Tubes in $PG(3, q)$

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Abstract. A tube (resp. an oval tube) in $PG(3, q)$ is a pair $\mathcal{T} = \{L, \mathcal{L}\}$, where $\{L\} \cup \mathcal{L}$ is a collection of mutually disjoint lines of $PG(3, q)$ such that for each plane π of $PG(3, q)$ containing L the intersection of π with the lines of $\mathcal{L}$ is a hyperoval (resp. an oval). The line L is called the axis of $\mathcal{T}$. We show that every tube for q even and every oval tube for q odd can be naturally embedded into a regular spread and hence admits a group of automorphisms which fixes every element of $\mathcal{T}$ and acts regularly on each of them. For q odd we obtain a classification of oval tubes up to projective equivalence. Furthermore, we characterize the reguli in $PG(3, q), q$ odd, as oval tubes which admit more than one axis.

1. Introduction

A *partial tube* in $PG(3, q)$ is a pair $\mathcal{T} = \{L, \mathcal{L}\}$, where $\{L\} \cup \mathcal{L}$ is a collection of mutually disjoint lines of $PG(3, q)$ such that for each plane π of $PG(3, q)$ containing L the intersection of π with the lines of $\mathcal{L}$ is an arc. $\mathcal{T}$ is called a *tube* if each of these arcs is complete. It follows that tubes exists only for q even and that $\mathcal{L}$ contains $q + 2$ lines if $\mathcal{T}$ is a tube. If $\mathcal{L}$ contains $q + 1$ lines then $\mathcal{T}$ is called an *oval tube*. An obvious example of an oval tube is obtained by taking for $\mathcal{L}$ the lines of a regulus and for L any exterior line of the underlying hyperbolic quadric of $\mathcal{L}$. An oval tube of this type is called a quadric tube. If q is even, then $\mathcal{L}$ can be extended by the line $L^\perp$ which is the image of L under the polarity associated with $\mathcal{L}$ to form a tube. The line L is called the *axis* of the partial tube $\mathcal{T}$.

Tubes were introduced in [3] in connection with a construction problem for flat $\pi.C_2$ geometries, cp. [9].

Examples of partial tubes can be obtained as follows. Let L be a line of a regular spread in $PG(3, q)$. Choose a plane π_0 through L and let $\Omega \subset \pi_0 \setminus L$ be an arc. If $\mathcal{L}$ denotes the lines of the regular spread passing through the points of Ω, then $\mathcal{T} = \{L, \mathcal{L}\}$ is a partial tube. If Ω is a hyperoval or an oval then $\mathcal{T}$ is a tube or an oval tube. We are going to prove that in fact all tubes for q even and all oval tubes for q odd are obtained is this way. Since for odd q all ovals are conics, we obtain a complete classification up to projective equivalence in the odd order case. It turnes out that there are precisely $\frac{3q-1}{4}$ or $\frac{3q-3}{4}$ equivalence classes if $q \equiv 3 \bmod 4$ or $q \equiv 1 \bmod 4$, respectively.

In order to describe these tubes algebraically it seems convenient to introduce coordinates from $GF(q^2)$. Let V be the 4-dimensional $GF(q)$-vector space $GF(q^2) \times GF(q^2)$. Put $L = \{0\} \times GF(q^2)$ and $L(a, b) = \{(z, az + b\overline{z}) | z \in GF(q^2)\}$ for $a, b \in GF(q^2)$. The sets $L(a,b), a,b \in GF(q^2)$, are precisely the 2-dimensional subspaces of V which are complementary to L. The set $\mathcal{B} = \{L\} \cup \{L(m,0) | m \in GF(q^2)\}$ is a regular spread. If we take the elements of this spread as points and all reguli contained in it as circles, then we get a model of the Miquelian inversive plane $I(q)$ with pointset $GF(q^2) \cup \{\infty\}$. If we map $L(m,0)$ to m and L to ∞, then the reguli contained in $\mathcal{B}$ which do not contain L are mapped to the circles. Here, a circle is a set of the form $\{aw+b | w \in GF(q^2), w\overline{w} = 1\}$ with $a,b \in GF(q^2), a \neq 0$.

Let Ω be an arc in the affine plane $GF(q^2)$ and put $\mathcal{L} = \{L(a,0) | a \in \Omega\}$. Then $\mathcal{T} = \{L, \mathcal{L}\}$ is a partial tube. $\mathcal{T}$ is a quadric tube if and only if Ω is a circle.

A partial tube is called central if it admits a group of automorphisms which fixes all elements of $\mathcal{L}$ and acts regularly on each of them.

