Is there a McLaughlin geometry?

Leonard H. Soicher
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road, London E1 4NS, UK
email: L.H.Soicher@qmul.ac.uk

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Dedicated to Charles Leedham-Green on the occasion of his 65th birthday

Abstract

It has long been an open problem whether or not there exists a partial geometry with parameters $(s, t, \alpha) = (4, 27, 2)$. Such a partial geometry, which we call a *McLaughlin geometry*, would have the McLaughlin graph as point graph. In this note we use tools from computational group theory and computational graph theory to show that a McLaughlin geometry cannot have certain automorphisms, nor can such a geometry satisfy the Axiom of Pasch.

1 Introduction

The McLaughlin graph M was constructed by J. McLaughlin for the construction of his sporadic simple group McL [10]. The graph M has full automorphism group $\mathrm{Aut}\,(M) \cong McL{:}\,2$, acting transitively with permutation rank 3 on the set of vertices. Indeed, M is a particularly fascinating strongly regular graph, with parameters

$$(v, k, \lambda, \mu) = (275, 112, 30, 56),$$

and in [7], it is shown that M is the unique strongly regular graph with these parameters.

Partial geometries were introduced by Bose [1] to generalize certain classes of finite geometries. Partial geometries and related concepts will be defined in the next section; a good concise introduction is contained in [5]. It has long been an open problem whether or not there exists a partial geometry with parameters $(s, t, \alpha) = (4, 27, 2)$ (see [3, 9, 5, 11]). Such a partial geometry, which we call a *McLaughlin geometry*, would have the McLaughlin graph as point graph.

In this note we use tools from computational group theory and computational graph theory to show that a McLaughlin geometry cannot have automorphisms in any of the conjugacy classes $2A, 3B, 5B, 9AB, 2B$ of $McL{:}\,2$, nor can such a geometry satisfy the Axiom of Pasch. The calculations used in this note were performed using GAP [6] and GRAPE [15], taking a total CPU-time of under 23 seconds on a 2.4 GHz PC running Linux. For the interested reader, a log-file of these calculations is available on the internet [16].

Combining the results of this note with those of Mathon [11], we deduce that the automorphism group of a McLaughlin geometry has order dividing 3, and we are led to conjecture (like some others) that there is no McLaughlin geometry. If this is so, then M would be a pseudogeometric (s, t, α)-graph attaining equality in the *Krein bound*, $(s + 1 - 2\alpha)t \leq (s - 1)(s + 1 - \alpha)^2$, but M would not be geometric, and the existence of such a graph would answer a long-standing question [3, Question 7.10]. However, we believe that determining whether or not there is a McLaughlin geometry remains a very difficult problem.

Throughout this note, we use ATLAS [4] notation for group structures and conjugacy classes.

2 Partial linear spaces, graphs and partial geometries

Let s and t be positive integers. A *partial linear space* with *parameters* (s, t), or a $\mathrm{pls}(s, t)$, is an ordered pair $(P, \mathcal{L})$, where P is a non-empty set (of *points*) and $\mathcal{L}$ is a nonempty set of $(s+1)$-subsets of P (called *lines*), such that every point is in exactly $t + 1$ lines, and every pair of distinct points are on at most one line (and so every pair of distinct lines meet in at most one point). The *point graph* of a partial linear space $(P, \mathcal{L})$ is the graph with vertex-set P where two points are joined by an edge if and only if they are distinct and on a common line. Two partial linear spaces $(P, \mathcal{L})$ and $(P', \mathcal{L}')$ are *isomorphic* if there is a bijection $\phi : P \to P'$ such that $\phi(\mathcal{L}) = \mathcal{L}'$. An *automorphism* of a partial linear space is an isomorphism onto itself. The set of all automorphisms of a partial linear space $\mathcal{S}$ forms a group, the *automorphism group* $\mathrm{Aut}\,(\mathcal{S})$ of $\mathcal{S}$. Since any automorphism of $\mathcal{S}$ permutes its points and preserves the property of two points being distinct and on a common line, we see that $\mathrm{Aut}\,(\mathcal{S})$ is a subgroup of the automorphism group of the point graph of $\mathcal{S}$.

