensional subspace $V_F \subset V_5$ by $V_F = \{x + FMx : x \in V_2\}$ and we denote by $\lambda_F \subset \mathrm{PG}(4,2)$ the corresponding projective line:

$$\lambda_F = \{x + FMx : x \in \lambda_0\}; \tag{6.3}$$

in particular, choosing $F = 0$, note that λ_0 is one of the eight lines thus defined.

Theorem 6.1 *(i) The set of eight lines*

$$\mathcal{S}_8 = \{\lambda_F : F \in \mathbb{F}\} \tag{6.4}$$

is a partial spread of type O*; moreover* $\psi(\mathcal{S}_8)^c$ *is the plane* $\alpha = \mathbf{P}V_3$.

(ii) If $\mathcal{S}_7 = \mathcal{S}_8 \diagdown \{\lambda_F\}$, $F \in \mathbb{F}$, *then* $\mathcal{S}_7$ *is of type* O *and* $\mathcal{G}(\mathcal{S}_7) < \mathcal{G}(\mathcal{S}_8)$.

(iii) If $\mathcal{S}_9 = \mathcal{S}_8 \cup \{\nu\}$, $\nu \subset \alpha$, *then* $\mathcal{S}_9$ *is of type* X *and* $\mathcal{G}(\mathcal{S}_9) < \mathcal{G}(\mathcal{S}_8)$.

Proof. (i) Suppose $v = x + y \in V_F \cap V_{F'}$, with $F \neq F'$. Then $FMx = F'Mx$, so $(F - F')Mx = 0$. But $\mathbb{F}$ is a field, and so $Mx = 0$, whence $x = 0$. So $V_F \cap V_{F'} = \{0\}$, that is $\lambda_F \cap \lambda_{F'} = \emptyset$, and $\mathcal{S}_8$ is indeed a partial spread of size 8. Since $\lambda_F \cap \alpha = \emptyset$, and since $|\psi(\mathcal{S}_8)^c| = 7$, it follows that $\psi(\mathcal{S}_8)^c$ is precisely α.

Concerning the absence of reguli in $\mathcal{S}_8$, consider the hyperplane σ generated by a pair $\lambda_{F'}, \lambda_{F''}$ of lines of $\mathcal{S}_8$. Now the $7 + (3 + 3 + \ldots + 3)$ partition $\mathrm{PG}(4,2) = \alpha \cup_{F\in\mathbb{F}} \lambda_F$ of the $31 (= 7 + 8 \times 3)$ points of $\mathrm{PG}(4,2)$ induces a corresponding partition of the 15 points of σ, from which we obtain

$$(15 =)\ |\sigma| = |\alpha \cap \sigma| + \textstyle\sum_{F\in\mathbb{F}} |\lambda_F \cap \sigma|. \tag{6.5}$$

But a hyperplane in $\mathrm{PG}(4,2)$ meets a plane in at least the 3 points of a line, and meets a line λ_F in at least one point. Since $|\lambda_{F'} \cap \sigma| = 3 = |\lambda_{F''} \cap \sigma|$, it follows from (6.5) that $|\alpha \cap \sigma| = 3$ and $|\lambda_F \cap \sigma| = 1$ for each the 6 lines λ_F with $F \neq F'$, $F \neq F''$. Hence $N_8 = 0$, that is $\mathcal{S}_8$ is indeed of type O.

(ii) Since $\mathcal{S}_8$ is of type O, so also is $\mathcal{S}_7$. Now $\psi(\mathcal{S}_7)^c$ can be expressed as the disjoint union of a plane and a line in precisely one way, namely as $\alpha \cup \lambda_F$, and so any element $T \in \mathcal{G}(\mathcal{S}_7)$ must stabilize λ_F, whence $T \in \mathcal{G}(\mathcal{S}_8)$.

(iii) Since $N_9 = 4$, any extension $\mathcal{S}_9 = \mathcal{S}_8 \cup \{\nu\}$ of an $\mathcal{S}_8$ of type O must be of type X, with the line ν having valency 4. Any element $T \in \mathcal{G}(\mathcal{S}_9)$ must stabilize the eight lines of S_9 of valency 0, whence $T \in \mathcal{G}(\mathcal{S}_8)$. ■

We now wish to determine the structure of $\mathcal{G}(\mathcal{S}_8(\mathsf{O}))$. For $A, C \in \mathrm{GL}(V_3)$ and $C_2 \in \mathrm{GL}(V_2)$ as before the last theorem, define $T_A, U_C \in \mathrm{GL}(V_5)$ by

$$T_A = I_2 \oplus A, \qquad U_C = C_2 \oplus C, \tag{6.6}$$

and note that T_A and U_C stabilize the line λ_0 and permute the remaining seven lines λ_F, $F \neq 0$, of $\mathcal{S}_8$ amongst themselves in the manner:

$$T_A : \lambda_F \mapsto \lambda_{AF}, \qquad U_C : \lambda_F \mapsto \lambda_{F^2}. \tag{6.7}$$

To see the last result, put $C_2 x = x'$ and observe that $U_C(x + FMx) = C_2 x + CFMx = x' + F^2 M x'$, after using $CF = F^2 C$ and, see (6.2), $CM = MC_2$. If $\mathcal{S}_7 = \mathcal{S}_8 \diagdown \{\lambda_0\}$ note therefore that both T_A and U_C lie in $\mathcal{G}(\mathcal{S}_7)$ as well as in $\mathcal{G}(\mathcal{S}_8)$. Since also $U_C T_A (U_C)^{-1} = T_{CAC^{-1}} = T_{A^2}$, we see that via $A \mapsto T_A$ and $C \mapsto U_C$ we inject a copy of $F_{21} = \langle A \rangle \rtimes \langle C \rangle$ into $\mathcal{G}(\mathcal{S}_7)$ and into $\mathcal{G}(\mathcal{S}_8)$.

Further, for $F \in \mathbb{F}$, let us define $J_F \in \mathrm{GL}(V_5)$ by

$$J_F(x + y) = x + y + FMx, \tag{6.8}$$

and note the properties

$$\begin{aligned} J_F J_{F'} = J_{F+F'} = J_{F'} J_F, \qquad (J_F)^2 = I, \\ J_F : \lambda_{F'} \mapsto \lambda_{F+F'}. \end{aligned} \tag{6.9}$$

Thus $J : F \mapsto J_F$ maps the additive group of $\mathbb{F}$, isomorphic to that of a V_3, into a subgroup $\mathcal{J} < \mathcal{G}(\mathcal{S}_8)$, where $\mathcal{J} \cong (Z_2)^3$. Note also the properties

$$T_A J_F (T_A)^{-1} = J_{AF}, \qquad U_C J_F (U_C)^{-1} = J_{F^2}. \tag{6.10}$$

In particular observe that with the aid of T_A we may generate all the seven involutions $J_F, F \neq 0$, from just one, for example from J_I.

Theorem 6.2 *(i) The stabilizer of* $\mathcal{S}_8(\mathsf{O})$ *is of order 168, with structure*

$$\mathcal{G}(\mathcal{S}_8) \cong (Z_2)^3 \rtimes F_{21}. \tag{6.11}$$

(ii) For any $F \in \mathbb{F}$ *the stabilizer* $\mathcal{G}(\mathcal{S}_7)$ *of the partial spread* $\mathcal{S}_7(\mathsf{O}) = \mathcal{S}_8(\mathsf{O}) \diagdown \{\lambda_F\}$ *is a subgroup of* $\mathcal{G}(\mathcal{S}_8)$ *isomorphic to* F_{21}*. Up to equivalence every* $\mathcal{S}_7$ *of type* O *is of this kind, and so is projectively unique.*

(iii) For any line $\nu \subset \alpha$ *the stabilizer* $\mathcal{G}(\mathcal{S}_9)$ *of the partial spread* $\mathcal{S}_9(\mathsf{X}) = \mathcal{S}_8(\mathsf{O}) \cup \{\nu\}$ *is a subgroup of* $\mathcal{G}(\mathcal{S}_8)$ *isomorphic to* $\mathrm{Alt}(4) \times Z_2$.

Proof. (i) From the lead-in to the theorem it follows that the subgroup $\langle T_A, U_C, J_I\rangle < \mathcal{G}(\mathcal{S}_8)$ has the structure $(Z_2)^3 \rtimes F_{21}$ and is of order $8 \times 21 = 168$. Since, see before theorem 5.3, $|\mathcal{G}(\mathcal{S}_8)| = 168$, we have

$$\mathcal{G}(\mathcal{S}_8) = \langle J_I, T_A, U_C\rangle \cong (Z_2)^3 \rtimes F_{21}. \tag{6.12}$$

(ii) Recall that in the case of $\mathcal{S}_7 = \mathcal{S}_8 \diagdown \{\lambda_0\}$ we already know that $\mathcal{G}(\mathcal{S}_7)$ contains a subgroup $\langle T_A, U_C\rangle \cong F_{21} = \langle A\rangle \rtimes \langle C\rangle$. But, see (6.9), $J_F(\mathcal{S}_8 \diagdown \{\lambda_0\}) = \mathcal{S}_8 \diagdown \{\lambda_F\}$ and so, for any $F \in \mathbb{F}$, $\mathcal{G}(\mathcal{S}_8 \diagdown \{\lambda_F\})$ contains a subgroup $\cong F_{21}$. But since $\mathcal{J}$, and therefore $\mathcal{G}(\mathcal{S}_8)$, is transitive on the 8 lines of $\mathcal{S}_8$, and since $\mathcal{G}(\mathcal{S}_7) < \mathcal{G}(\mathcal{S}_8)$, see theorem 6.1(ii), we have $|\mathcal{G}(\mathcal{S}_7)| = |\mathcal{G}(\mathcal{S}_8)| \div 8 = 21$ for any partial spread $\mathcal{S}_7 = \mathcal{S}_8(\mathsf{O}) \diagdown \{\lambda_F\}$. Hence $\mathcal{G}(\mathcal{S}_7) \cong F_{21}$.

Any $\mathcal{S}_7$ of type O (indeed any non-maximal $\mathcal{S}_7$) can be obtained from an $\mathcal{S}_9$ by deleting two lines. But recall that an $\mathcal{S}_9$ is of type X, E or $\mathsf{I\Delta}$; so to obtain an $\mathcal{S}_7(\mathsf{O})$ from an $\mathcal{S}_9$ we need to start out from an $\mathcal{S}_9(\mathsf{X})$, one deleted line being the line of valency 4. The $\mathcal{S}_7(\mathsf{O})$ thus obtained is of the form $\mathcal{S}_7 = \mathcal{S}_8(\mathsf{O}) \diagdown \{\lambda\}$. But an $\mathcal{S}_8(\mathsf{O})$ is projectively unique and, as just noted, its stabilizer $\mathcal{G}(\mathcal{S}_8)$ is transitive on the lines $\lambda \in \mathcal{S}_8(\mathsf{O})$. Hence an $\mathcal{S}_7(\mathsf{O})$ is projectively unique.

(iii) The group $\mathcal{G}(\mathcal{S}_8)$, indeed any Z_7 subgroup of $\mathcal{G}(\mathcal{S}_8)$, is transitive on the set of seven extensions $\mathcal{S}_8(\mathsf{O}) \cup \{\nu\}$, $\nu \subset \alpha$, of $\mathcal{S}_8$. Thus the subgroup $\mathcal{G}(\mathcal{S}_9)$ of $\mathcal{G}(\mathcal{S}_8)$, see theorem 6.1(iii), which stabilizes an extension has order $168/7 = 24$. Also it suffices to consider the particular extension $\mathcal{S}_9 = \mathcal{S}_8 \cup \{\nu_0\}$, where, see (6.1), ν_0 is that line of α which is stabilized by our chosen $C \in F_{21}$. Now $\mathcal{G}(\mathcal{S}_9)$ contains $\mathcal{J} \cong (Z_2)^3$ as a subgroup, since each of the involutions J_F fixes the plane α, and in particular ν_0, pointwise. But also $\mathcal{G}(\mathcal{S}_9)$ contains $\langle U_C \rangle \cong Z_3$, since C stabilizes ν_0. Now from (6.10) we see that J_F commutes with U_C if and only if $F = 0$ or $F = I$. It follows that

$$\mathcal{G}(\mathcal{S}_9) \cong ((Z_2)^2 \rtimes Z_3) \times Z_2 \cong \mathrm{Alt}(4) \times Z_2, \tag{6.13}$$

where, in this case of $\mathcal{S}_9 = \mathcal{S}_8 \cup \{\nu_0\}$, the central Z_2 is generated by J_I. ∎

Remark 6.3 *The constructions in this section using (6.4) can be tied in with spreads and partial spreads of planes in* $\mathrm{PG}(5,2)$, *see [24, section 3.2.3] and especially [22, section 4.2], — by taking a* $\mathrm{PG}(4,2)$ *section* π *of a 2-spread* $\Sigma_9 = \{\alpha_0, \ldots, \alpha_7, \alpha\}$ *in* $\mathrm{PG}(5,2)$ *such that* $\alpha \subset \pi$. *(One of the* Z_7 *subgroups in [22, section 4.2] is not relevant in the* $\mathrm{PG}(4,2)$ *set-up, since it does not preserve the set of lines* $\{\lambda_i = \pi \cap \alpha_i\}$. *But the* $\mathcal{G}_1$ *and* $\Gamma\mathcal{G}$ *groups in [22, eqs. (24), (10)] can be tied in with the groups* $\mathcal{G}(\mathcal{S}_8)$, $\mathcal{G}(\mathcal{S}_9)$ *in (6.11), (6.13) above.)*

6.2 Partial spreads $\mathcal{S}_6$ of type O

The group $\mathcal{G}_8 = \mathcal{G}(\mathcal{S}_8)$ in (6.11), for an $\mathcal{S}_8$ of type O, is transitive on the lines of $\mathcal{S}_8$, and the subgroup $\mathcal{G}_7$ in theorem 6.1(ii) which fixes a line of $\mathcal{S}_8$ is transitive on the lines of $\mathcal{S}_7$. Hence $\mathcal{G}_8$ is 2-transitive on $\mathcal{S}_8$. Since there exist unique classes of partial spreads $\mathcal{S}_7$ and $\mathcal{S}_8$ of type O it follows that those regulus-free partial spreads $\mathcal{S}_6$ *which possess an extension to a regulus-free* S_7 form a single class, say VIa.1, whose members are of the form $\mathcal{S}_6 = \mathcal{S}_8(\mathsf{O}) \diagdown \{\lambda, \lambda'\}$ for some unique $\mathcal{S}_8(\mathsf{O})$. Note that $\mathcal{G}_6 = \mathcal{G}(\mathcal{S}_6)$ is a subgroup of $\mathcal{G}_8$, and let $\mathcal{G}_*$ be that subgroup of $\mathcal{G}_8$ (and of $\mathcal{G}_6$) which fixes separately the lines λ, λ'. By 2-transitivity we have $|\mathcal{G}_8| = 8.7.|\mathcal{G}_*|$, whence $|\mathcal{G}_*| = 3$. But there exists an involution $J \in \mathcal{G}_8$ which effects the interchange $\lambda \rightleftarrows \lambda'$, whence $|\mathcal{G}_6| = 6$. In fact $\mathcal{G}_6 \cong Z_6$. To see this, consider $\mathcal{S}_6 = \mathcal{S}_8(\mathsf{O}) \diagdown \{\lambda_0, \lambda_I\}$ with $\mathcal{S}_8$ as in (6.4). Then, see (6.7), $\mathcal{G}_* = \langle U_C \rangle \cong Z_3$ and, see (6.9), J_I effects $\lambda_0 \rightleftarrows \lambda_I$. Moreover, see (6.10), U_C commutes with J_I. Hence

$$\mathcal{G}(\mathcal{S}_6) = \langle U_C, J_I \rangle \cong Z_6. \tag{6.14}$$

(Alternatively, observe that $\mathcal{S}_6$ is also of the form $\mathcal{S}_6 = \mathcal{S}_7(\mathsf{Y}) \diagdown \{\nu\}$, where ν is the line of $\mathcal{S}_7(\mathsf{Y})$ of valency 3; hence $\mathcal{G}(\mathcal{S}_6) = \mathcal{G}(\mathcal{S}_7(\mathsf{Y})) \cong Z_6$, see section 4.4.)