Proposition 1.1. *A partial tube is central if and only if it is isomorphic to one of the examples just described.*

Proof. This is proved in [3], Theorem 3.2 for tubes, but the argument carries over to partial tubes. □

2. Tubes of even order

For any three mutually skew lines L_1, L_2, L_3 of $PG(3,q)$ we denote the regulus spanned by them by $\mathcal{R}(L_1, L_2, L_3)$

Lemma 2.1. *Let $\mathcal{T} = \{L, \mathcal{L}\}$ be a partial tube and let $\mathcal{L} = \{L_0, \ldots, L_m\}$. Then $\mathcal{A}_n = \cup_{i \neq n} \mathcal{R}(L, L_i, L_n)$ is a partial spread of $PG(3,q)$ for $n = 0, \ldots, m$.*

Proof. Let G be any line of $PG(3,q)$ which intersects L and L_n and let π be the plane spanned by L and G. Since G intersects L_n and the intersection of π with the lines of $\mathcal{L}$ is an arc there is at most one $i \in \{0, \ldots, m\} \setminus \{n\}$ with $G \cap L_i \neq \emptyset$. It follows that there is at most one $i \in \{0, \ldots, m\} \setminus \{n\}$ such that G is a transversal of $\mathcal{R}(L, L_i, L_n)$ and hence $\mathcal{A}_n$ is a partial spread. □

Proposition 2.2. *Let $\mathcal{T}$ be a tube with q even. Then the partial spread $\mathcal{A}_n$ is a regular spread for $n = 0, \ldots, q+2$ and these spreads all coincide.*

Proof. By the preceding lemma $\mathcal{A}_n$ is a partial spread. Since $\mathcal{A}_n$ contains $2 + (q+1)(q-1) = q^2 + 1$ lines it is actually a spread.

Consider the following incidence structure $\mathcal{M}$. Points of $\mathcal{M}$ are the lines of $PG(3,q)$ that intersect L and L_n and circles of $\mathcal{M}$ are the reguli of $PG(3,q)$ that admit L and L_n as transversals. Then it is well-known that $\mathcal{M}$ is isomorphic to the Miquelian Minkowski plane over $GF(q)$, cp. eg. [1: III 4. Satz 5.1].

The reguli opposite to the $\mathcal{R}(L, L_i, L_n), i \in \{0, \ldots, q+2\} \setminus \{n\}$, constitute a flock of this Minkowski plane and $\mathcal{A}_n$ is the spread associated with this flock, cp. [4]. By a result of Thas [10], the flock is linear since q is even and hence $\mathcal{A}_n$ is regular.

Under the Plücker mapping the regular spreads correspond to intersections of the Klein quadric with certain projective spaces of rank 3. It follows that two regular spreads which have 4 common lines that are not contained in a regulus are the same. Since each of the spreads $\mathcal{A}_n, n = 0, \dots, q+2$ contains the lines $L, L_0, \dots, L_{q+2}$, they all coincide. $\square$

This result immediately implies the following

Theorem 2.3. *Every tube of even order is central.*

3. Oval tubes of odd order

For oval tubes of odd order the situation is more complicated. In this case, every partial spread $\mathcal{A}_n$ is associated with a partial flock of a hyperbolic quadric of deficiency one. Partial flocks of this type have been investigated by Johnson [5], and there are examples of such partial flocks which cannot be extended to a flock. In any case, the partial spread $\mathcal{A}_n \setminus \{L, L_n\}$ can be extended to a spread by a collection of transversals to L and L_n. The partial flock can be extended to a flock if and only if these transversals form a regulus. There are known counterexamples for $q = 4, 5, 9$, cp. [5], [6], [2].

In [3] the following description for partial tubes has been given. Let V be the 4-dimensional vector space $GF(q)^2 \times GF(q)^2$ and put $L = \{0\} \times GF(q)^2$. Then every line of $PG(V)$ which is disjoint from L is the graph of a unique linear mapping from $GF(q)^2$ to $GF(q)^2$, which we identify with its matrix. A collection $A_0, \dots, A_m$ of 2×2 matrices defines a partial tube with axis L if and only if the following two conditions are satisfied:

(i) $A_i - A_j$ is nonsingular for $i \neq j$

(ii) for any vector $v \neq 0$ and any distinct $i, j, k \in \{0, \dots, m\}$ the points $A_i v, A_j v, A_k v$ are affine independent.

Note that (ii) is equivalent to $\Omega_v = \{A_0 v, \dots, A_m v\}$ being an $(m+1)$-arc (in particular an oval if $m = q$) for every $v \neq 0$.

Lemma 3.1. *A collection $A_0, \dots, A_m$ of 2×2 matrices defines a partial tube with axis L if and only if they satisfy (i) and*

(ii') For all $(\lambda, \mu) \in GF(q)^2 \setminus \{(0,0)\}$ and all distinct $i, j, k \in \{0, \dots, m\}$ the matrix $\lambda(A_j - A_i) + \mu(A_k - A_i)$ is nonsingular.

Proof. Condition (ii) is equivalent to the following requirement. For all $(\lambda, \mu) \in GF(q)^2 \setminus \{(0,0)\}$, for all $v \neq 0$ and for all distinct i, j, k there holds $\lambda(A_j - A_i)v + \mu(A_k - A_i)v \neq 0$. But this is also equivalent to (ii'). $\square$

This lemma is in fact equivalent to Proposition 2.1 since it easy to see that $\{A_n + \lambda(A_i - A_n) | \lambda \in GF(q), i \neq n\}$ is a (partial) matrix spread set for the partial spread $\mathcal{A}_n$.

This follows from the fact that the reguli of $PG(V)$ which contain L correspond to the affine lines of the space of 2×2 matrices, cp. e.g. [8] or [7: Lemma 4.11].