Instead of starting with a partial linear space, we may start with a regular (simple) graph $\Gamma = (V, E)$ of degree $s(t + 1)$ and ask whether it is the point graph of some $\mathrm{pls}(s, t)$, or more generally, we may wish to determine all $\mathrm{pls}(s, t)$, up to isomorphism, having Γ as point graph. Now such a Γ is the point graph of a $\mathrm{pls}(s, t)$ if and only if there is a set $\mathcal{C}$ of $(s+1)$-cliques of Γ forming a *spread* of E (*i.e.* every edge of Γ is contained in exactly one element of $\mathcal{C}$), in which case $(V, \mathcal{C})$ is a $\mathrm{pls}(s, t)$ with point graph Γ. Indeed, it is not difficult to see that (the line sets of) a transversal of the isomorphism classes of the $\mathrm{pls}(s, t)$ with point graph Γ is given by a transversal of the $\mathrm{Aut}\,(\Gamma)$-orbits on the set of all sets $\mathcal{C}$ of $(s + 1)$-cliques such that $\mathcal{C}$ is a spread of E. Of course such spreads need not exist; it may even be that Γ contains no $(s + 1)$-clique.

For positive integers s, t, α, a *partial geometry* with *parameters* (s, t, α), or a $\mathrm{pg}(s, t, \alpha)$, is a $\mathrm{pls}(s, t)$ for which every line L and every point p not on L, there are exactly α lines through p meeting L. It is well-known that a $\mathrm{pg}(s, t, \alpha)$ has a strongly regular point graph, with parameters

$$((s + 1)(st + \alpha)/\alpha,\ s(t + 1),\ s - 1 + t(\alpha - 1),\ (t + 1)\alpha) \qquad (1)$$

(see [1]). A strongly regular graph with parameters of the form (1), with $1 \leq s, t$ and $1 \leq \alpha \leq \min\{s + 1, t + 1\}$, is called a *pseudogeometric* (s, t, α)-*graph*, and such a graph is called *geometric* if it is the point graph of at least one $\mathrm{pg}(s,t,\alpha)$. It is known that in a pseudogeometric (s,t,α)-graph Γ having an $(s+1)$-clique C, that every vertex not in C is adjacent to exactly α elements of C, and so a pseudogeometric (s,t,α)-graph Γ is geometric if and only if it is the point graph of a $\mathrm{pls}(s,t)$, and every $\mathrm{pls}(s,t)$ having Γ as point graph is a $\mathrm{pg}(s,t,\alpha)$ (see [8]).

The GRAPE package [15] for GAP [6] contains a function `PartialLinearSpaces` to classify the partial linear spaces with given point graph and parameters. (The algorithm used is similar to that independently devised by Reichard [12].) In general this can be an extremely difficult problem, but this function has been used successfully with certain pseudogeometric (s,t,α)-graphs to classify the $\mathrm{pg}(s,t,\alpha)$ having these as point graphs. One such application, described fully in the GRAPE documentation [15], shows that the Haemers partial geometry [8], a certain $\mathrm{pg}(4,17,2)$, is uniquely determined (up to isomorphism) by its point graph, as is the dual of the Haemers partial geometry, and each of these geometries has automorphism group isomorphic to A_7. Unfortunately, the problem of classifying the $\mathrm{pg}(4,27,2)$ (whose point graph would be the McLaughlin graph) seems much too difficult to handle in the same way.

3 The 5-cliques of the McLaughlin graph

In this section we collect some results about the 5-cliques of the McLaughlin graph. These facts can be easily established using the information on McL and $McL:2$ contained in the ATLAS [4, pp. 100–101], together with straightforward calculation in GAP and GRAPE, which the interested reader can find in [16], or do for herself using the copy of the McLaughlin graph (in GRAPE format) in the GRAPE package file `grh/McL`. The automorphism group of this graph can be obtained using the GRAPE command `AutGroupGraph`.