Now $\mathcal{G}(\mathcal{S}_6)$ is seen to be transitive on the six lines $\in \mathcal{S}_6$, and hence *the partial spread* $\mathcal{S}_6$ *is cyclic,* of the form

$$\mathcal{S}_6 = \{\lambda_1, \lambda_2, \ldots, \lambda_6\}, \qquad \text{with } \lambda_{i+1} = A^i \lambda_1, \tag{6.15}$$

for any choice of line $\lambda_1 \in \mathcal{S}_6$ and generator A of $\mathcal{G}(\mathcal{S}_6)$. There are two classes, 6A and 6B (see [4, p.70]), of elements of order 6 in $\mathrm{GL}(5,2)$. Since an element $A \in$ class 6A only has two 6-cycles in its action upon $\mathrm{PG}(4,2)$, see [9, Table 4], it follows that A in (6.15) must belong to class 6B. If in (6.15) we choose $A : e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto e_5 \mapsto u \mapsto e_1$ and $\lambda_1 = \{1, 34, 134\}$ we obtain the following explicit example of a cyclic $\mathcal{S}_6(\mathsf{O}) = \{\lambda_1, \ldots, \lambda_6\}$:

$$\begin{aligned} &\lambda_1 = \{1, 34, 134\},\ \lambda_2 = \{2, 45, 245\},\ \lambda_3 = \{3, 5u, 35u = 124\}, \\ &\lambda_4 = \{4, u1, 4u1 = 235\},\ \lambda_5 = \{5, 12, 512\},\ \lambda_6 = \{u, 23, u23 = 145\}. \end{aligned} \tag{6.16}$$

Here $\psi = \psi(\mathcal{S}_6)$ has for its complement ψ^c the disjoint union of a plane α and two lines λ, λ' :

$$\psi^c = \alpha \cup \lambda \cup \lambda', \quad \text{where } \alpha = \prec 14, 25, 135 \succ, \ \lambda = \{35, 13, 15\}, \ \lambda' = \{2u, 4u, 24\}, \quad (6.17)$$

and $\mathcal{S}_6 \cup \{\lambda, \lambda'\}$ is an $\mathcal{S}_8(\mathsf{O})$. Under the action of $\mathcal{G}_6 \cong Z_6$ the 20 transversals τ_{ijk} of $\mathcal{S}_6(\mathsf{O}) = \{\lambda_1, \dots, \lambda_6\}$ in (6.16) fall into four orbits of lengths $6, 6, 6$ and 2, with respective representatives $\tau_{123} = \{134, 2, 5u\}$, $\tau_{125} = \{1, 2, 12\}$, $\tau_{124} = \{34, 245, 235\}$ and $\tau_{135} = \{34, 5u, 12\}$. So the points $1, 34, 134$ of λ_1 lie on $3, 4, 3$ transversals, respectively, and hence the profile of $\mathcal{S}_6$ is $(4, 5, 4)^6$.

To justify our assignment of $\mathcal{S}_6 = \mathcal{S}_8(\mathsf{O}) \setminus \{\lambda, \lambda'\}$ to class VIa.1 we need to show that $\psi(\mathcal{S}_6)$ belongs to the orbit VIa in table B.1. To this end let us compute the signature of the disjoint union $\alpha \cup \lambda \cup \lambda'$ in (6.17), where α is the plane $\psi(\mathcal{S}_8)^c$. Let the solid $\langle \lambda, \lambda' \rangle$ meet α in the line ν, and so $\rho = \{\nu, \lambda, \lambda'\}$ is a regulus. Then the 13-set $\alpha \cup \lambda \cup \lambda'$ has 12 lines, namely λ, λ', the seven lines of α and the three lines of ρ. The signature of $\psi(\mathcal{S}_6)^c$ is then easily seen to be $(13, 12, 1)(6^2 4^3 3^4)(6^0 7^1)$, with the 6 points of $\lambda \cup \lambda'$ contributing the 6^2 and the 3 points of ν contributing the 3^4. Hence the orbit of $\psi(\mathcal{S}_6)$ is indeed VIa.

Theorem 6.4 *(i) Those partial spreads $\mathcal{S}_6$ of type O which possess extensions to an $\mathcal{S}_7(\mathsf{O})$ form a single class, listed as VIa.1 in table B.2a. A partial spread $\mathcal{S}_6 \in$ VIa.1 is cyclic, with stabilizer $\mathcal{G}(\mathcal{S}_6) \cong Z_6$ and profile $(445)^6$.*

(ii) If $\mathcal{S}_6(\mathsf{O}) \in$ class VIa.1 then each $\mathcal{S}_5 = \mathcal{S}_6(\mathsf{O}) \setminus \{\lambda\}$ is of class Vd.1.

Proof. (i) See the lead-in to the theorem.

(ii) Consider $\mathcal{S}_5 = \mathcal{S}_6(\mathsf{O}) \setminus \{\lambda\}$ with $\mathcal{S}_6(\mathsf{O})$ as in (6.16). Because of the cyclic symmetry we may as well take $\lambda = \lambda_6$. The 10 transversals of $\mathcal{S}_5$ are a subset of the 20 transversals of $\mathcal{S}_6$ considered above, see after (6.17), and we see that the profile of $\mathcal{S}_5$ is $(234)^3(333)^2$, with λ_2 and λ_3 being balanced and λ_1, λ_4 and λ_5 being $(2, 3, 4)$-unbalanced. Hence, see table B.2a, $\mathcal{S}_5 \in$ class Vd.1. ∎

We now construct an example of a partial spread $\mathcal{S}_6$ of type O which does *not* possess any extensions to an $\mathcal{S}_7(\mathsf{O})$. Choose an $\mathcal{S}_5 \in$ class Vb.1; so $\mathcal{S}_5$ is uniquely of the form $\mathcal{S}_5 = \mathcal{S}_4 \cup \{\lambda_5\}$, with profile $(333)^4(135)$, where $\mathcal{S}_4 = \mathcal{S}_4(\mathsf{O})$ is cyclic and contributes the $(333)^4$. Let n be the nucleus of the parabolic quadric $\mathcal{P}_4$ associated with $\mathcal{S}_4$, and let λ be that line on $\mathcal{P}_4$ such that $\mathcal{S}_4 \cup \{\lambda\}$ is a spread on $\mathcal{P}_4$. Since $\mathcal{S}_5 \in$ class Vb.1, λ_5 is a nuclear tangent to $\mathcal{P}_4$, of the form $\lambda_5 = \{n, p, p + n\}$, for some $p \in \lambda$, with the points $n, p, p+n$ contributing respectively $1, 3, 5$ to the profile (135) of λ_5. Consider an extension $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\lambda_6\}$ of $\mathcal{S}_5$. There are 20 choices for λ_6, but under the action of $\mathcal{G}(\mathcal{S}_5) \cong D_8$ only one orbit Ω, of length 4, gives rise to a regulus-free $\mathcal{S}_6$. If $\lambda_6 \in \Omega$ then $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\lambda_6\}$ is of type O but, by theorem 6.4(ii), it does not belong to the class VIa.1.

In fact $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\lambda_6\} = \mathcal{S}_4 \cup \{\lambda_5, \lambda_6\}$ is easily seen to have profile $(445)^4(355)^2$, with $\{\lambda_5, \lambda_6\}$ contributing the $(355)^2$. So any $T \in \mathcal{G}(\mathcal{S}_6)$ must stabilize separately $\mathcal{S}_4$ and $\{\lambda_5, \lambda_6\}$. Since $|\mathcal{G}(\mathcal{S}_5)|/|\Omega| = 2$, there is an involution $J \in \mathcal{G}(\mathcal{S}_5)$ which fixes λ_6, and so $\mathcal{G}(\mathcal{S}_6)$ contains a subgroup $\langle J \rangle \cong Z_2$. A straightforward check in fact shows that $\mathcal{G}(\mathcal{S}_6) = \langle J \rangle \cong Z_2$. (To carry out this check it helps to use the fact that $T \in \mathcal{G}(\mathcal{S}_6)$ must permute amongst themselves the four points, one on each line of $\mathcal{S}_4$, responsible for the 5 in the $(445)^4$, and also permute the two points, one on each of λ_5 and λ_6, responsible for the 3 in the $(355)^2$.)

The $\mathcal{S}_6(\mathsf{O})$ just constructed is listed under class VIb.1 in table B.2a. Let us outline a proof that *the two classes VIa.1 and VIb.1 include all the regulus-free partial spreads of size 6.* Now any $\mathcal{S}_6$ of type O arises as an extension $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\lambda_6\}$ where $\mathcal{S}_5$ is regulus-free. So we need to consider the five cases (a)-(e) which arise from the five choices Va.1-Ve.1 of class for the regulus-free $\mathcal{S}_5$. In fact only the four cases (b)-(e) need to be considered, since an $\mathcal{S}_5$ of class Va.1 has no regulus-free extensions. The proof now proceeds in two steps:

Step 1. In the cases (b), (c) and (e) one finds that there is only one $\mathcal{G}(\mathcal{S}_5)$-orbit for λ_6 which gives rise to a regulus-free $\mathcal{S}_6$, and that in each of these cases $\mathcal{S}_6$ has profile $(445)^4(355)^2$, whence $\mathcal{S}_6$ is not of class VIa.1. In case (d), where $\mathcal{S}_5$ belongs to class Vd.1, then one finds there exist two $\mathcal{G}(\mathcal{S}_5)$-orbits for λ_6 which yield a regulus-free $\mathcal{S}_6$. For one of these $\mathcal{S}_6$ is of class VIa.1, in conformity with theorem 6.4(ii), while for λ_6 on the other orbit $\mathcal{S}_6$ is not of class VIa.1 since its profile is found to be $(445)^4(355)^2$.

Step 2. Thus there are at most four regulus-free classes of $\mathcal{S}_6$ other than the class VIa.1, the profile in each case (b)-(e) being $(445)^4(355)^2$. The proof is completed by checking for one, *and hence for all four*, of the cases (b)-(e) that $\mathcal{S}_6$ contains partial spreads $\mathcal{S}_5(\mathsf{O})$ belonging to each of the classes Vb.1-Ve.1. This check is straightforward but rather tedious. One finds that the four classes for $\mathcal{S}_5(\mathsf{O}) = \mathcal{S}_6 \diagdown \{\lambda\}$ arise from the four choices of $\mathcal{G}(\mathcal{S}_6)$-orbit for $\lambda \in \mathcal{S}_6$, of lengths 2, 2, 1 and 1. (The check is perhaps most easily done by starting out from $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\lambda_6\}$ with $\mathcal{S}_5 \in$ class Ve.1, since one may then use the cyclic symmetry of $\mathcal{S}_5$ to quickly display 10 out of the 20 transversals of $\mathcal{S}_6$.)

Theorem 6.5 *There are just two equivalence classes VIa.1 and VIb.1 of regulus-free partial spreads of size 6 in* $\mathrm{PG}(4,2)$, *where the class VIa.1 is as considered in theorem 6.4. A* $\mathcal{S}_6$ *of class VIb.1 has stabilizer* $\mathcal{G}(\mathcal{S}_6) \cong Z_2$ *and profile* $(445)^4(355)^2$, *and does not extend to an* $\mathcal{S}_7(\mathsf{O})$*; moreover an* $\mathcal{S}_6 \in$ *class VIb.1 can be obtained as an extension of an* $\mathcal{S}_5$ *belonging to any of the four classes Vb.1-Ve.1.* ■

7 The remaining classes of partial spreads

In this section we deal with the remaining classes of partial spreads $\mathcal{S}_r$, $r > 5$, namely those of types $\binom{4}{3}$, II and I.

7.1 The three maximal partial spreads of size 7

We already know, see corollary 4.8, that a partial spread $\mathcal{S}_7$ is maximal if and only if it is of type $\binom{4}{3}$, that is if and only if it is of the form

$$\mathcal{S}_7 = \mathcal{S}_4(\tbinom{4}{3}) \cup \mathcal{S}_3(\mathsf{O}). \tag{7.1}$$

We now show that the maximal partial spreads (7.1) of size 7 fall into three different GL(5,2) orbits. Given $\mathcal{S}_7 = \mathcal{S}_4(\binom{4}{3}) \cup \mathcal{S}_3(\mathsf{O})$, with $\mathcal{S}_4(\binom{4}{3}) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\mathcal{S}_3(\mathsf{O}) = \{\mu_1, \mu_2, \mu_3\}$, let σ be the hyperplane containing the lines λ_i and let $\tau(\subset \sigma)$ be the transversal of the lines μ_i. Hence $\Sigma_5 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \tau\}$ is a spread for σ, and σ is the even hyperplane determined by $\mathcal{S}_3(\mathsf{O})$. So σ comes equipped with a distinguished null polarity, see sections 3.2 and A.2.1. By lemma A.4 the 56 spreads of lines in PG(3,2) fall into three Sp(4,2)-orbits (of lengths 6, 20 and 30) consisting of spreads of the respective polar types $(5,0)$, $(3,2)$ and $(1,4)$. Since τ is self-polar, the partial spread $\mathcal{S}_4 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ for σ is of polar type $(4,0)$, $(2,2)$ or $(0,4)$. These three possible polar types for $\mathcal{S}_4$ lead to three different orbits for a maximal $\mathcal{S}_7$. We denote a partial spread in (7.1) by $\mathcal{S}_7(\binom{4}{3}_{(4,0)})$, $\mathcal{S}_7(\binom{4}{3}_{(2,2)})$ or $\mathcal{S}_7(\binom{4}{3}_{(0,4)})$ according as $\mathcal{S}_4(\binom{4}{3}) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ contains four, two or no self-polar lines for the null polarity for σ determined by $\mathcal{S}_3(\mathsf{O})$.