Lemma 3.2. *Let $\Omega = \{v_0, \ldots, v_q\} \subset GF(q)^2$ be a conic. Then $\sum_{i=0}^{q} v_i$ is the midpoint of Ω.*

Proof. This is obviously true if the midpoint of Ω is the origin.

Let v be the midpoint of Ω, then $\Omega - v$ is a conic whose midpoint is the origin. Thus we get

$$0 = \sum_{i=0}^{q} (v_i - v) = (\sum_{i=0}^{q} v_i) - (q+1)v = (\sum_{i=0}^{q} v_i) - v$$

and the result follows. □

Corollary 3.3. *The midpoint of Ω_v is given by $m_v = (\sum_{j=0}^{q} A_j)v$.*

Lemma 3.4. *For every $i \in \{0, \ldots, q\}$ there exists $k = \sigma(i) \in \{0, \ldots, q\}$ such that $A_i + A_{\sigma(i)} = 2(\sum_{j=0}^{q} A_j)$.*

Proof. Given $i \in \{0, \ldots, q\}$ and $v \in GF(q)^2 \setminus \{0\}$ there exists $k \in \{0, \ldots, q\} \setminus \{i\}$ such that $A_i v + A_k v = 2(\sum_{j=0}^{q} A_j)v$, but k is dependent on v. Since there are $q+1$ mutually linear independent vectors in $GF(q)^2$ but only q choices for k, there are linear independent vectors v and w such that $A_i v + A_k v = 2(\sum_{j=0}^{q} A_j)v$ and $A_i w + A_k w = 2(\sum_{j=0}^{q} A_j)w$. It follows that $A_i + A_k = 2(\sum_{j=0}^{q} A_j)$. □

If we replace every A_i by $A_i - (\sum_{j=0}^{q} A_j)$ we may assume that our collection of matrices is closed unter taking additive inverses and that all conics Ω_v are centrally symmetric with respect to the origin. Moreover, we may renumber the matrices A_i with indices from $\{\pm 1, \ldots, \pm \frac{q+1}{2}\}$ such that $A_{-i} = -A_i$ for all $i \in \{\pm 1, \ldots, \pm \frac{q+1}{2}\}$.

By identifying $GF(q)^2$ and $GF(q^2)$ we may assume that for each $i \in \{\pm 1, \ldots, \pm \frac{q+1}{2}\}$ there are $a_i, b_i \in GF(q^2)$ such that $A_i v = a_i v + b_i \overline{v}$ for all $v \in GF(q^2)$. We may also assume that $a_1 = 1, b_1 = 0$ and $b_2 = 0$, i.e. that A_1 is the identity and that A_2 is linear over $GF(q^2)$. Note that the elements of $\mathcal{L}$ are now precisely the sets $L(a_i, b_i), i \in \{\pm 1, \ldots, \pm \frac{q+1}{2}\}$. Since division by non-zero elements of $GF(q^2)$ is linear over $GF(q)$, the sets

$$\Omega'_v = \{\frac{A_i v}{v} | i \in \{\pm 1, \ldots, \pm \frac{q+1}{2}\}\} = \{a_i + b_i \frac{\overline{v}}{v} | i \in \{\pm 1, \ldots, \pm \frac{q+1}{2}\}\}$$

are conics for all $v \in GF(q^2) \setminus \{0\}$. By Hilbert's theorem 90, the elements of the form $\frac{\overline{v}}{v}, v \in GF(q^2)$, are precisely the elements of norm 1. Hence the conics $\Omega'_v, v \in GF(q^2) \setminus \{0\}$, coincide with the conics

$$\Omega^*_w = \{a_i + b_i w | i \in \{\pm 1, \ldots, \pm \frac{q+1}{2}\}\}, w \in GF(q^2), w\overline{w} = 1.$$

Note that each of these conics passes through the four points ± 1 and $\pm a_2$, while the other points are moving on circles.

Theorem 3.5. *Every oval tube of odd order is central.*

Proof. We have to show that $b_i = 0$ for $i \in \{\pm 1, \ldots, \pm \frac{q+1}{2}\}$. Since a conic is uniquely determined by five points, it is actually sufficient to show that $b_i = 0$ for one $i \in \{\pm 3, \ldots, \pm \frac{q+1}{2}\}$.

If $q = 3$ there is nothing to prove.

If $q = 5$, then the union of the six lines connecting any two of the four points $\pm 1, \pm a_2$ contains 21 points. The remaining set of 4 points cannot contain a proper circle and hence $\pm b_3 = 0$.

If $q = 7$, then the union of the six lines connecting any two of the four points $\pm 1, \pm a_2$ contains 33 points. The remaining set of 16 points cannot contain four distinct circles since their union contains at least 20 points.

From now on we assume $q \geq 9$.

Choose $i, j \in \{\pm 3, \ldots, \pm \frac{q+1}{2}\}, i \neq \pm j$, and $w \in GF(q^2)$ with $w\overline{w} = 1$. Put $x = a_i + b_i w$ and $y = a_j + b_j w$.

We are going to apply Pascal's theorem to the hexagon $\{1, -1, y, -a_2, a_2, x\}$, which lies on the conic Ω_w^*.