If $\mathcal{S}_7 = \mathcal{S}_4 \cup \mathcal{S}_3$ is of the form (7.1) then clearly $\mathcal{G}(\mathcal{S}_7)$ is a subgroup of the group $\mathcal{G}(\mathcal{S}_3) \cong \mathrm{Sym}(4) \times Z_2$ considered in lemma 3.1, with the central Z_2 being generated by the involution $J = J(\mathcal{S}_3)$. Now, see eqs. (A.3)-(A.11), $\tau = \lambda(s_{23})$ belongs to precisely
(i) two spreads, $\Sigma_5(T_2)$ and $\Sigma_5(T_3)$, of polar type $(5,0)$,
(ii) two spreads, see eq. (A.11), of polar type $(1,4)$, and
(iii) four spreads, $\Sigma_5(23k), k = 1, 4, 5, u$, of polar type $(3,2)$.
According as $\Sigma_5 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \tau\}$ belongs to case (i), (ii) or (iii) then $\mathcal{S}_7$ is of type $\binom{4}{3}_{(4,0)}$, $\binom{4}{3}_{(0,4)}$ or $\binom{4}{3}_{(2,2)}$. Since in each case $\mathcal{G}(\mathcal{S}_3)$ acts transitively, the index of $\mathcal{G}(\mathcal{S}_7)$ in $\mathcal{G}(\mathcal{S}_3)$ is accordingly (i) 2, (ii) 2, and (iii) 4. In cases (i) and (ii), the involution J effects the interchange of the two spreads; so, since $J \notin \mathcal{G}(\mathcal{S}_7)$, we must have $\mathcal{G}(\mathcal{S}_7) \cong \mathrm{Sym}(4)$. In fact one sees that

$$\mathcal{G}(\mathcal{S}_7) = \begin{cases} \mathcal{G}_0(\mathcal{S}_3)', & \text{if } \mathcal{S}_7 \text{ is of type } \binom{4}{3}_{(4,0)}, \\ \mathcal{G}_0(\mathcal{S}_3), & \text{if } \mathcal{S}_7 \text{ is of type } \binom{4}{3}_{(0,4)}, \end{cases}$$

where $\mathcal{G}_0(\mathcal{S}_3) \cong \mathrm{Sym}(4)$ and $\mathcal{G}_0(\mathcal{S}_3)' \cong \mathrm{Sym}(4)$ are as in remark 3.2. On the other hand in the case (iii), where $\mathcal{S}_7$ is of type $\binom{4}{3}_{(2,2)}$, we see that $J \in \mathcal{G}(\mathcal{S}_7)$, and hence that $\mathcal{G}(\mathcal{S}_7) \cong \mathrm{Sym}(3) \times Z_2$. (In this last case the Z_3 subgroup of $\mathcal{G}(\mathcal{S}_7)$ fixes each line of Σ_5.)

Theorem 7.1 *There are precisely three classes of maximal partial spreads $\mathcal{S}_7$, represented by $\mathcal{S}_7(\binom{4}{3}_{(4,0)})$, $\mathcal{S}_7(\binom{4}{3}_{(0,4)})$ and $\mathcal{S}_7(\binom{4}{3}_{(2,2)})$ as just described. These three classes are listed as VIIf.1, VIIf.2 and VIIf.3, respectively, in table B.2b, the stabilizer groups being isomorphic to* Sym(4), Sym(4) *and* $\mathrm{Sym}(3) \times Z_2$, *of respective orders 24, 24 and 12.* ∎

Suppose $\mathcal{S}_7 = \mathcal{S}_4(\binom{4}{3}) \cup \mathcal{S}_3(\mathsf{O}) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \cup \{\mu_1, \mu_2, \mu_3\}$ is as above, of type $\binom{4}{3}$. For fixed $\mathcal{S}_6 = \mathcal{S}_4 \cup \{\mu_1, \mu_2\}$ there are $(L' =)4$ choices for μ_3, leading therefore to four different Sp(4,2) geometries for the hyperplane σ. It is straightforward to check that $\mathcal{S}_4 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is of polar type $(4,0)$, $(0,4)$ and $(2,2)$ for respectively one, one and two of these choices. Since a partial spread $\mathcal{S}_6$ of type $\binom{4}{3}$ is necessarily of the kind $\mathcal{S}_6 = \mathcal{S}_7(\binom{4}{3}) \setminus \{\mu_3\}$, it follows that *there is just one class of partial spread S_6 of type $\binom{4}{3}$*. By considering either the case when $\mathcal{S}_7$ is of type $\binom{4}{3}_{(4,0)}$ or the case when $\mathcal{S}_7$ is of type

$\binom{4}{3}_{(0,4)}$, it also follows that $\mathcal{G}(\mathcal{S}_6)$ has index 3 in $\mathcal{G}(\mathcal{S}_7) \cong \mathrm{Sym}(4)$, and hence that $\mathcal{S}_6(\binom{4}{3})$ has stabilizer $\mathcal{G}(\mathcal{S}_6) \cong D_8$.

Lemma 7.2 *There is just one class, VIj.1 in table B.2b, of partial spread $\mathcal{S}_6$ of type $\binom{4}{3}$, with stabilizer isomorphic to D_8.* ■

7.2 Partial spreads of type II

7.2.1 The projectively unique $\mathcal{S}_6(\mathsf{II})$

Suppose $\mathcal{S}_6 = \rho \cup \rho'$ is of type II, with reguli $\rho = \{\lambda_1, \lambda_2, \lambda_3\}$ and $\rho' = \{\lambda_4, \lambda_5, \lambda_6\}$. Then, with respect to an appropriate basis, we may cast $\mathcal{S}_6$ into the form

$$\begin{aligned} &\lambda_1 = \{1, 25, 125\},\ \lambda_2 = \{3, 345, 45\},\ \lambda_3 = \{13, 234, 124\},\\ &\lambda_4 = \{2, 35, 235\},\ \lambda_5 = \{4, 145, 15\}\ \lambda_6 = \{24, 134, 123\}. \end{aligned} \tag{7.2}$$

Here the basis has been chosen such that the ambient hyperplanes $\sigma = \sigma_{123},\ \sigma' = \sigma_{456}$ of the two reguli $\rho = \rho_{123},\ \rho' = \rho_{456}$ are $\sigma = \langle 1, 25, 3, 45\rangle,\ \sigma' = \langle 15, 2, 35, 4\rangle$. The quadric $\mathcal{H} = \psi(\rho) = \lambda_1 \cup \lambda_2 \cup \lambda_3$ intersects the plane $\alpha = \sigma \cap \sigma'$ in the conic $\kappa = \{13, 125, 345\}$ and the quadric $\mathcal{H}' = \psi(\rho') = \lambda_4 \cup \lambda_5 \cup \lambda_6$ intersects the plane α in the conic $\kappa' = \{24, 145, 235\}$. Note that $\alpha = \kappa \cup \kappa' \cup \{5u\}$, the point $5u$ being privileged, since it is the only point of α not belonging to $\psi(\mathcal{S}_6)$. The pair of points $\{13, 24\}$, and hence also the pair of lines $\{\lambda_3, \lambda_6\}$, is distinguished, since 13 is the nucleus of the conic κ' and 24 is the nucleus of the conic κ. Also note that $\sigma = \mathcal{H} \cup \mu \cup \nu$ and $\sigma' = \mathcal{H}' \cup \mu' \cup \nu'$, where

$$\begin{aligned} \mu &= \{5u, 145, 235\}, &\qquad \nu &= \{24, 4u, 2u\},\\ \mu' &= \{5u, 125, 345\}, &\qquad \nu' &= \{13, 1u, 3u\}. \end{aligned}$$

Here $\mu,\ \mu'$ are the external lines of the conics $\kappa,\ \kappa'$, and the lines $\nu,\ \nu'$ intersect α in the nuclei 24, 13 of the conics $\kappa,\ \kappa'$.

Let A be that element of $\mathrm{GL}(5, 2)$, belonging in fact to class 4C, see [4], which is defined on the chosen basis by

$$A : 1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1,\ 5 \mapsto 5 \quad \text{(and hence } u \mapsto u\text{)}. \tag{7.3}$$

Then A effects the permutation $(\lambda_1\lambda_4\lambda_2\lambda_5)(\lambda_3\lambda_6)$ of the lines of $\mathcal{S}_6$, whence $\mathcal{G}(\mathcal{S}_6)$ contains $\langle A\rangle \cong Z_4$ as a subgroup. In fact $\mathcal{G}(\mathcal{S}_6)$ is no larger:

$$\mathcal{G}(\mathcal{S}_6) = \langle A\rangle \cong Z_4. \tag{7.4}$$

This can be seen as follows. If $T \in \mathcal{G}(\mathcal{S}_6)$ then T must fix the privileged point $5u$ and also the pair $\{24, 13\}$ of nuclei. If T fixes 13, and hence also 24, then T must fix $\tau_{123} = \{1, 3, 13\} \in \rho^{\mathrm{opp}}$ and $\tau_{456} = \{2, 4, 24\} \in (\rho')^{\mathrm{opp}}$ and so fixes the pairs $\{1, 3\}$ and $\{2, 4\}$.. If instead T effects $24 \rightleftarrows 13$, then T effects $\tau_{123} \rightleftarrows \tau_{456}$ and so interchanges $\{1, 3\}$ and $\{2, 4\}$. In the latter case we quickly see from (7.2) that either $T = A$ or $T = A^{-1} = A^3$, and in the former case $T = A^2$.

Partial spreads $\mathcal{S}_6$ of type II are classed as VIg.1 in table B.2a. Concerning the profile of $\mathcal{S}_6$ in (7.2), the distinguished lines λ_3 and λ_6 each contribute $(4,4,7)$, with the distinguished points 13 and 24 each contributing a 7. Each of the remaining four lines has profile $(4,5,6)$, with the remaining points $125, 345, 145, 235$ on the conics contributing a 6, and the points $1,3,2,4$ on the transversals through 13 and 24 contributing a 4. So $\text{profile}(\mathcal{S}_6(\mathsf{II})) = (4,4,7)^2(4,5,6)^4$.

7.2.2 The three classes of $\mathcal{S}_7$ of type II

Consider an extension $\mathcal{S}_7 = \mathcal{S}_6(\mathsf{II}) \cup \{\lambda\}$. Under the action of $\mathcal{G}(\mathcal{S}_6) \cong Z_4$, see (7.4), the $L' = 8$ choices for λ fall into four orbits Ω, Ω', Ω'' and Ω''', of lengths 4, 1, 1 and 2, with respective representatives

$$\lambda_7 = \{1u, 2u, 12\}, \qquad \lambda_7' = \{5u, u, 5\},$$
$$\lambda_7'' = \{5u, 135, 245\}, \qquad \lambda_7''' = \{5u, 12, 34\}.$$

The extension $\mathcal{S}_7''' = \mathcal{S}_6(\mathsf{II}) \cup \{\lambda'''\}$ is seen to be of type F, with $\{\lambda_3, \lambda_6, \lambda_7\}$ the extra regulus; it is projectively unique, of class VIIe.5, see theorem 4.7. The remaining three extensions $\mathcal{S}_7$, $\mathcal{S}_7'$, $\mathcal{S}_7''$, using $\lambda_7, \lambda_7', \lambda_7''$, are all of type II, and represent the three classes VIId.1, VIIc.1, VIIc.2, respectively, in table B.2b. For VIIc.1, VIIc.2, but not for VIId.1, the line λ of zero valency passes through the privileged point ($= 5u$, for $\mathcal{S}_6(\mathsf{II})$ as in the previous section) of $\mathcal{S}_6(\mathsf{II})$. For $\mathcal{S}_7 \in$ VIId.1 the line λ is balanced, with profile $(6,6,6)$, while for class VIIc.1 and class VIIc.2 the line λ is $(5,5,8)$-unbalanced. It seems quite hard to distinguish between the classes VIIc.1 and VIIc.2! One distinction is that for one of them, say VIIc.1, but not for the other, say VIIc.2, the line λ of zero valency is fixed *pointwise* by the stabilizer group (the latter being $\cong Z_4$, as in the next lemma).