The lines $1 \vee -1$ and $a_2 \vee -a_2$ intersect in the origin.

The intersection point of the lines $1 \vee x$ and $-a_2 \vee y$ is determined by the equation

$$x + t_1(1 - x) = y + t_2(-a_2 - y), t_1, t_2 \in GF(q).$$

This yields

$$t_2 = \frac{x - y + t_1(1 - x)}{-a_2 - y} = \overline{t_2} = \frac{\overline{x} - \overline{y} + t_1(1 - \overline{x})}{-\overline{a_2} - \overline{y}}$$

and hence

$$t_1 = \frac{(\overline{x} - \overline{y})(a_2 + y) - (x - y)(\overline{a_2} + \overline{y})}{(1 - x)(\overline{a_2} + \overline{y}) - (1 - \overline{x})(a_2 + y)}.$$

Similarly, the intersection point of the lines $a_2 \vee x$ and $-1 \vee y$ is determined by the equation

$$x + t_3(a_2 - x) = y + t_4(-1 - y), t_3, t_4 \in GF(q).$$

This yields

$$t_3 = \frac{y - x - t_4(1 + y)}{a_2 - x} = \overline{t_3} = \frac{\overline{y} - \overline{x} - t_4(1 + \overline{y})}{\overline{a_2} - \overline{x}}$$

and hence

$$t_4 = \frac{(\overline{y} - \overline{x})(a_2 - x) - (y - x)(\overline{a_2} - \overline{x})}{-(1 + y)(\overline{a_2} + \overline{x}) + (1 + \overline{y})(a_2 - x)}.$$

Pascal's theorem now says that the line joining the two points $x + t_1(1 - x)$ and $y + t_4(-1 - y)$ passes through the origin. This is equivalent to the requirement $\frac{x+t_1(1-x)}{y-t_4(1+y)} \in GF(q)$, which is in turn equivalent to

$$(x + t_1(1 - x))\overline{(y - t_4(1 + y))} = \overline{(x + t_1(1 - x))}(y - t_4(1 + y)). \qquad (*)$$

From our computations of t_1 and t_4 we get

$$\begin{aligned}
x + t_1(1 - x) &= \frac{x((1 - x)(\overline{a_2} + \overline{y}) - (1 - \overline{x})(a_2 + y)) + (1 - x)((\overline{x} - \overline{y})(a_2 + y) - (x - y)(\overline{a_2} + \overline{y}))}{(1 - x)(\overline{a_2} + \overline{y}) - (1 - \overline{x})(a_2 + y)} \\
&= \frac{(x - (x - y))(1 - x)(\overline{a_2} + \overline{y}) + ((1 - x)(\overline{x} - \overline{y}) - x(1 - \overline{x}))(a_2 + y)}{(1 - x)(\overline{a_2} + \overline{y}) - (1 - \overline{x})(a_2 + y)} \\
&= \frac{y(1 - x)(\overline{a_2} + \overline{y}) + (\overline{x} - x - (1 - x)\overline{y})(a_2 + y)}{(1 - x)(\overline{a_2} + \overline{y}) - (1 - \overline{x})(a_2 + y)} \\
&= \frac{(\overline{x} - x)(a_2 + y) + y(1 - x)\overline{a_2} - (1 - x)\overline{y}a_2}{(1 - x)(\overline{a_2} + \overline{y}) - (1 - \overline{x})(a_2 + y)}
\end{aligned}$$

and

$$\begin{aligned}
y - t_4(1 + y) &= \frac{y(-(1 + y)(\overline{a_2} - \overline{x}) + (1 + \overline{y})(a_2 - x)) - (1 + y)((\overline{y} - \overline{x})(a_2 - x) - (y - x)(\overline{a_2} - \overline{x}))}{-(1 + y)(\overline{a_2} - \overline{x}) + (1 + \overline{y})(a_2 - x)} \\
&= \frac{(-y + (y - x))(1 + y)(\overline{a_2} - \overline{x}) + (-(1 + y)(\overline{y} - \overline{x}) + y(1 + \overline{y}))(a_2 - x)}{-(1 + y)(\overline{a_2} - \overline{x}) + (1 + \overline{y})(a_2 - x)} \\
&= \frac{-x(1 + y)(\overline{a_2} - \overline{x}) + (y - \overline{y} + (1 + y)\overline{x})(a_2 - x)}{-(1 + y)(\overline{a_2} - \overline{x}) + (1 + \overline{y})(a_2 - x)} \\
&= \frac{(y - \overline{y})(a_2 - x) - x(1 + y)\overline{a_2} + (1 + y)\overline{x}a_2}{-(1 + y)(\overline{a_2} - \overline{x}) + (1 + \overline{y})(a_2 - x)}
\end{aligned}$$

Note that the denominators of the expressions obtained for the points $y - t_4(1 + y)$ and $x + t_1(1 - x)$ are both skew with respect to the conjugation mapping and hence cancel in equation $(*)$. This equation now becomes

$$\begin{aligned}
&((\overline{x} - x)(a_2 + y) + (x - 1)(\overline{y}a_2 - y\overline{a_2}))\,((\overline{y} - y)(\overline{a_2} - \overline{x}) + (\overline{y} + 1)(x\overline{a_2} - \overline{x}a_2)) \\
&\quad = ((x - \overline{x})(\overline{a_2} + \overline{y}) + (\overline{x} - 1)(y\overline{a_2} - \overline{y}a_2))\,((y - \overline{y})(a_2 - x) + (y + 1)(\overline{x}a_2 - x\overline{a_2}))
\end{aligned}$$

which leads to

$$\begin{aligned}
&(\overline{x} - x)(\overline{y} - y)(a_2 + y)(\overline{a_2} - \overline{x}) + (x - 1)(\overline{y} + 1)(\overline{y}a_2 - y\overline{a_2})(x\overline{a_2} - \overline{x}a_2) \\
&\quad + (\overline{x} - x)(\overline{y} + 1)(a_2 + y)(x\overline{a_2} - \overline{x}a_2) + (x - 1)(\overline{y} - y)(\overline{a_2} - \overline{x})(\overline{y}a_2 - y\overline{a_2}) \\
=&(x - \overline{x})(y - \overline{y})(\overline{a_2} + \overline{y})(a_2 - x) + (\overline{x} - 1)(y + 1)(y\overline{a_2} - \overline{y}a_2)(\overline{x}a_2 - x\overline{a_2}) \\
&\quad + (x - \overline{x})(y + 1)(\overline{a_2} + \overline{y})(\overline{x}a_2 - x\overline{a_2}) + (\overline{x} - 1)(y - \overline{y})(a_2 - x)(y\overline{a_2} - \overline{y}a_2).
\end{aligned}$$

Subtracting the right hand side from the left yields

$$\begin{aligned}
0 &= (\overline{x}-x)(\overline{y}-y)(y\overline{a_2}-a_2\overline{x}-y\overline{x}-\overline{y}a_2+\overline{a_2}x+\overline{y}x)\\
&\quad+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x\overline{y}+x-\overline{y}-\overline{x}y-\overline{x}+y)\\
&\quad+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(\overline{y}a_2+a_2+y-y\overline{a_2}-\overline{a_2}-\overline{y})\\
&\quad+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{a_2}+\overline{x}-\overline{x}a_2+a_2-x)\\
&=(\overline{x}-x)(\overline{y}-y)(\overline{y}x-y\overline{x})+(\overline{x}-x)(\overline{y}-y)(\overline{a_2}x-a_2\overline{x})+(\overline{x}-x)(\overline{y}-y)(y\overline{a_2}-\overline{y}a_2)\\
&\quad+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x\overline{y}-\overline{x}y)+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x-\overline{x}+y-\overline{y})\\
&\quad+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(y-\overline{y})+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(\overline{y}a_2-y\overline{a_2})+(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)(a_2-\overline{a_2})\\
&\quad+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(\overline{x}-x)+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)+(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})(a_2-\overline{a_2})\\
&=(\overline{x}-x)(\overline{y}-y)(\overline{y}x-y\overline{x})+(\overline{y}a_2-y\overline{a_2})(x\overline{a_2}-\overline{x}a_2)(x\overline{y}-\overline{x}y)\\
&\quad+(a_2-\overline{a_2})(\overline{x}-x)(x\overline{a_2}-\overline{x}a_2)+(a_2-\overline{a_2})(\overline{y}-y)(\overline{y}a_2-y\overline{a_2})
\end{aligned}$$

Since $x=a_i+b_iw$ and $y=a_j+b_jw$ the last expression is a polynomial $P(w,\overline{w})$ of degree 3 in w and $\overline{w}$. Since $w\overline{w}=1$ the term $P(w,\overline{w})w^3$ is a polynomial of degree 6 in w. This polynomial has at least the $q+1$ elements $w\in GF(q^2)$ with $w\overline{w}=1$ as zeroes, and so all coefficients are zero since $q\geq 9$. We have

$$\begin{aligned}
\overline{x}-x &= -b_iw+\overline{a_i}-a_i+\overline{b_i}\overline{w},\\
\overline{y}-y &= -b_jw+\overline{a_j}-a_j+\overline{b_j}\overline{w},\\
\overline{y}x-y\overline{x} &= (\overline{a_j}+\overline{b_j}\overline{w})(a_i+b_iw)-(a_j+b_jw)(\overline{a_i}+\overline{b_i}\overline{w})\\
&= (\overline{a_j}b_i-b_j\overline{a_i})w+\overline{a_j}a_i+\overline{b_j}b_i-a_j\overline{a_i}-b_j\overline{b_i}+(\overline{b_j}a_i-a_j\overline{b_i})\overline{w},\\
x\overline{a_2}-\overline{x}a_2 &= (a_i+b_iw)\overline{a_2}-(\overline{a_i}+\overline{b_i}\overline{w})a_2\\
&= b_i\overline{a_2}w+a_i\overline{a_2}-\overline{a_i}a_2-\overline{b_i}a_2\overline{w},\\
\overline{y}a_2-y\overline{a_2} &= (\overline{a_j}+\overline{b_j}\overline{w})a_2-(a_j+b_jw)\overline{a_2}\\
&= -b_j\overline{a_2}w+\overline{a_j}a_2-a_j\overline{a_2}+\overline{b_j}a_2\overline{w}.
\end{aligned}$$