Lemma 7.3 *Up to equivalence, there exist precisely three partial spreads $\mathcal{S}_7$ of regulus type II, listed as VIIc.1, VIIc.2 and VIId.1 in table B.2b. For VIIc.1 andVIIc.2, $\mathcal{G}(\mathcal{S}_7) \cong Z_4$, while for VIId.1, $\mathcal{G}(\mathcal{S}_7)$ is trivial.*

Proof. Since any $\mathcal{S}_7$ of type II is of the form $\mathcal{S}_6(\mathsf{II}) \cup \{\lambda\}$, and since there is only one class of $\mathcal{S}_6(\mathsf{II})$, we have already proved the first part of the lemma. Let classes VIId.1, VIIc.1 and VIIc.2 be represented by $\mathcal{S}_7$, $\mathcal{S}_7'$, and $\mathcal{S}_7''$, as in the lead-in to the lemma. Since $|\Omega| = 4$ the order of $\mathcal{G}(\mathcal{S}_7)$ is $|\mathcal{G}(\mathcal{S}_6)| \div 4 = 1$, and since $|\Omega'| = |\Omega''| = 1$, we have $\mathcal{G}(\mathcal{S}_7') = \mathcal{G}(\mathcal{S}_7'') = \mathcal{G}(\mathcal{S}_6) \cong Z_4$. ■

7.3 Partial spreads of type I

7.3.1 The seven classes of partial spreads $\mathcal{S}_6$ of type I

If $\mathcal{S}_6$ is of type I then it has a unique decomposition of the form

$$\mathcal{S}_6 = \mathcal{S}_3(\mathsf{I}) \cup \mathcal{S}_3(\mathsf{O}). \tag{7.5}$$

Let σ denote the ambient hyperplane of the regulus $\rho = \mathcal{S}_3(\mathsf{I})$, and so $\sigma = \mathcal{H} \cup \nu \cup \nu'$, where ν, ν' are the external lines of the quadric $\mathcal{H} = \psi(\rho) \subset \sigma$. Let τ denote the unique transversal of the three lines of $\mathcal{S}_3(\mathsf{O})$. Two cases arise

$$\text{Case (a): } \tau \subset \sigma; \qquad \text{Case (b): } \tau \text{ meets } \sigma \text{ in a point } p. \tag{7.6}$$

If case (a) holds then σ is the even hyperplane of $\mathcal{S}_3(\mathsf{O})$ and comes along equipped with a distinguished null polarity, see section 3.2. Four possibilities arise, according as the three lines of ρ are of polar type (see section A.2.2 of the appendix) $(3,0)$, $(2,1)$, $(1,2)$ or $(0,3)$. The corresponding four classes of $\mathcal{S}_6(\mathsf{I})$ are listed in table B.2a as VId.2, VIc.2, VId.1, and VIc.1, respectively, and representatives will be denoted $\mathcal{S}_6(\mathsf{I})_{(3,0)}$, $\mathcal{S}_6(\mathsf{I})_{(2,1)}, \mathcal{S}_6(\mathsf{I})_{(1,2)}$ and $\mathcal{S}_6(\mathsf{I})_{(0,3)}$. These four classes can just as well be arrived at by removing a line from the $\mathcal{S}_4(\binom{4}{3})$ in the maximal $\mathcal{S}_7$ in eq. (7.1) and theorem 7.1:

$$\begin{aligned} &\text{VIc.1}: \ \mathcal{S}_6(\mathsf{I})_{(0,3)} = \mathcal{S}_7(\tbinom{4}{3}_{(0,4)}) \setminus \{\lambda_1\}, \quad \text{VIc.2: } \mathcal{S}_6(\mathsf{I})_{(2,1)} = \mathcal{S}_7(\tbinom{4}{3}_{(2,2)}) \setminus \{\lambda^{\text{nonpolar}}\}, \\ &\text{VId.1}: \ \mathcal{S}_6(\mathsf{I})_{(1,2)} = \mathcal{S}_7(\tbinom{4}{3}_{(2,2)}) \setminus \{\lambda^{\text{self-polar}}\}, \quad \text{VId.2: } \mathcal{S}_6(\mathsf{I})_{(3,0)} = \mathcal{S}_7(\tbinom{4}{3}_{(4,0)}) \setminus \{\lambda_1\}. \end{aligned} \tag{7.7}$$

Incidentally, see lemma A.10, every extension of a parabolic $\mathcal{S}_5(\mathsf{O})$ is of class VIc.2.

Concerning the stabilizer groups of these four classes of $\mathcal{S}_6(\mathsf{I})$, recall the discussion of the stabilizers of maximal spreads $\mathcal{S}_7(\binom{4}{3})$ in the lead-in to theorem 7.1. First of all let us deal with classes VIc.2 and VId.1 in (7.7). If $\mathcal{S}_3(\mathsf{O})$ in (7.5) is as in eq. (3.2) then, cf. $\Sigma_5(234)$ in eq. (A.10), the spread Σ_5 in σ defined by

$$\begin{aligned} &\Sigma_5 = \{\tau, \lambda(s_{24}), \lambda(s_{34}), \nu, \nu'\} = \{\tau\} \cup \mathcal{S}_4(\tbinom{4}{3}), \qquad \text{where} \\ &\tau = \lambda(s_{23}) = \{12, 34, 5u\}, \ \nu = \lambda_{135} = \{13, 15, 35\}, \ \nu' = \lambda_{24u} = \{24, 2u, 4u\}, \end{aligned}$$

is of polar type $(3,2)$. So an $\mathcal{S}_6 = \rho \cup \mathcal{S}_3(\mathsf{O})$ of the kind $\mathcal{S}_6(\mathsf{I})_{(2,1)}$ may be realized by the choice $\rho = \{\lambda(s_{24}), \lambda(s_{34}), \nu'\}$, and of the kind $\mathcal{S}_6(\mathsf{I})_{(1,2)}$ by the choice $\rho = \{\lambda(s_{34}), \nu, \nu'\}$. Now the Z_3 subgroup of the stabilizer $\mathcal{G}(\mathcal{S}_7) \cong \text{Sym}(3) \times Z_2$ of $\mathcal{S}_7(\binom{4}{3}_{(2,2)}) = \mathcal{S}_4(\binom{4}{3}) \cup \mathcal{S}_3(\mathsf{O})$ is generated by $A_1 = (3,4,5,u,1)$, since A_1, as in eq. (3.6), not only stabilizes $\mathcal{S}_3(\mathsf{O})$ but also fixes each line of Σ_5. Moreover the element $L = (3,4,1,2,5)$ of $\mathcal{G}(\mathcal{S}_3(\mathsf{O}))$ fixes ν' and effects the interchange $\lambda(s_{24}) \rightleftarrows \lambda(s_{34})$, while $L' = (4,3,2,1,u)$ fixes $\lambda(s_{34})$ and effects the interchange $\nu \rightleftarrows \nu'$. So, since $J \notin \mathcal{G}(\mathcal{S}_6)$, the stabilizer of $\mathcal{S}_6(\mathsf{I})_{(2,1)}$ is seen to be $\langle A_1, L\rangle \cong \text{Sym}(3)$, and the stabilizer of $\mathcal{S}_6(\mathsf{I})_{(1,2)}$ is $\langle A_1, L'\rangle \cong \text{Sym}(3)$. The classes VIc.1 and VId.2 are dealt with more simply. The stabilizer $\mathcal{G}(\mathcal{S}_7) \cong \text{Sym}(4)$ of either $\mathcal{S}_7(\binom{4}{3}_{(4,0)})$ or $\mathcal{S}_7(\binom{4}{3}_{(0,4)})$ effects all permutations of the four lines of $\mathcal{S}_4(\binom{4}{3})$, and on removing one, say λ_1, of these lines the resulting $\mathcal{S}_6(\mathsf{I})_{(3,0)}$ or $\mathcal{S}_6(\mathsf{I})_{(0,3)}$ has stabilizer $\mathcal{G}(\mathcal{S}_6) \cong \text{Sym}(3)$ effecting all permutations of the three lines of the regulus $\rho = \mathcal{S}_4(\binom{4}{3}) \setminus \{\lambda_1\}$. *So for all four classes in (7.7) we have* $\mathcal{G}(\mathcal{S}_6) \cong \text{Sym}(3)$.

Suppose now that case (b) holds in (7.6). Let the three lines of $\mathcal{S}_3(\mathsf{O})$ intersect σ in points p_1, p_2 and p_3. Because τ does not lie inside σ, these points are not collinear; so we may as well label things so that $p_1 \in \nu$ and p_2, $p_3 \in \nu'$. Two subcases arise: (b$_1$): $p = p_1$

and (b_2): $p = p_2$ (say). Subcase (b_1) yields the class VIf.1 in table B.2a, with stabilizer $\mathcal{G}(\mathcal{S}_6) \cong Z_2$ which effects the interchange $p_2 \rightleftarrows p_3$. Subcase (b_2) yields the classes VIe.1 and VIe.2 in table B.2a. For either of these classes there are so many distinguished points that one finds that the stabilizer is trivial. The classes VIe.1 and VIe.2 are quite hard to distinguish. One distinction is as follows: in the labelling with $p = p_2$, let λ be the line of ρ which contains the point $p_1 + p_2$; then profile(λ) = $(5, 5, 5)$ for one class, say VIe.1, and profile(λ) = $(4, 5, 6)$ for the other class VIe.2.

Lemma 7.4 *There exist precisely seven equivalence classes of partial spreads $\mathcal{S}_6$ of regulus type* I*. Each of the four classes VIc.1, VIc.2, VId.1 and VId.2, see table B.2a, has stabilizer* $\mathcal{G}(\mathcal{S}_6) \cong \mathrm{Sym}(3)$. *For class VIf.1, $\mathcal{G}(\mathcal{S}_6) \cong Z_2$, while for both of the classes VIe.1 and VIe.2, $\mathcal{G}(\mathcal{S}_6)$ is trivial.* ∎

7.3.2 The three classes of partial spreads $\mathcal{S}_7$ of type I

Any non-maximal $\mathcal{S}_7$ can be expressed in the form $\mathcal{S}_7 = \mathcal{S}_9 \setminus \{\lambda, \lambda'\}$. In order for $\mathcal{S}_7$ to be of type I then $\mathcal{S}_9$ must be of type E or IΔ, with at least one of λ, λ' of valency 2; hence, if λ has valency 2, note that $\mathcal{S}_8 = \mathcal{S}_9 \setminus \{\lambda\}$ is of type II. But recall from section 5.2 that there is a unique class of partial spreads $\mathcal{S}_8(\mathsf{II})$. Hence any $\mathcal{S}_7 = \mathcal{S}_7(\mathsf{I})$ can be expressed in the form

$$\mathcal{S}_7 = \mathcal{S}_9(\mathsf{E}) \setminus \{\lambda, \lambda'\}, \qquad \text{for suitable } \lambda, \lambda' \in \mathcal{S}_9(\mathsf{E}). \tag{7.8}$$

Without loss of generality we may suppose that $\mathcal{S}_9(\mathsf{E})$ is as in theorem 5.1, with regulus pattern $R_{123}, R_{145}, R_{267}, R_{389}$. Then under the action of $\mathcal{G}(\mathcal{S}_9) = \langle E \rangle \cong Z_6$, where $E = (2u, 45, 5, 1u, 35)$ effects the permutation $(\lambda_1\lambda_3\lambda_2)\ (\lambda_4\lambda_8\lambda_7\lambda_5\lambda_9\lambda_6)$, the relevant pairs $\{\lambda, \lambda'\}$ of lines of $\mathcal{S}_9(\mathsf{E})$ fall into just three orbits Ω_1, Ω_2 and Ω_3, of lengths 3, 6 and 6, with representatives

$$\text{(i) } \{\lambda_1, \lambda_2\}, \quad \text{(ii) } \{\lambda_1, \lambda_9\}, \quad \text{(iii) } \{\lambda_1, \lambda_7\}.$$

We now show that the resulting three partial spreads $\mathcal{S}_7^{\mathrm{i}}$, $\mathcal{S}_7^{\mathrm{ii}}$ and $\mathcal{S}_7^{\mathrm{iii}}$ of type I are inequivalent, and so conclude that *there are precisely three classes of partial spreads S_7 of type* I. These are listed as VIIb.1, VIIb.2 and VIIb.3 in table B.2b.

For each of the choices (i), (ii) and (iii), $\mathcal{S}_7 = \mathcal{S}_7(\mathsf{I})$ is of the form $\mathcal{S}_3(\mathsf{I}) \cup \mathcal{S}_4(\mathsf{O})$; in each case let us denote by ν, ν' the two lines such that $\mathcal{S}_3(\mathsf{I}) \cup \{\nu, \nu'\}$ is a spread in the ambient hyperplane σ of the regulus $\mathcal{S}_3(\mathsf{I})$. In the case (i), $\mathcal{S}_7^{\mathrm{i}} = \mathcal{S}_9(\mathsf{E}) \setminus \{\lambda_1, \lambda_2\}$, we see that $\mathcal{S}_4(\mathsf{O})$ is cyclic and that its four lines meet the lines ν, ν' in a $2+2$ pattern. In contrast, for each of the choices (ii) and (iii) we find that $\mathcal{S}_4(\mathsf{O})$ is non-cyclic and that its four lines meet the lines ν, ν' in a $3+1$ pattern. So, for both choices (ii) and (iii), we have $\mathcal{S}_4(\mathsf{O}) = \mathcal{S}_3(\mathsf{O}) \cup \{\mu\}$, where the transversal τ of $\mathcal{S}_3(\mathsf{O})$ is (say) ν and the even hyperplane of $\mathcal{S}_3(\mathsf{O})$ is σ, and where μ, meeting ν' in a point, is one of the unbalanced members of $\mathcal{S}_4(\mathsf{O})$. Moreover $\mathcal{S}_7^{\mathrm{ii}}$ and $\mathcal{S}_7^{\mathrm{iii}}$ share the same profile $(5, 5, 8)(5, 6, 7)^2(6, 6, 6)(6, 7, 7)^3$, with $\mathcal{S}_3(\mathsf{I})$ contributing in each case the $(6, 7, 7)^3$. Nevertheless $\mathcal{S}_7^{\mathrm{ii}}$ and $\mathcal{S}_7^{\mathrm{iii}}$ are inequivalent. To see this let us, in each of the two cases, make use of the null polarity induced in the even hyperplane σ of $\mathcal{S}_3(\mathsf{O})$ to examine the polar type of the regulus $\mathcal{S}_3(\mathsf{I})$:

for $\mathcal{S}_7^{\text{ii}} = \mathcal{S}_9(\mathsf{E})\diagdown\{\lambda_1, \lambda_9\}$, the regulus ρ_{267} is of polar type $(1, 2)$;
for $\mathcal{S}_7^{\text{iii}} = \mathcal{S}_9(\mathsf{E})\diagdown\{\lambda_1, \lambda_7\}$, the regulus ρ_{389} is of polar type $(3, 0)$.

For each of the partial spreads $\mathcal{S}_7^{\text{i}}$, $\mathcal{S}_7^{\text{ii}}$ and $\mathcal{S}_7^{\text{iii}}$ one finds that the stabilizer is of order 2. In the case of $\mathcal{S}_7^{\text{i}} = \mathcal{S}_9(\mathsf{E})\diagdown\{\lambda_1, \lambda_2\}$, with $\mathcal{S}_9(\mathsf{E})$ is as in theorem 5.1, the stabilizer is $\langle E^3\rangle \cong Z_2$. A quick way to see that class VIIb.3 has stabilizer of order 2 is to represent the class by $\mathcal{S}_7 = \mathcal{S}_9(\mathsf{I}^\rho\Delta)\diagdown\{\lambda_1, \lambda_7\}$, with $\mathcal{S}_9(\mathsf{I}^\rho\Delta)$ as in theorem 4.5, when one sees that the stabilizer is $\langle K\rangle$, where $K = (2, 1, 34, 4, 5)$.

7.4 Conclusion

All classes of partial spreads of lines in $\mathrm{PG}(4, 2)$ have now been found, and are as listed in tables B.2a and B.2b. In particular there are precisely eight classes of maximal partial spreads, namely

$$\text{Vj.1}; \quad \text{VIIf.1, VIIf.2, VIIf.3}; \quad \text{IXa.1, IXa.2, IXa.3, IXa.4.}$$

A Appendix: Aspects of $\mathrm{Sym}(6)$, $\mathrm{Sp}(4, 2)$ and $\mathrm{O}(5, 2)$

In this appendix we treat the well-known group isomorphisms

$$\text{(i)}\ \mathrm{Sp}(4, 2) \cong \mathrm{Sym}(6), \qquad \text{(ii)}\ \mathrm{O}(5, 2) \cong \mathrm{Sym}(6), \tag{A.1}$$

stressing those features which are relevant to the main body of the paper.

A.1 Aspects of $\mathrm{Sym}(6)$

A.1.1 Synthemes, totals and near-totals

Consider the symmetric group $\mathrm{Sym}(6)$ on the six symbols $\{1, 2, 3, 4, 5, 6\}$, and let $ijklmn$ denote any permutation of 123456. From the six symbols we may form not only fifteen *duads* ij but also fifteen *synthemes* $ij\,kl\,mn$. Just as each syntheme contains three duads, each duad is contained in three synthemes. From the six symbols we may also form six *totals.* (A total is defined to be a set of five synthemes such that each duad occurs in one syntheme of the total — just as a symbol i could be interpreted as a pentad $\{ij : j \neq i\}$ of five duads such that each syntheme contains one of the duads of the pentad.) That there exist precisely six totals was noticed by Sylvester [28] in 1861, and he pointed out that any two totals overlap in precisely one syntheme (just as any two symbols are shared by one duad). The six totals are displayed in the following symmetric 6×6 array, the ith total T_i consisting of the five synthemes in the ith row, equally the ith column, of the array.

	T_1	T_2	T_3	T_4	T_5	T_6
T_1	–	15 23 46	14 35 26	13 24 56	12 45 36	25 34 16
T_2	15 23 46	–	12 34 56	14 25 36	24 35 16	13 45 26
T_3	14 35 26	12 34 56	–	23 45 16	13 25 46	15 24 36
T_4	13 24 56	14 25 36	23 45 16	–	15 34 26	12 35 46
T_5	12 45 36	24 35 16	13 25 46	15 34 26	–	14 23 56
T_6	25 34 16	13 45 26	15 24 36	12 35 46	14 23 56	–

(A.2)

Observe therefore that the synthemes can be given a duadic labelling, with $s_{ij}(= s_{ji})$ denoting that syntheme, in the ij entry of the array (A.2), which is contained in the two totals

$$T_i = \{s_{ik} : k \neq i\} \qquad \text{and} \qquad T_j = \{s_{jk} : k \neq j\}. \tag{A.3}$$

Remark A.1 *We ought to point out that the numbers* $1, 2, 3, 4, 5, 6$ *labelling the six totals can be assigned in a quite arbitrary manner. In the array (A.2) we chose to number the totals so as to satisfy* $i6 \in s_{i6}$, *for* $i = 1, 2, 3, 4, 5$. *In fact our array has the property that* $ab \in s_{cd}$ *if and only if* $cd \in s_{ab}$; *here* c, d *are not necessarily distinct from* a, b.

If from a total T_i we remove a syntheme s_{ij} the resulting set $T_{i,j} = T_i \diagdown \{s_{ij}\} = \{s_{ik} : k \neq i, k \neq j\}$ of four synthemes will be referred to as a *near-total.* If D denotes the set of 15 duads, consider the subset $D_{ij} = D \diagdown s_{ij}$ of 12 duads obtained by removing from D the three duads in the syntheme s_{ij}. Then observe that the 12-set $D_{ij} = D_{ji}$ can be partitioned as a near-total in precisely two ways, namely as $T_{i,j}$ and as $T_{j,i}$. Consequently the 12-set D_{ij} may be conveniently exhibited as a *double-four array*, with the four synthemes of $T_{i,j}$ in the four columns and the four synthemes of $T_{j,i}$ in the four rows. For example, in the numbering scheme (A.2) for which $s_{56} = \{14, 23, 56\}$, we may exhibit D_{56} as the double-four array

$$D_{56} = \begin{pmatrix} - & 16 & 25 & 34 \\ 45 & - & 13 & 26 \\ 36 & 24 & - & 15 \\ 12 & 35 & 46 & - \end{pmatrix}. \tag{A.4}$$

A.1.2 An outer automorphism θ of Sym(6)

As is well-known, the group Sym(6), alone amongst the symmetric groups Sym(n), possesses outer automorphisms. There are 6! of these, and they map, see lemma A.2, proof (i), one class of six Sym(5) subgroups of Sym(6) on to another class of six Sym(5) subgroups. (The existence of outer automorphisms lies behind the duality, hinted at in the opening paragraph of section A.1.1, between on the one hand the 6 symbols/pentads and 15 duads, and on the other the 6 totals and 15 synthemes: in particular, see (A.6) below, outer automorphisms map the class of 15 involutions of the kind (ij) to the class of 15 involutions of the kind $(ij)(kl)(mn)$.) A particular outer automorphism θ, *arising from our particular numbering of the totals in (A.2),* is obtained as follows. Each permutation

$\pi \in \mathrm{Sym}(6)$ acts, via $ij \mapsto \pi(i)\pi(j)$, on the fifteen duads, hence on the fifteen synthemes, and hence induces a permutation of the 6 totals T_1, T_2, ... T_6. Consequently there exists an automorphism θ of $\mathrm{Sym}(6)$ which maps $\pi \in \mathrm{Sym}(6)$ onto $\theta(\pi) = \rho$, where $\rho \in \mathrm{Sym}(6)$ is that permutation such that

$$\pi(T_i) = T_{\rho(i)}, \qquad \text{and hence} \qquad \pi(s_{ij}) = s_{\rho(i)\rho(j)} \ . \tag{A.5}$$

The effect of θ on the transpositions $(ab) \in \mathrm{Sym}(6)$ is as follows:

$$\text{if} \quad s_{ab} = \{ij,\ kl,\ mn\} \quad \text{then} \quad \theta(ab) = (ij)(kl)(mn)\,, \tag{A.6}$$

this last result confirming that θ is indeed outer. Using (A.6) one can show that θ is involutory: $\theta^2 = id$. (See ([20]) for further details and references.)

Lemma A.2 *(i) The subgroup of* $\mathrm{Sym}(6)$ *which stabilizes a total is* $\cong \mathrm{Sym}(5)$.
(ii) The subgroup which stabilizes a syntheme is $\cong \mathrm{Sym}(4) \times Z_2$.
(iii) The subgroup which stabilizes a near-total is $\cong \mathrm{Sym}(4)$.

Proof. (i) $\mathrm{stab}(T_i) = \theta(\mathrm{stab}\{i\})$, and $\mathrm{stab}\{i\} \cong \mathrm{Sym}(5)$.
(ii) $\mathrm{stab}(s_{23}) = \theta(\mathrm{stab}\{2,3\})$, and $\mathrm{stab}\{2,3\} = \mathrm{Sym}(\{1,4,5,6\}) \times \langle(23)\rangle$.
(iii) In part (ii) the $\mathrm{Sym}(4)$ subgroup of $\mathrm{stab}(s_{23})$ stabilizes the near-totals $T_{2,3}$ and $T_{3,2}$ but the Z_2 subgroup effects $T_{2,3} \rightleftarrows T_{3,2}$. ■

Remark A.3 *Just as* $\mathrm{stab}\{i\}$ *effects all* $5!$ *permutations of the five symbols* $\neq i$*, so* $\mathrm{stab}(T_i)$ *effects all* $5!$ *permutations of the five totals* $\neq T_i$*. Similarly both* $\mathrm{stab}(s_{ij}) = \mathrm{stab}(D_{ij})$ *and* $\mathrm{stab}(T_{i,j})$ *effect all* $4!$ *permutations of the four totals* $T_k, k \neq i, k \neq j$.

A.2 Aspects of $\mathrm{Sym}(6)$ and $\mathrm{Sp}(4,2)$

A.2.1 $\mathrm{Sp}(4,2)$ geometry in even hyperplanes

Let the 6-set $\mathcal{B}^* = \{p_1, \dots, p_6\}$ be a *"hyperbasis"* (= 6-arc) for $V_5 = V(5,2)$. That is, the only non-trivial linear relation satisfied by the p_i is $\Sigma_i p_i = 0$; so, for each of the six choices of $u \in \mathcal{B}^*$, we have a decomposition $\mathcal{B}^* = \mathcal{B} \cup \{u\}$ where $\mathcal{B} = \{e_1, \dots, e_5\}$ is a basis and $u = \Sigma_{i=1}^5 e_i$ is the "all-one" vector (projectively, the *unit point)* in that basis. Setting $p_{ij} = p_i + p_j$ and $p_{ijk} = p_i + p_j + p_k$, take note that $p_{ijk} = p_{lmn}$ holds for any permutation $ijklmn$ of 123456. Observe further that, in the projective space $\mathrm{PG}(4,2) = \mathbf{P}V_5$, the 15 points $\sigma = \{p_{ij}\}$, of even weight in a(ny) basis $\mathcal{B} = \mathcal{B}^* \diagdown \{u\}$, constitute a $\mathrm{PG}(3,2)$, to be referred to as the *even hyperplane* σ of $\mathcal{B}^*$, or of $\mathcal{B}$.

For $\pi \in \mathrm{Sym}(6)$ we may define an element $A(\pi)$ of $\mathrm{GL}(5,2)$ by $A(\pi)p_i = p_{\pi(i)}$ and thereby embed $\mathrm{Sym}(6)$ as a subgroup $\{A(\pi) : \pi \in \mathrm{Sym}(6)\}$ of $\mathrm{GL}(5,2)$. Under the action A of $\mathrm{Sym}(6)$ the 31 points of $\mathrm{PG}(4,2)$ split into the three orbits $\mathcal{B}^*$, σ and $\Omega = \{p_{ijk}\} = \sigma^c \diagdown \mathcal{B}^*$, of respective lengths 6, 15 and 10. Now the hyperbasis $\mathcal{B}^*$ gives rise to a *distinguished* $\mathrm{Sp}(4,2)$ *geometry* for the *even subspace* $V_4 = \sigma \cup \{0\}$ of V_5, namely that determined by the scalar product $x.y$ on V_4 which satisfies

$$p_{ij}.p_{ik} = 1, \qquad p_{ij}.p_{kl} = 0, \tag{A.7}$$

whenever i, j, k, l are distinct. If $B(\pi) : p_{ij} \mapsto p_{\pi(i)\,\pi(j)}$ denotes the restriction of $A(\pi)$ to the invariant subspace V_4 of V_5, then note that $\pi \mapsto B(\pi)$ embeds Sym(6) as a subgroup of $\mathrm{GL}(V_4) \cong \mathrm{GL}(4,2)$ which preserves the scalar product (A.7). Since Sym(6) has the same order $720 = 6!$ as $\mathrm{Sp}(4,2)$, we thus arrive at the isomorphism (i) in (A.1).

In projective terms the even hyperplane σ of $\mathcal{B}^*$ thus comes equipped with a *distinguished null polarity.* Of the 35 lines of the polar space σ, fifteen are *self-polar* ($\lambda^\perp = \lambda$) and the remaining twenty are *nonpolar* ($\lambda^\perp \cap \lambda = \emptyset$), with the latter comprising ten *polar pairs* $\{\lambda, \lambda^\perp\}$. Explicitly the self-polar and nonpolar lines are of the respective forms

$$\text{(i) } \lambda_{ij\,kl\,mn} = \{p_{ij}, p_{kl}, p_{mn}\} \quad \text{and} \quad \text{(ii) } \lambda_{ijk} = \{p_{ij}, p_{ik}, p_{jk}\} = (\lambda_{lmn})^\perp, \tag{A.8}$$

the former being labelled by one of the 15 synthemes $ij\,kl\,mn$, and the latter by one of the 20 triples ijk. *Relative to our choice of scheme (A.2),* take note that the fifteen self-polar lines may just as well be written $\lambda_{s_{ij}}$, and so labelled by the fifteen duads ij rather than by the fifteen synthemes.

A.2.2 Symplectic classification of spreads in PG(3, 2)

There are precisely 56 spreads of lines in PG(3, 2), all equivalent under the action of GL(4, 2), see [11, Section 17.1]. However, for our present purposes, we need to know the split of this single GL(4, 2)-orbit into Sp(4, 2)-orbits. To this end, for a given choice of symplectic geometry, let us say that a spread Σ_5 in PG(3, 2) is of *polar type* (n, n') if n lines of Σ_5 are self-polar and $n' (= 5 - n)$ lines are nonpolar. (The polar type $(n, r - n)$ of an $\mathcal{S}_r$ in PG(3, 2) is similarly defined for $r < 5$.) Clearly spreads of different type belong to different Sp(4, 2)-orbits.

Since there exist precisely six totals, from (A.8i) there must exist precisely six spreads of polar type $(5, 0)$, namely $\Sigma_5(T_i)$, $i = 1, \ldots, 6$, where

$$\Sigma_5(T_i) = \{\lambda_{s_{ij}} : j \neq i\}. \tag{A.9}$$

Moreover, by use of the isomorphism $\mathrm{Sp}(4,2) \cong \mathrm{Sym}(6)$, we see that these six spreads belong to the same Sp(4, 2)-orbit.

Next there exist precisely $\binom{6}{3} = 20$ spreads of polar type $(3, 2)$, namely those of the form $\Sigma_5(ijk)$, for some choice of triple ijk, where

$$\Sigma_5(ijk) = \{\lambda_{s_{ij}}, \lambda_{s_{ik}}, \lambda_{s_{jk}}, \mu, \nu\}; \tag{A.10}$$

here the lines $\{\mu, \nu\}$ are determined uniquely by the requirement that they extend the three self-polar lines $\mathcal{S}_3 = \{\lambda_{s_{ij}}, \lambda_{s_{ik}}, \lambda_{s_{jk}}\}$ to a spread for PG(3, 2). Observe that the lines of $\mathcal{S}_3$ are indeed pairwise skew; for example $\lambda_{s_{ij}}$ is skew to $\lambda_{s_{ik}}$ since s_{ij} and s_{ik} belong to the same total T_i. *Moreover the lines* μ, ν *are necessarily nonpolar*, since a fourth self-polar line would not be skew to at least one member of $\mathcal{S}_3$, for example $\lambda_{s_{il}}$ meets $\lambda_{s_{jk}}$, since s_{il} and s_{jk} do not lie in a common total. Also one sees that $\nu = \mu^\perp$.