For the coefficient of w^3 in $P(w,\overline{w})$ we get

$$(-b_i)(-b_j)(\overline{a_j}b_i-b_j\overline{a_i})+(-b_j\overline{a_2})(b_i\overline{a_2})(\overline{a_j}b_i-b_j\overline{a_i})=b_ib_j(\overline{a_j}b_i-b_j\overline{a_i})(1-\overline{a_2}^2).$$

If $b_i=0$ for at least one $i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}$ the theorem is proved. So we may assume that $b_i\neq 0$ for all $i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}$. Since $\overline{a_2}^2\neq 1$ this yields

$$\overline{a_j}b_i-b_j\overline{a_i}=0 \quad \text{for all} \quad i,j\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}, i\neq\pm j$$

It follows that $b_i=c\overline{a_i}$ for all $i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}$ for some constant c. For the conics Ω_w^* this implies

$$\Omega_w^*=\{a_i+c\overline{a_i}w|i\in\{\pm3,\ldots,\pm\frac{q+1}{2}\}\}\cup\{\pm1,\pm a_2\} \quad \text{for all} \quad w\in GF(q^2), w\overline{w}=1.$$

Consider now the $GF(q)$-linear mapping $\lambda_w : GF(q^2) \to GF(q^2)$ which maps $z \in GF(q^2)$ to $z + cw\overline{z}$. The inverse of λ_w is given by $\lambda_w^{-1}(z) = \frac{1}{1-c\overline{c}}(z - cw\overline{z})$. For each $w \in GF(q^2), w\overline{w} = 1$ the set

$$\lambda_w^{-1}(\Omega_w^*) = \{a_i | i \in \{\pm 3, \ldots, \pm\frac{q+1}{2}\}\} \cup \{\pm\frac{1-cw}{1-c\overline{c}}, \pm\frac{a_2 - cw\overline{a_2}}{1-c\overline{c}}\}$$

is a conic in the affine plane $GF(q^2)$. Since $q \geq 9$ these conics are all the same and hence $1 - cw$ and $a_2 - cw\overline{a_2}$ are independent of w. It follows that $c = 0$ and hence $b_i = 0$ for all $i \in \{\pm 3, \ldots, \pm\frac{q+1}{2}\}$. □

Theorem 3.6. *Let q be odd. Let $b \in GF(q^2)$ with $b\overline{b} \neq 1$ and set $\mathcal{L}(b) = \{L(w + b\overline{w}, 0) | w \in GF(q^2), w\overline{w} = 1\}$. Then $\mathcal{T}(b) = \{L, \mathcal{L}(b)\}$ is an oval tube of $PG(V)$. Every oval tube of $PG(3, q)$ is projectively equivalent to some $\mathcal{T}(b)$. The tubes $\mathcal{T}(b_1)$ and $\mathcal{T}(b_2)$ are projectively equivalent if and only $b_2 = b_1c^2, b_2 = b_1^{-1}c^2, b_2 = \overline{b}_1c^2$ or $b_2 = \overline{b}_1^{-1}c^2$ for some $c \in GF(q^2)$ with $c\overline{c} = 1$. If $q \equiv 3 \mod 4$ there are precisely $\frac{3q-1}{4}$ equivalence classes of oval tubes and if $q \equiv 1 \mod 4$ there are $\frac{3q-3}{4}$ equivalence classes. $\mathcal{T}(b)$ is a quadric tube if and only if $b = 0$.*

Proof. By Theorem 3.5 we know that every oval tube is projectively equivalent to an oval tube $\mathcal{T} = \{L, \mathcal{L}\}$ which has axis $L = \{0\} \times GF(q^2)$ and is contained in the regular spread $\mathcal{B} = \{L\} \cup \{L(m, 0) | m \in GF(q^2)\}$. We also know that the set $\Omega = \{m \in GF(q^2) | L(m, 0) \in \mathcal{L}\}$ is a conic in the affine plane $GF(q^2)$, and we may assume that this conic is centered at the origin. It follows that there are $a, b \in GF(q^2)$ with $a\overline{a} \neq b\overline{b}$ such that $\Omega = \Omega(a, b) = \{az + b\overline{z} | z \in GF(q^2), z\overline{z} = 1\}$. Let the corresponding tube be called $\mathcal{T}(a, b)$. Note that a and b are not uniquely determined by Ω, but that $\Omega(a, b) = \Omega(ac, b\overline{c}) = \Omega(b\overline{c}, ac)$, and hence also $\mathcal{T}(a, b) = \mathcal{T}(ac, b\overline{c}) = \mathcal{T}(b\overline{c}, ac)$, for all $c \in GF(q^2), c\overline{c} = 1$.