If instead we start out from one of the 15 self-polar lines and seek extensions using only nonpolar lines, then it is easy to see there are precisely two such extensions, leading to a Sp$(4,2)$-orbit of $15 \times 2 = 30$ spreads of type $(1,4)$. For example the self-polar line $\lambda(12\,34\,56)$ belongs to just the following two spreads of type $(1,4)$:

$$\{\lambda_{12\,34\,56}, \lambda_{135}, \lambda_{146}, \lambda_{236}, \lambda_{245}\}, \quad \{\lambda_{12\,34\,56}, \lambda_{246}, \lambda_{235}, \lambda_{145}, \lambda_{136}\}. \tag{A.11}$$

It is easy to check directly that there are no further spreads in PG$(4,2)$. Alternatively the $6+20+30$ spreads just described already account for the known total number 56 of spreads. So we have proved the following lemma:

Lemma A.4 *Under the action of* Sp$(4,2)$ *the* 56 *spreads of lines in* PG$(3,2)$ *fall into three orbits of lengths* $6, 20$ *and* 30, *which consist of spreads of the respective polar types* $(5,0), (3,2)$ *and* $(1,4)$. ■

A.3 Aspects of Sym(6) and O(5, 2)

A.3.1 O(5, 2) geometry and subspaces

Let us start from a 6-dimensional symplectic space V_6 and choose, cf. [21], a basis $\{a_1, \ldots, a_6\}$ whose vectors are pairwise non-perpendicular, satisfying that is $a_i.a_j = 1$ for $i \neq j$. Set $V_5 = \prec n \succ^{\perp}$, where $n = a_1 + \ldots + a_6$, so that V_5 consists of those vectors of V_6 of even weight in the chosen basis. The 31 points of the associated projective space PG$(4,2)$ are then: 15 points $a_{ij} (= a_{ji})$, 15 points $v_{ij} (= v_{ji})$ and n, where we define, for $i \neq j$,

$$a_{ij} = a_i + a_j, \quad v_{ij} = n + a_{ij}.$$

Using $x.y$ for the scalar product on V_5 (whose kernel is $\prec n \succ$), observe that the a_{ij} enjoy the metrical properties

$$a_{ij}.a_{ik} = 1, \qquad a_{ij}.a_{kl} = 0, \qquad n.a_{ij} = 0, \tag{A.12}$$

whence we also have $a_{ij}.v_{ik} = 1 = v_{ij}.v_{ik}$ and $a_{ij}.v_{kl} = 0 = v_{ij}.v_{kl}$.

Let Q be that quadratic form on V_5 such that $Q(x) = 1$ for x a vector of the basis $\{a_{16}, a_{26}, \ldots, a_{56}\}$ and whose associated alternating form is $x.y$. It follows that $Q(a_{ij}) = 1$, $Q(n) = 1$, $Q(v_{ij}) = 0$. *Thus $Q = 0$ is the equation of a parabolic quadric $\mathcal{P}_4$ in* PG$(4,2)$, *whose points are the 15 points $\{v_{ij},\ i \neq j\}$, whose nucleus is n and whose invariance group is* O$(5,2)$. Now the symmetric group Sym(6) acts linearly upon the space V_6 via $\pi \mapsto C(\pi)$, where $C(\pi)a_i = a_{\pi(i)}$, and upon restriction to the invariant subspace V_5 we obtain a monomorphism Sym$(6) \to$ O$(5,2) : \pi \mapsto D(\pi)$, where $D(\pi)a_{ij} = a_{\pi(i)\pi(j)}$. Since O$(5,2)$ has the same order 720 as Sym(6), we thus arrive at the isomorphism (ii) in (A.1).

Remark A.5 *We have encountered two subgroups of* GL$(5,2)$ *which are isomorphic to* Sym(6), *namely the subgroup* $\{A(\pi) : \pi \in \text{Sym}(6)\}$ *of section A.2.1 and the subgroup* $\{D(\pi) : \pi \in \text{Sym}(6)\}$ *of the present section. These two subgroups are* not *conjugate in* GL$(5,2)$, *since, when acting on* $V_5 \setminus \{0\}$, *the former has orbits of lengths* $6, 15$ *and* 10, *while the latter has, see the next lemma, orbits of lengths* $15, 15$ *and* 1. *(By using* $E(\pi) =$

$B(\pi) \oplus I_1$ on $V_5 = V_4 \oplus V_1$ we obtain a third subgroup $\{E(\pi) : \pi \in \mathrm{Sym}(6)\} \cong \mathrm{Sym}(6)$ of $\mathrm{GL}(5,2)$; this is not conjugate to either of the previous two subgroups, since unlike them it has a decomposable action on V_5.)

By using the foregoing 6-dimensional notation for the points, and by appealing to the isomorphism $\mathrm{O}(5,2) \cong \mathrm{Sym}(6)$, it is an easy matter, [20], [21], to classify, and list quite explicitly, all subspaces of the orthogonal space (V_5, Q). Let $ijklmn$ denote, as usual, an arbitrary permutation of 123456, and let us adopt projective language. Then, restricting our attention to the points and lines of $\mathrm{PG}(4,2)$, we have:

Lemma A.6 *Under the action of $\mathrm{O}(5,2) \cong \mathrm{Sym}(6)$ the 31 points of $\mathrm{PG}(4,2)$ fall into the three orbits of lengths 15, 1 and 15, namely the 15 points $\{v_{ij}\}$ on the quadric $\mathcal{P}_4$, the nucleus $\{n\}$ of $\mathcal{P}_4$ and the remaining 15 points $\{a_{ij}\}$ off $\mathcal{P}_4$. The 155 lines of $\mathrm{PG}(4,2)$ fall into the five orbits*

$$\begin{aligned}
\Lambda_1 &: \textit{the 15 lines on } \mathcal{P}_4, \textit{ namely } \lambda(ij\,kl\,mn) = \{v_{ij}, v_{kl}, v_{mn}\};\\
\Lambda_2 &: \textit{the 15 nuclear tangents to } \mathcal{P}_4, \textit{ namely } \lambda(ij) = \{n, v_{ij}, a_{ij}\};\\
\Lambda_3 &: \textit{the 45 other tangents to } \mathcal{P}_4, \textit{ namely } \lambda(ij,kl) = \{a_{ij}, a_{kl}, v_{mn}\};\\
\Lambda_4 &: \textit{the 20 external lines } \lambda(ijk) = \{a_{ij}, a_{ik}, a_{jk}\};\\
\Lambda_5 &: \textit{the 60 bisecants } \lambda(i,jk) = \{v_{ij}, v_{ik}, a_{jk}\}. \qquad \blacksquare
\end{aligned}$$

A.3.2 The six spreads on a parabolic quadric $\mathcal{P}_4$

Since the 15 internal lines Λ_1 of $\mathcal{P}_4$ are labelled by the 15 synthemes $ij\,kl\,mn$, we can immediately deduce features of $\mathcal{P}_4$ from the $\mathrm{Sym}(6)$ material of section A.1.1. Since a *spread $\mathcal{S}_5$ on a $\mathcal{P}_4$* is, by definition, a partition of the 15 points of $\mathcal{P}_4$ into five skew lines, and since each duad belongs to precisely three synthemes, we deduce:

Lemma A.7 *There exist precisely six spreads on a $\mathcal{P}_4$, each $\mathcal{S}_5$ having profile $(3,3,3)^5$, the five lines $\lambda(ij\,kl\,mn) = \{v_{ij}, v_{kl}, v_{mn}\}$ of the rth spread having their synthemic labels $ij\,kl\,mn$ drawn from the rth total T_r in the array (A.2). Each of the 15 lines λ on $\mathcal{P}_4$ belongs to precisely two of these spreads, the 12-set $\psi = \mathcal{P}_4^- = \mathcal{P}_4 \diagdown \lambda$ being a non-degenerate double-four, see section 3.3.1, partitioning in precisely two ways to yield partial spreads $\mathcal{S}_4$ and $\mathcal{S}_4^{opp}$, each of profile $(2,2,2)^4$, on $\mathcal{P}_4$.* $\blacksquare$

Thus if $\mathcal{S}_5 = \{\lambda(s_{i6}) : i \neq 6\}$ is the spread on $\mathcal{P}_4$ which uses the total T_6 in the array (A.2), and if $\mathcal{S}_4 = \mathcal{S}_5 \diagdown \{\lambda(s_{56})\}$, then, see (A.4), $\psi_4 = \psi(\mathcal{S}_4)$ is the double-four

$$\psi_4 = \begin{pmatrix} - & v_{16} & v_{25} & v_{34} \\ v_{45} & - & v_{13} & v_{26} \\ v_{36} & v_{24} & - & v_{15} \\ v_{12} & v_{35} & v_{46} & - \end{pmatrix} \tag{A.13}$$

whose rows are the lines $\lambda(s_{i6})$, $i = 1,2,3,4$, of $\mathcal{S}_4$ and whose columns are the lines $\lambda(s_{5j})$, $j = 1,2,3,4$,of $\mathcal{S}_4^{\mathrm{opp}}$.

Also from lemma A.2 and remark A.3 we obtain:

Lemma A.8 *If $\mathcal{S}_5$, ψ_4 and $\mathcal{S}_4$ are as in the preceding paragraph, then*

$$\text{(i) } \mathcal{G}(\mathcal{S}_5) \cong \mathrm{Sym}(5) \quad \text{(ii) } \mathcal{G}(\psi_4) \cong \mathrm{Sym}(4) \times Z_2 \quad \text{(iii) } \mathcal{G}(\mathcal{S}_4) \cong \mathrm{Sym}(4).$$

Moreover $\mathcal{G}(\mathcal{S}_5)$, $\mathcal{G}(\mathcal{S}_4)$ effect all permutations of the lines of $\mathcal{S}_5$, $\mathcal{S}_4$, respectively, and in particular both $\mathcal{S}_5$ and $\mathcal{S}_4$ are cyclic. ■

(Incidentally there are three kinds of cyclic $\mathcal{S}_5$ s in PG(4, 2), the other two belonging to classes Ve.1 and Vj.1.) Let $\mathcal{S}_5 = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ again be that spread on $\mathcal{P}_4$ corresponding to the total T_6 in (A.2):

$$\begin{aligned} &\lambda_1 = \lambda(25\,34\,16),\ \lambda_2 = \lambda(13\,45\,26),\ \lambda_3 = \lambda(15\,24\,36), \\ &\lambda_4 = \lambda(12\,35\,46),\ \lambda_5 = \lambda(14\,23\,56). \end{aligned} \tag{A.14}$$

Then for the 5-cycle $\pi = (12345)(6)$ the element $D(\pi) \in \mathrm{O}(5, 2)$ effects the cyclic permutation $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)$ of the lines of $\mathcal{S}_5$. If we set $e_i = a_{i6}$ then $\{e_1, ..., e_5\}$ is a basis with $u = n$, and $D(\pi)$ effects $(e_1e_2e_3e_4e_5)$. In this basis the five lines of $\mathcal{S}_5$ in (A.14) are, in abbreviated notation,

$$\begin{aligned} &\lambda_1 = \{134, 125, 1u\},\ \lambda_2 = \{245, 231, 2u\},\ \lambda_3 = \{351, 342, 3u\}, \\ &\lambda_4 = \{412, 453, 4u\},\ \lambda_5 = \{523, 514, 5u\}, \end{aligned} \tag{A.15}$$

and the double-four $\psi_4 = \mathcal{P}_4 \setminus \lambda_5$ in (A.13) is

$$\psi_4 = \begin{pmatrix} - & 1u & 134 & 125 \\ 231 & - & 245 & 2u \\ 3u & 351 & - & 342 \\ 453 & 412 & 4u & - \end{pmatrix}. \tag{A.16}$$

Starting from this 4×4 array we may recover λ_5, and the nucleus $n = u$, with the aid of lemma 3.4. By taking advantage of the Z_5-symmetry of $\mathcal{S}_5$ in (A.15), we see that the equations $f_{r\,s} = 0$ of the hyperplanes $\sigma_{rs} = \langle \lambda_r, \lambda_s \rangle$ are given by the ten linear forms:

$$f_{r\,r+1} = x_r + x_{r+1} + x_{r+3}, \qquad f_{r\,r+2} = x_r + x_{r+1} + x_{r+2}, \tag{A.17}$$

where the coordinates x_1, ..., x_5 are relative to the chosen basis, and where r runs through $1, 2, 3, 4, 5 \bmod 5$.

Lemma A.9 *A partial spread $\mathcal{S}_5$ in* PG(4, 2) *which is a spread on a $\mathcal{P}_4$ is necessarily of type O (and hence so is $\mathcal{S}_4 = \mathcal{S}_5 \setminus \{\lambda\}$).*

Proof. The ten $f_{r\,s}$ in (A.17) are distinct, whence so are the ten hyperplanes $\langle \lambda_r, \lambda_s \rangle$. ■

(Alternative proof: if one regulus exists then, by the cyclic symmetry, at least five exist; but, from (1.5) or theorem 1.6(b), $N_5 > 4$ implies $N_5 = 10$, which only occurs when $\mathcal{S}_5$ is a spread in a hyperplane.)

A.3.3 Extending an $\mathcal{S}_5$ on a $\mathcal{P}_4$ to an $\mathcal{S}_r^{\max}$

Let $ijklmn$ be any permutation of 123456. Then take note that the twenty lines $\lambda(ijk) = \{a_{ij}, a_{ik}, a_{jk}\}$ external to $\mathcal{P}_4$, see lemma A.6, partition into ten pairs of the kind $\{\lambda, \lambda^*\}$, where for $\lambda = \lambda(ijk)$ we define $\lambda^* = \lambda(lmn)$. In the coordinates $x_i,\ 1 \le i \le 5$, used in eq. (A.17), let the hyperplane $\sigma_{rs} = \langle\lambda_r, \lambda_s\rangle$ have equation $x_i + x_j + x_k = 0$, and observe that the ten equations (A.17) make use of all ten linear forms which are of weight 3 in the coordinates. Also note that σ_{rs} contains precisely two of the twenty lines which are external to $\mathcal{P}_4$, namely $\lambda(ijk) = \{ij, ik, jk\}$ and $\lambda(lm6) = \{l, m, lm\}$. So we have part (i) of the next lemma.

Lemma A.10 *(i) For an $\mathcal{S}_5$ on a $\mathcal{P}_4$ each hyperplane $\sigma_{rs} = \langle\lambda_r, \lambda_s\rangle$ contains precisely two lines which are external to $\mathcal{P}_4$. The two lines form a pair $\{\lambda, \lambda^*\}$, with all ten such pairs arising from the ten hyperplanes σ_{rs}.*

(ii) Every extension $\mathcal{S}_6$ of $\mathcal{S}_5$ is of class VIc.2.

Proof. (ii) Consider the extensions $\mathcal{S}_6 = \mathcal{S}_5 \cup \{\lambda_{ijk}\}$ and $\mathcal{S}_6^* = \mathcal{S}_5 \cup \{\lambda_{lm6}\}$ of $\mathcal{S}_5$. Both extensions are of the form $\mathcal{S}_3(\mathsf{I}) \cup \mathcal{S}_3(\mathsf{O})$ where, see the lead-in to the lemma, $\mathcal{S}_3(\mathsf{O}) = \mathcal{S}_5 \setminus \{\lambda_r, \lambda_s\}$ and $\mathcal{S}_3(\mathsf{I})$ is a regulus in the hyperplane $\sigma_{rs} = \langle\lambda_r, \lambda_s\rangle$; here $\mathcal{S}_3(\mathsf{I}) = \{\lambda_r, \lambda_s, \lambda(ijk)\}$ for $\mathcal{S}_6$ and $\mathcal{S}_3(\mathsf{I}) = \{\lambda_r, \lambda_s, \lambda(lm6)\}$ for $\mathcal{S}_6^*$. Observe that $\{\lambda_r, \lambda_s, \lambda(ijk), \lambda(lm6), \tau\}$ is a spread for σ_{rs} whose fifth member τ can only be the transversal of the non-regulus $\mathcal{S}_3(\mathsf{O})$. Consequently $\mathcal{S}_6$ and $\mathcal{S}_6^*$ are of type I and moreover *case (a) of (7.6) applies.* Upon inspection we see that, for the null polarity induced in the even hyperplane σ_{rs} of $\mathcal{S}_3(\mathsf{O})$, both λ_r and λ_s are self-polar, and both $\lambda(ijk)$ and $\lambda(lm6)$ are nonpolar lines. Since for both $\mathcal{S}_6$ and $\mathcal{S}_6^*$ the regulus $\mathcal{S}_3(\mathsf{I})$ is thus of polar type $(2, 1)$, each extension is, see section 7.3.1, of class VIc.2. ■

If $\mathcal{S}_5$ is a spread on a $\mathcal{P}_4$ then the 16-set $(\psi_5)^c$ complementary to $\psi_5 = \psi(\mathcal{S}_5)$ consists of the nucleus n of $\mathcal{P}_4$ and the remaining 15 points a_{ij} off $\mathcal{P}_4$. If $\mathcal{S}_r = \mathcal{S}_{5+t} = \mathcal{S}_5 \cup \mathcal{S}_t$ is an extension of $\mathcal{S}_5$ then $\mathcal{S}_t$ must consist of mutually skew lines chosen from the 20 lines $\lambda(ijk)$. external to $\mathcal{P}_4$. If $\mathcal{S}_r$ is maximal then two cases arise: either (i) $t = 2$, or else (ii) $t = 4$. In the former case it is easily seen that $\mathcal{S}_2$ must be one of the ten pairs $\{\lambda, \lambda^*\}$ just described.

So let us consider the possibilities for $\mathcal{S}_4$ in case (ii), where we deal with an extension $\mathcal{S}_9 = \mathcal{S}_5 \cup \mathcal{S}_4$ of $\mathcal{S}_5$. The 4-set $(\psi_9)^c$ complementary to $\psi_9 = \psi(\mathcal{S}_9)$ consists of the nucleus n of $\mathcal{P}_4$ together with three of the points a_{ij}, these three being of the form $\{a_{ij}, a_{kl}, a_{mn}\}$ for some syntheme $ij\,kl\,mn$. Observe that the 4-set $(\psi_9)^c$ is thus of the form $\alpha \setminus \lambda(ij\,kl\,mn)$ where α is a plane — in conformity with lemma 1.2(iii). Two subcases arise:

$$\text{(iia) } \lambda(ij\,kl\,mn) \in \mathcal{S}_5, \qquad \text{(iib) } \lambda(ij\,kl\,mn) \notin \mathcal{S}_5. \tag{A.18}$$

Without loss of generality we may suppose that $\mathcal{S}_5 = \{\lambda(s_{i6}) : i \neq 6\}$ is the spread on $\mathcal{P}_4$ which uses the total T_6 in the array (A.2). Since $\mathcal{G}(\mathcal{S}_5)$ is transitive both on the five lines of $\mathcal{S}_5$ and on the ten other internal lines $\Lambda_1 \setminus \mathcal{S}_5$, we may also suppose that the two subcases to consider are:

$$\text{(a) } (\psi_9)^c = \alpha \setminus \lambda(s_{16}), \qquad \text{(b) } (\psi_9)^c = \alpha \setminus \lambda(s_{23}). \tag{A.19}$$

Consider $\psi_4 = \psi(\mathcal{S}_4) = \psi_9 \diagdown \mathcal{P}_4$. Since $\lambda(s_{16}) = \lambda(25\,34\,16)$, if (A.19a) holds we see that $\psi_4 = \{a_{ij}\}_{1\le i<j\le 6} \diagdown \{a_{25}, a_{34}, a_{16}\}$ has the double-four structure

$$\psi_4 = \begin{pmatrix} - & a_{12} & a_{23} & a_{13} \\ a_{56} & - & a_{36} & a_{35} \\ a_{45} & a_{14} & - & a_{15} \\ a_{46} & a_{24} & a_{26} & - \end{pmatrix}. \tag{A.20}$$

Consequently the spread $\mathcal{S}_5$ on $\mathcal{P}_4$ has precisely two extensions to an $\mathcal{S}_9$ such that (A.19a) holds, namely $\mathcal{S}_9 = \mathcal{S}_5 \cup \mathcal{S}_4$ and $\mathcal{S}'_9 = \mathcal{S}_5 \cup \mathcal{S}_4^{\mathrm{opp}}$, where

$$S_4 = \{\lambda(123), \lambda(356), \lambda(145), \lambda(246)\}, \quad S_4^{\mathrm{opp}} = \{\lambda(456), \lambda(124), \lambda(234), \lambda(135)\} \tag{A.21}$$

are given by the rows and columns of the double-four array (A.20), and so are both of class IVa.1. Since there are five choices for $\lambda \in \mathcal{S}_5$ in (A.18a), case (iia) gives rise to ten extensions of $\mathcal{S}_5$ to an $\mathcal{S}_9$. Similarly, since $\lambda(s_{23}) = \lambda(12\,34\,56)$, if (A.19b) holds we see that $\psi_4 = \{a_{ij}\}_{1\le i<j\le 6} \diagdown \{a_{12}, a_{34}, a_{56}\}$ has a double-four structure and that the spread $\mathcal{S}_5$ on $\mathcal{P}_4$ has precisely two extensions to an $\mathcal{S}_9$ such that (A.19b) holds, namely $\mathcal{S}_9 = \mathcal{S}_5 \cup \mathcal{S}_4$ and $\mathcal{S}'_9 = \mathcal{S}_5 \cup \mathcal{S}_4^{\mathrm{opp}}$, where

$$S_4 = \{\lambda(135), \lambda(146), \lambda(236), \lambda(245)\}, \quad S_4^{\mathrm{opp}} = \{\lambda(246), \lambda(235), \lambda(145), \lambda(136)\} \tag{A.22}$$

are of class IVa.1. Since there are ten choices for $\lambda \notin \mathcal{S}_5$ in (A.18b), case (iib) gives rise to twenty extensions of $\mathcal{S}_5$ to an $\mathcal{S}_9$.

Theorem A.11 *If $\mathcal{S}_5$ is a spread on a parabolic quadric $\mathcal{P}_4$ then its extensions to a maximal partial spread are precisely those listed below:*
(i) there are ten extensions of $\mathcal{S}_5$ to an $\mathcal{S}_7^{max}$, each of class VIIf.3;
(iia) there are ten extensions of $\mathcal{S}_5$ to an $\mathcal{S}_9(\mathsf{X})$;
(iib) there are twenty extensions of $\mathcal{S}_5$ to an $\mathcal{S}_9(\mathsf{I}^\kappa\Delta)$.

Proof. This follows from our lead-in upon noting the following.

(i) Consider the extension $\mathcal{S}_7^{\max} = \mathcal{S}_5 \cup \{\lambda(123), \lambda(456)\}$, and recall the proof of lemma A.10. We have $\mathcal{S}_7^{\max} = \mathcal{S}_4(\binom{4}{3}) \cup \mathcal{S}_3(\mathsf{O})$ where $\mathcal{S}_3(\mathsf{O}) = \{\lambda_2, \lambda_4, \lambda_5\}$ and where $\mathcal{S}_4(\binom{4}{3}) = \{\lambda_1, \lambda_3, \lambda(123), \lambda(456)\}$ is of polar type $(2,2)$. Hence, see theorem 7.1, $\mathcal{S}_7^{\max}$ is of class VIIf.3.

(iia) If $\mathcal{S}_9 = \mathcal{S}_5 \cup \mathcal{S}_4$ with $\mathcal{S}_4$ as in (A.21) then the four reguli of $\mathcal{S}_9$ are, by use of (A.17), seen to be $\{\lambda_1, \lambda_2, \lambda(356)\}$, $\{\lambda_1, \lambda_3, \lambda(123)\}$, $\{\lambda_1, \lambda_4, \lambda(145)\}$ and $\{\lambda_1, \lambda_5, \lambda(246)\}$, and so $\mathcal{S}_9$ is of type X. By lemma A.10(i), the same regulus pattern holds if each $\lambda(ijk) \in \mathcal{S}_4$ is replaced by $\lambda(ijk)^* \in \mathcal{S}_4^{\mathrm{opp}}$, and so $\mathcal{S}'_9 = \mathcal{S}_5 \cup \mathcal{S}_4^{\mathrm{opp}}$, with $\mathcal{S}_4^{\mathrm{opp}}$ as in (A.21), is also of type X.

(iib) If $\mathcal{S}_9 = \mathcal{S}_5 \cup \mathcal{S}_4$ with $\mathcal{S}_4$ as in (A.22) then the four reguli of $\mathcal{S}_9$ are, by use of (A.17), seen to be $\{\lambda_2, \lambda_3, \lambda(146)\}$, $\{\lambda_1, \lambda(236), \lambda_4\}$, $\{\lambda_4, \lambda(245), \lambda_5\}$ and $\{\lambda_5, \lambda(135), \lambda_1\}$, and so, see section 4.3, $\mathcal{S}_9$ is seen to be of type $\mathsf{I}^\kappa\Delta$. Again the same regulus pattern holds if each $\lambda(ijk) \in \mathcal{S}_4$ is replaced by $\lambda(ijk)^* \in \mathcal{S}_4^{\mathrm{opp}}$, and so $\mathcal{S}'_9 = \mathcal{S}_5 \cup \mathcal{S}_4^{\mathrm{opp}}$, with $\mathcal{S}_4^{\mathrm{opp}}$ as in (A.22), is also seen to be of type $\mathsf{I}^\kappa\Delta$. ∎

B Appendix: Tables of results

B.1 Point-set orbits

Table B.1: the orbits of $\psi(\mathcal{S}_r)$, $1 \le r \le 9$

$\Theta(\psi)$	$\lvert\mathcal{G}(\psi)\rvert$	Signature (ψ)	Signature (ψ^c)
Ia	64512	$(3,1,0)(3^1)(3^0)$	$(28,112,64)(28^{12})(28^{16})$
IIa	1152	$(6,2,0)(6^1)(6^0)$	$(25,78,24)(16^9 9^{10})(16^6 9^8)$
IIIa	48	$(9,4,0)(6^1 3^2)(9^0)$	$(22,52,8)(4^6 12^7 6^8)(16^2 6^4)$
IIIb	1152	$(9,6,0)(9^2)(9^0)$	$(22,50,8)(16^6 6^9)(16^2 6^4)$
IVa	48	$(12,8,0)(12^2)(12^0)$	$(19,33,4)(1^3 15^5 3^7)(13^1 3^2 3^3)$
IVb	6	$(12,8,0)(3^1 6^2 3^3)(12^0)$	$(19,33,2)(3^4 9^5 7^6)(6^0 12^1 1^2)$
IVc	24	$(12,10,0)(2^1 3^2 6^3 1^4)(12^0)$	$(19,31,2)(2^3 6^4 6^5 2^6 3^7)(8^0 8^1 3^2)$
IVd	9216	$(12,16,0)(12^4)(12^0)$	$(19,25,4)(16^3 3^9)(16^1 3^4)$
Va	720	$(15,15,0)(15^3)(15^0)$	$(16,20,0)(1^0 15^4)(16^0)$
Vb	16	$(15,15,0)(1^1 13^3 1^5)(15^0)$	$(16,20,0)(2^2 14^4)(16^0)$
Vc	48	$(15,15,0)(15^3)(15^0)$	$(16,20,2)(3^2 12^4 1^6)(3^0 12^1 1^2)$
Vd	6	$(15,15,0)(3^2 9^3 3^4)(15^0)$	$(16,20,1)(7^3 6^4 3^5)(9^0 7^1)$
Ve	10	$(15,15,0)(5^2 5^3 5^4)(15^0)$	$(16,20,0)(5^3 10^4 1^5)(16^0)$
Vf	8	$(15,17,0)(2^2 6^3 6^4 1^5)(15^0)$	$(16,18,1)(4^2 6^3 3^4 2^5 1^6)(9^0 7^1)$
Vg	4	$(15,17,1)(1^1 2^2 4^3 6^4 2^5)(8^0 7^1)$	$(16,18,0)(3^2 6^3 5^4 2^5)(16^0)$
Vh	48	$(15,19,1)(9^3 6^5)(8^0 7^1)$	$(16,16,0)(1^0 6^2 9^4)(16^0)$
Vi	384	$(15,23,4)(2^1 12^5 1^7)(2^0 12^2 1^4)$	$(16,12,0)(2^0 12^2 2^6)(16^0)$
Vj	322560	$(15,35,15)(15^7)(15^7)$	$(16,0,0)(16^0)(16^0)$
VIa	48	$(18,26,0)(12^4 6^5)(18^0)$	$(13,12,1)(6^2 4^3 3^4)(6^0 7^1)$
VIb	16	$(18,26,0)(2^3 8^4 8^5)(18^0)$	$(13,12,0)(4^2 8^3 1^4)(13^0)$
VIc	36	$(18,28,0)(3^3 15^5)(18^0)$	$(13,10,0)(1^0 9^2 3^4)(13^0)$
VId	48	$(18,28,0)(6^4 12^5)(18^0)$	$(13,10,1)(6^1 4^3 3^4)(6^0 7^1)$
VIe	4	$(18,28,2)(2^3 6^4 6^5 4^6)(7^0 8^1 3^2)$	$(13,10,0)(2^1 6^2 4^3 1^4)(13^0)$
VIf	16	$(18,28,2)(4^3 13^5 1^7)(7^0 8^1 3^2)$	$(13,10,0)(11^2 2^4)(13^0)$
VIg	32	$(18,30,4)(8^4 4^5 4^6 2^7)(8^1 10^2)$	$(13,8,0)(4^1 8^2 1^4)(13^0)$
VIh	8	$(18,30,3)(1^3 4^4 8^5 4^6 1^7)(3^0 10^1 4^2 1^3)$	$(13,8,0)(1^0 4^1 4^2 4^3)(13^0)$
VIi	144	$(18,32,3)(15^5 3^7)(3^0 12^1 3^3)$	$(13,6,0)(4^0 9^2)(13^0)$
VIj	192	$(18,34,8)(4^3 8^6 6^7)(4^0 14^4)$	$(13,4,0)(4^0 8^1 1^4)(13^0)$
VIIa	1008	$(21,42,0)(21^6)(21^0)$	$(10,8,1)(3^1 7^3)(3^0 7^1)$
VIIb	32	$(21,44,4)(4^5 8^6 8^7 1^8)(8^0 12^2 1^4)$	$(10,6,0)(4^1 4^2 2^3)(10^0)$
VIIc	16	$(21,46,7)(2^5 8^6 8^7 3^8)(2^0 12^2 5^3 2^5)$	$(10,4,0)(2^0 4^1 4^2)(10^0)$
VIId	48	$(21,46,8)(15^6 6^8)(1^0 12^2 8^4)$	$(10,4,0)(9^1 1^3)(10^0)$
VIIe	48	$(21,48,9)(6^6 12^7 3^8)(6^2 12^3 3^5)$	$(10,2,0)(4^0 6^1)(10^0)$
VIIf	720	$(21,50,15)(6^5 15^8)(6^0 15^7)$	$(10,0,0)(10^0)(10^0)$
VIIIa	64512	$(24,64,0)(24^8)(24^0)$	$(7,7,1)(7^3)(7^1)$
VIIIb	768	$(24,68,16)(18^8 6^{10})(2^0 16^4 6^8)$	$(7,3,0)(6^1 1^3)(7^0)$
VIIIc	144	$(24,70,21)(9^8 12^9 3^{10})(9^4 12^7 3^9)$	$(7,1,0)(4^0 3^1)(7^0)$
IXa	9216	$(27,101,54)(24^{11} 3^{13})(24^{13} 3^{22})$	$(4,0,0)(4^0)(4^0)$