From now on we assume that $\mathcal{T} = \mathcal{T}(a, b)$ is not the quadric tube, i.e that $a \neq 0 \neq b$. Then $\mathcal{B}$ is the only regular spread which contains $\mathcal{T}$. It follows that two such tubes are projectively equivalent if and only if one is mapped onto the other by a $GF(q)$-linear mapping of the vector space $GF(q^2)^2$ which fixes the spread $\mathcal{B}$ and the lines L and $L(0, 0)$. These linear mappings are of the form $A(d_1, d_2) : (z, w) \mapsto (d_1z, d_2w)$ or $B(d_1, d_2) : (z, w) \mapsto (d_1\overline{z}, d_2\overline{w}), d_1, d_2 \in GF(q^2), d_1 \neq 0 \neq d_2$. A short calculation shows that $A(d_1, d_2)$ and $B(d_1, d_2)$ map $\mathcal{T}(a, b)$ to $\mathcal{T}(da, db)$ and $\mathcal{T}(d\overline{b}, d\overline{a})$, respectively, where $d = \frac{d_2}{d_1}$. It follows that $\mathcal{T}(a, b)$ is projectively equivalent to $\mathcal{T}(1, \frac{b}{a}) = \mathcal{T}(\frac{b}{a})$.

Assume now that $\mathcal{T}(b_1)$ and $\mathcal{T}(b_2), b_1 \neq 0 \neq b_2$, are projectively equivalent. Then we get that $(1, b_2)$ is equal to one of $(dc, db_1\overline{c}), (db_1\overline{c}, dc), (d\overline{b}_1c, d\overline{c})$ or $(d\overline{c}, d\overline{b}_1c)$ for some $c, d \in GF(q^2), c\overline{c} = 1, d \neq 0$. These four cases lead to $b_2 = b_1\overline{c}^2, b_2 = b_1^{-1}c^2, b_2 = \overline{b}_1^{-1}\overline{c}^2$ or $b_2 = \overline{b}_1c^2$, respectively.

It remains to determine the number of isomorphism classes.

This is essentially the problem of counting the orbits of a group of order $2(q + 1)$ acting on the set $M = \{b \in GF(q^2) | 0 \neq b\overline{b} \neq 1\}$, which contains $(q + 1)(q - 2)$ elements. The group consists of the following mappings

(i) $b \mapsto bc^2, c\overline{c} = 1,$

(ii) $b \mapsto b^{-1}c^2, c\overline{c} = 1$,

(iii) $b \mapsto \overline{b}^{-1}c^2, c\overline{c} = 1$,

(iv) $b \mapsto \overline{b}c^2, c\overline{c} = 1$.

Since $b\overline{b} \neq 1$ the mappings of type (ii) have no fixed points, and so the dihedral group of order $q+1$ which comprises the mappings of type (i) and (ii) acts freely on M. So the stabilizer of any point contains at most two elements.

Assume that a mapping of type (iii) fixes $b \in M$, then we get $b\overline{b} = c^2$. Since $1 \neq b\overline{b} \in GF(q)$ and $c\overline{c} = 1$ this yields $b\overline{b} = -1 = c^2$. The equation $c^2 = -1$ has a solution with $c\overline{c} = 1$ if and only if $4|q+1$. In this case there are precisely $q+1$ elements of M which have a mapping of type (iii) in their stabilizer.

Assume now that a mapping of type (iv) fixes a point $b \in M$, then we get $\frac{b}{\overline{b}} = c^2$ with $c\overline{c} = 1$. There are precisely $\frac{(q+1)(q-1)}{2}$ elements $b \in GF(q^2) \setminus \{0\}$ for which this equation has a solution, but those with $b\overline{b} = 1$ are among them. So there are $\frac{(q+1)(q-3)}{2}$ elements of M which have a mapping of type (iv) in their stabilizer.

Now we can count as follows. If $q \equiv 3 \bmod 4$ there are $q+1+\frac{(q+1)(q-3)}{2} = \frac{(q+1)(q-1)}{2}$ elements with a stabilizer of order 2 and hence $(q+1)(q-2) - \frac{(q+1)(q-1)}{2} = \frac{(q+1)(q-3)}{2}$ elements with trivial stabilizer. The number of orbits thus becomes

$$\frac{\frac{(q+1)(q-1)}{2}}{(q+1)} + \frac{\frac{(q+1)(q-3)}{2}}{2(q+1)} = \frac{3q-5}{4}.$$

Taking into account the quadric tube we have to add one to this number and so we have $\frac{3q-1}{4}$ equivalence classes of oval tubes.

If $q \equiv 1 \bmod 4$ a similar computation shows that there are $\frac{3q-3}{4}$ equivalence classes. □

Remarks 3.7. a) If $q \equiv 1 \bmod 4$ it is possible to construct a system of representatives for the projective equivalence classes of oval tubes as follows. Choose an element $\mu \in GF(q^2) \setminus GF(q)$ with $\mu^2 \in GF(q)$. Then every oval tube is equivalent to some $\mathcal{T}(b)$ with $b \in GF(q) \setminus \{\pm 1\} \cup \mu \cdot GF(q)$. If $b \in GF(q) \setminus \{0, \pm 1\}$ there are two representatives, namely b and b^{-1}, and if $b \in \mu \cdot GF(q) \setminus \{0\}$ there are four, namely $b, -b = \overline{b}, b^{-1}$ and $-b^{-1} = \overline{b}^{-1}$.

If $q \equiv 3 \bmod 4$ there seems to be no obvious choice for a system of representatives.

b) Our classification is in fact equivalent to the classification of the conics in the affine plane under the group of similarities.

c) In principle, it is also possible to classify tubes up to equivalence under $P\Gamma L(4,q)$. The field automorphisms just act in their standard way on M.

Lemma 3.8. *Let Ω be a subset of the Miquelian inverse plane $I(q)$, q odd, and assume that there are two distinct points a, b of $I(q)$ such that Ω is a conic in the affine planes $I(q)_a$ and $I(q)_b$. Then Ω is a circle of $I(q)$.*

Proof. We may identify $I(q)_a$ with the affine plane $GF(q)^2$ and we may assume that $b = (0, 0)$. The conics in the affine plane which do not pass through b are the sets $\{(x, y) \in GF(q)^2 | a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y + c = 0\}$ for $a_{ij}, b_i, c \in GF(q), c \neq 0, a_{12}^2 - a_{11}a_{22}$ a nonsquare in $GF(q)$.

We may also fix a nonsquare ϱ in $GF(q)$ such that $I(q)$ is the inversive plane associated with the field extension $GF(q)[x]/(x^2 - \varrho) : GF(q)$. Then the circles are precisely the conics with $a_{12} = 0$ and $a_{22} = -\varrho a_{11}$. The mapping $\sigma : I(q) \to I(q)$ which exchanges a and b and maps $(x, y) \neq (0, 0)$ to $(\frac{x}{x^2-\varrho y^2}, \frac{y}{x^2-\varrho y^2})$ is an involutorial automorphism of $I(q)$.

It is sufficient to show that if Ω and $\sigma(\Omega)$ are both conics in $GF(q)^2 \setminus \{(0, 0)\}$, then Ω is a circle.

Assume that

$$\Omega = \{(x, y) \in GF(q)^2 | a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y + c = 0\},$$

then we get

$$\sigma(\Omega) = \{(x, y) \in GF(q)^2 | a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + (b_1x + b_2y)(x^2 - \varrho y^2) + c(x^2 - \varrho y^2)^2 = 0\}.$$

This set is a conic if and only if the polynomial

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + (b_1x + b_2y + c(x^2 - \varrho y^2))(x^2 - \varrho y^2)$$

is a product of two polynomials of degree 2. Since this polynomial contains no terms of degree 0 and 1, one of the factors must be the polynomial $a_{11}x^2 + 2a_{12}xy + a_{22}y^2$. It follows that $a_{11}x^2 + 2a_{12}xy + a_{22}y^2$ divides $x^2 - \varrho y^2$ and hence $a_{12} = 0$ and $a_{22} = -\varrho a_{11}$, i.e. Ω is a circle. □

Proposition 3.9. *An oval tube of odd order which has more than one axis is isomorphic to the quadric tube.*

Proof. Let $\mathcal{T} = \{L, \mathcal{L}\}$ be an oval tube and let L' be a line of $PG(3, q)$ such that $\{L', \mathcal{L}\}$ is also an oval tube.

If $\mathcal{T}$ is not the quadric tube then the regular spread containing $\mathcal{L}$ is uniquely determined. It follows that L' is also contained in this spread. Since the lines of this regular spread and the reguli contained in it form a model for the inversive plane $I(q)$ the result now follows from the preceding lemma. □

This proposition yields the following characterization of reguli in $PG(3, q)$.

Corollary 3.10. *Let $\mathcal{L}$ be a collection of $q + 1$ mutually skew lines in $PG(3, q), q$ odd. Assume that there are two distinct lines $L_1, L_2 \notin \mathcal{L}$ such that each line which intersects L_1 or L_2 meets at most two distinct lines of $\mathcal{L}$. Then $\mathcal{L}$ is a regulus.*

References

[1] Benz, W.: Vorlesungen über Geometrie der Algebren. Berlin–Heidelberg– New York. Springer 1973

[2] Biliotti, M. and Johnson, N. L.: Maximal Baer groups in translation planes and compatibility with homology groups. Geom Dedicata **59**, 65–101 (1996)

[3] Cameron, P. J. and Ghinelli, D.: Tubes of even order and flat $\pi.C_2$ geometries. Geom. Dedicata **55**, 265–278 (1995)

[4] Fisher, J. C. and Thas, J. A.: Flocks in $PG(3, q)$. Math. Z. **169**, 1–11 (1979)

[5] Johnson, N. L.: Flocks of hyperbolic quadrics and translation planes admitting affine homologies. J. Geom. **34**, 50–73 (1989)

[6] Johnson, N. L. and Pomareda, R.: A maximal partial flock of deficiency one of the hyperbolic quadric in $PG(3, 9)$. Simon Stevin **64**, 169–177 (1990)

[7] Knarr, N.: Translation Planes. Lecture Notes in Mathematics **1611**, 1995.

[8] Metz, R.: Der affine Raum verallgemeinerter Reguli. Geom. Dedicata **10**, 337–367 (1981)

[9] Pasini, A.: Some geometries of type $c.C_2$ and $C_2.Af$. Atti Semin. Mat. Fis. Univ. Modena **45**, 1–8 (1997)

[10] Thas, J. A.: Flocks of nonsingular ruled quadrics in $PG(3, q)$. Atti Accad. Naz. Lincei Rend. **59**, 83–85 (1975)

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