B.2 Partial spread orbits

Table B.2a: the classes of partial spreads $\mathcal{S}_r,\ 1 \le r \le 6$

$\Theta(\mathcal{S}_r)$	N_r	Type	$\mathcal{G}(\mathcal{S}_r)$	profile($\mathcal{S}_r$)	Notes
Ia.1	0	O	64512	(111)	$2^6{:}(\mathrm{L}_2(2)\times \mathrm{L}_3(2))$
IIa.1	0	O	1152	$(111)^2$	$\{2^4{:}(\mathrm{L}_2(2)\times \mathrm{L}_2(2))\}.2$
IIIa.1	0	O	$\mathrm{Sym}(4)\times Z_2$	$(112)^3$	§3.2, non-regulus, cyclic
IIIb.1	1	I	576	$(222)^3$	§3.1, regulus, cyclic
IVa.1	0	O	Sym(4)	$(222)^4$	§3.3.1, cyclic
IVb.1	0	O	Sym(3)	$(123)^3(222)$	§3.3.2
IVc.1	1	I	$\mathrm{Sym}(3)\times Z_2$	$(233)^3(114)$	§3.1, §3.5, eq. (3.28)
IVd.1	4	$\binom{4}{3}$	1152	$(444)^4$	§3.1, cyclic
Va.1	0	O	Sym(5)	$(333)^5$	§3.4, §A.3.2, cyclic
Vb.1	0	O	D_8	$(333)^4(135)$	§3.4
Vc.1	0	O	$\mathrm{Sym}(3)\times Z_2$	$(333)^5$	§3.4
Vd.1	0	O	Z_3	$(234)^3(333)^2$	§3.4
Ve.1	0	O	Z_5	$(234)^5$	§3.4, cyclic
Vf.1	1	I	Z_2	$(344)^2(335).(234)^2$	§3.5
Vg.1	1	I	Z_2	$(344)^3.(135)(225)$	§3.5
Vh.1	2	L	D_8	$(355).(335)^4$	§3.5, lemma 4.1
Vi.1	4	$\binom{4}{3}$	$\mathrm{Sym}(4)\times Z_2$	$(555)^4.(117)$	§3.1
Vj.1	10	$\binom{5}{3}$	$2^4{:}\Gamma\,\mathrm{L}(2,4)$	$(777)^5$	§3.1, **maximal**, cyclic
VIa.1	0	O	Z_6	$(445)^6$	§6.2, cyclic
VIb.1	0	O	Z_2	$(355)^2(445)^4$	§6.2
VIc.1	1	I	Sym(3)	$(555)^3.(355)^3$	§7.3.1
VIc.2	1	I	Sym(3)	$(555)^3.(355)^3$	§7.3.1
VId.1	1	I	Sym(3)	$(555)^3.(445)^3$	§7.3.1
VId.2	1	I	Sym(3)	$(555)^3.(445)^3$	§7.3.1
VIe.1	1	I	1	$(456)^2(555).(346)^2(445)$	§7.3.1
VIe.2	1	I	1	$(456)^2(555).(346)^2(445)$	§7.3.1
VIf.1	1	I	Z_2	$(555)^3.(355)^2(337)$	§7.3.1
VIg.1	2	II	Z_4	$2\times(456)^2(447)$	§7.2.1
VIh.1	2	L	1	$(566).(456)^2(447)(555).(355)$	§4.2, lemma 4.2
VIh.2	2	L	Z_2	$(557).(456)^4.(355)$	§4.2, lemma 4.2
VIi.1	3	Δ	Sym(3)	$(557)^3.(555)^3$	§4.2, lemma 4.3
VIj.1	4	$\binom{4}{3}$	D_8	$(667)^4.(337)^2$	§7.1, lemma 7.2

Table B.2b: the classes of partial spreads $\mathcal{S}_r,\ r > 6$

$\Theta(\mathcal{S}_r)$	N_r	Type	$\mathcal{G}(\mathcal{S}_r)$	profile($\mathcal{S}_r$)	Notes
VIIa.1	0	O	7:3	$(666)^7$	§6.1, cyclic
VIIb.1	1	I	Z_2	$(668)(677)^2.(567)^4$	§7.3.2
VIIb.2	1	I	Z_2	$(677)^3.(558)(567)^2(666)$	§7.3.2
VIIb.3	1	I	Z_2	$(677)^3.(558)(567)^2(666)$	§7.3.2
VIIc.1	2	II	Z_4	$\{2 \times (668)(677)^2\}.(558)$	§7.2.2, lemma 7.3
VIIc.2	2	II	Z_4	$\{2 \times (668)(677)^2\}.(558)$	§7.2.2, lemma 7.3
VIIc.3	2	L	Z_2	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIIc.4	2	L	Z_2	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIIc.5	2	L	1	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIIc.6	2	L	1	$(778).(668)^2(677)^2.(567)^2$	§4.4
VIId.1	2	II	1	$(668)^6.(666)$	§7.2.2, lemma 7.3
VIIe.1	3	Δ	Sym(3)	$(778)^3.(677)^3.(666)$	§4.3, theorem 4.5
VIIe.2	3	Δ	Z_3	$(778)^3.(677)^3.(666)$	§4.3, theorem 4.5
VIIe.3	3	Δ	Z_2	$(778)^3.(677)^3.(666)$	§4.3, theorem 4.5
VIIe.4	3	Y	Z_6	$(888).(677)^6$	§4.4
VIIe.5	3	F	Z_2	$(778)^2.(668)(677)^4$	§4.4
VIIf.1	4	$\binom{4}{3}_{(4,0)}$	Sym(4)	$(888)^4.(558)^3$	§7.1, **maximal**
VIIf.2	4	$\binom{4}{3}_{(0,4)}$	Sym(4)	$(888)^4.(558)^3$	§7.1, **maximal**
VIIf.3	4	$\binom{4}{3}_{(2,2)}$	$\mathrm{Sym}(3) \times Z_2$	$(888)^4.(558)^3$	§7.1, **maximal**
VIIIa.1	0	O	$2^3{:}\,\mathrm{F}_{21}$	$(888)^8$	§6.1, transitive
VIIIb.1	2	II	Z_2	$(88\mathtt{t})^6.(888)^2$	§5.2
VIIIc.1	3	Δ	Sym(3)	$(99\mathtt{t})^3.(899)^3.(888)^2$	§4.3, thm 4.5, §5.2
VIIIc.2	3	Δ	Z_3	$(99\mathtt{t})^3.(899)^3.(888)^2$	§4.3, thm 4.5, §5.2
VIIIc.3	3	Δ	Z_2	$(99\mathtt{t})^3.(899)^3.(888)^2$	§4.3, thm 4.5, §5.2
VIIIc.4	3	Y	Z_3	$(\mathtt{ttt}).(899)^6.(888)$	§5.2
VIIIc.5	3	$\mathsf{I}^\rho\mathsf{L}$	Z_2	$(99\mathtt{t}).(899)^2(88\mathtt{t})^2 \vert (899)^3$	§5.2
VIIIc.6	3	$\mathsf{I}^\kappa\mathsf{L}$	Z_2	$(99\mathtt{t}).(899)^2(88\mathtt{t})^2 \vert (899)^3$	§5.2
VIIIc.7	3	F	1	$(99\mathtt{t})^2.(888)(899)^4.(888)$	§5.2
IXa.1	4	X	$\mathrm{Alt}(4) \times Z_2$	$(13,13,13).(11,11,11)^8$	§5.1, thms 5.1, 6.2, **maximal**
IXa.2	4	$\mathsf{I}^\rho\Delta$	Sym(3)	$(11,11,13)^3.(11,11,11)^6$	§5.1, thm 5.1, **maximal**
IXa.3	4	$\mathsf{I}^\kappa\Delta$	Sym(3)	$(11,11,13)^3.(11,11,11)^6$	§5.1, thm 5.1, **maximal**
IXa.4	4	E	Z_6	$(11,11,13)^3.(11,11,11)^6$	§5.1, thm 5.1, **maximal**

B.3 Notes to the tables

B.3.1 Table B.1

(i) By appeal to table B.1, the GL(5, 2)-orbit of the underlying set $\psi = \psi(\mathcal{S}_r)$ of a given partial spread $\mathcal{S}_r$ can be found as follows. Compute either signature(ψ) or signature(ψ^c), whichever is easiest, and then, with one exception, the orbit may be read off from table B.1. The one exception is of orbits Va and Vc, both of which have signature(ψ) = $(15, 15, 0)(15^3)(15^0)$; but these two orbits are still distinguished by ψ^c having different signatures.

(ii) A point-set $\psi(\mathcal{S}_r)$ defines uniquely a homogeneous polynomial f of reduced degree ≤ 3 in the coordinates $x_1, \ldots, x_5$ such that if r is odd then ψ has equation $f(x) = 0$, while if r is even then ψ^c has equation $f(x) = 0$; see [23], [19]. In [7], *all* point-sets ψ defined in terms of a homogeneous polynomial f of reduced degree ≤ 3 were classified into GL(5, 2)-orbits; also for each orbit the signature, along with a representative polynomial f, was computed. From [7, Table 8.4.6] it is seen that, *even for these more general point-sets ψ, the pair* {signature(ψ), signature(ψ^c)} *suffices to distinguish between the various* GL(5, 2)*-orbits.* (It should be mentioned that most of the computations in [7] were carried out using a combination of specially written Pascal code and Mathematica [29]; for details, see [7, Section 8.4] and [8].)

For $|\psi| = 24$ there are three orbits, all being represented by a ψ of the form $\psi(\mathcal{S}_8)$, for $|\psi| = 21$ there are seven orbits, all but one being represented by a ψ of the form $\psi(\mathcal{S}_7)$, and for $|\psi| = 18$ there are twelve orbits, all but two being represented by a ψ of the form $\psi(\mathcal{S}_6)$. (Of the eighteen orbits in [7] for $|\psi| = 15$, eight do not arise from a $\psi(\mathcal{S}_5)$, and of the twelve orbits for $|\psi| = 12$, eight do not arise from a $\psi(\mathcal{S}_4)$.)

B.3.2 Tables B.2a and B.2b

A profile $(1, 2, 3)^3(2, 2, 2)$, see section 1.1, is written $(123)^3(222)$. We use `t` as an abbreviation for 10. Dots are used to separate groups of lines of the same valency, with the valencies occurring in descending order. Thus for an $\mathcal{S}_6$ of type L a profile $(557).(456)^4.(355)$ conveys the information that the line of valency 2 has profile $(5, 5, 7)$, each of the four lines of valency 1 has profile $(4, 5, 6)$ and the line of valency 0 has profile $(3, 5, 5)$. The entry $2\times(456)^2(447)$ for the profile of class VIg.1, of type II, indicates that both reguli contribute $(4, 5, 6)^2(4, 4, 7)$ to the overall profile.

VIIe.5. There are five lines of valency 1, and one of these forms a regulus with the two lines of valency 2; it is this line which has profile $(6, 6, 8)$, and the other four which have profile $(6, 7, 7)$.

VIIIc.7. A similar remark applies concerning the line of valency 1 and profile $(8, 8, 8)$.

VIIIc.5, c.6. In the case of an $\mathcal{S}_8$ of type IL the 7 lines of valency 1 have a natural $4+3$ split, with the 3 forming the stand-alone regulus. A vertical line | is used to separate the profile of the 4 from that of the 3.

Reguli reversals. If a partial spread $\mathcal{S}_r$ contains a regulus ρ then we may obtain another partial spread $\mathcal{S}'_r$ by *reversing* the regulus ρ, that is by replacing ρ by ρ^{opp}. Since $\mathcal{S}_r$ and $\mathcal{S}'_r$ share the same point-set $\psi(\mathcal{S}_r) = \psi(\mathcal{S}'_r)$, it follows from eq. (1.4) that *reguli reversals do not change the number of reguli*: $N(\mathcal{S}_r) = N(\mathcal{S}'_r)$. Moreover in the cases $r \leq 5$ the partial spreads $\mathcal{S}_r$ and $\mathcal{S}'_r$ necessarily belong to the same class since, as pointed out in section 2.1, the orbit $\Theta(\psi)$ of the underlying point-set $\psi(\mathcal{S}_r)$ suffices to distinguish the classes of $\mathcal{S}_r$ if $r \leq 5$. For $r > 5$ some reguli reversals $\rho \rightleftarrows \rho^{\text{opp}}$ do result in $\mathcal{S}_r$ and $\mathcal{S}'_r$ belonging to different classes. Examples include the following pairs of classes which are interchanged under a (suitable) regulus reversal:

Type I: VIc.1 & VIc.2; VId.1 & VId.2; VIe.1 & VIe.2; VIIb.2 & VIIb.3.
Type L: VIh.1 & VIh.2. Type II: VIIc.1 & VIIc.2.
Type IL: VIIIc.5 & VIIIc.6. Type IΔ: IXa.2 & IXa.3.
Types X & E: IXa.1 & IXa.4.

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