Theorem 3.1 *Let M be the McLaughlin graph, and let $G = \mathrm{Aut}\,(M) \cong McL:2$. Then M has exactly 15400 maximal cliques, all of size 5, forming just one G-orbit. The stabilizer in G of a 5-clique C is a group H of shape $3^{1+4}_{+}:4.S_5$, the normalizer in G of a $3A$-generated subgroup T of order 3, corresponding to C. This subgroup T is the centre of the derived subgroup of H, and C is the set of fixed points of T. In its action on the set of 5-cliques of M, H has just five orbits, of sizes*

$$1,\ 90,\ 1215,\ 2430,\ 11664.$$

Representative cliques in these orbits have respective intersection sizes

$$5,\ 2,\ 1,\ 0,\ 0$$

with C, and the cyclic $3A$-subgroups corresponding to these representatives generate with T respective groups of shapes

$$3,\ 3^2,\ 2{\cdot}A_4,\ 2{\cdot}A_5,\ 5^{1+2}_{+}:3.$$

4 A group-theoretical condition equivalent to the existence of a McLaughlin geometry

This short section is independent of the remainder of the paper, but contains a result we think is worth pointing out.

A McLaughlin geometry would have exactly $(275\times 28)/5 = 1540$ lines, and so if the McLaughlin graph M is geometric it would contain 1540 5-cliques meeting pairwise in at most 1 vertex. Conversely, if M has a set $\mathcal{C}$ of 1540 5-cliques meeting pairwise in at most 1 vertex, then each vertex of M would have to be contained in exactly 28 elements of $\mathcal{C}$, and so $(V(M),\mathcal{C})$ would be a $\mathrm{pls}(4,27)$ with point graph M, and so a McLaughlin geometry. From Theorem 3.1, we see that two distinct 5-cliques of M meet in more than one point (necessarily in two points) if and only if their corresponding subgroups of order 3 commute. We have thus proved:

Theorem 4.1 *There is a McLaughlin geometry if and only if the McLaughlin group has a set of 1540* $3A$*-generated subgroups of order 3, no distinct pair of which commute.*

5 A McLaughlin geometry has a small automorphism group

In this section we consider possible groups of automorphisms of a McLaughlin geometry $\mathcal{M} = (P,\mathcal{L})$. Such a group must be a subgroup of the automorphism group of the McLaughlin graph M. Moreover, if $H \leq \mathrm{Aut}\,(\mathcal{M})$ and $g \in \mathrm{Aut}\,(M)$, then $H^g \leq \mathrm{Aut}\,(\mathcal{M}^g)$, where $H^g = g^{-1}Hg$, and $\mathcal{M}^g = (P,\mathcal{L}^g)$ is a McLaughlin geometry isomorphic to $\mathcal{M}$. Hence, we restrict our attention to conjugacy class representatives of subgroups of $McL\colon 2$.

We shall apply the GRAPE package functionality which can classify the cliques with given vertex-weight sum in a vertex-weighted G-graph (see [14] and the GRAPE documentation [15]), in order to show that a McLaughlin geometry cannot have an automorphism of order 2, nor an automorphism in class $3B$ or $5B$ of $McL\colon 2$. (This technique has applicability to other pseudogeometric (s,t,α)-graphs, even very large ones whose $(s+1)$-cliques cannot be explicitly listed.) We also show that a McLaughlin geometry has no automorphism in class $9AB$ of $McL\colon 2$ (the class $9AB$ is the union of McL-classes $9A$ and $9B$).

In [11], Mathon describes the computational techniques he used to show that a McLaughlin geometry has no automorphism of order 5, 7 or 11. Combining our results with those of Mathon, it follows (see [4, p. 101]) that the automorphism group of a McLaughlin geometry is trivial or a 3-group whose nontrivial elements are all in class $3A$. Consideration of the five possibilities of subgroups generated by two $3A$-elements (see Theorem 3.1) shows that the only possibility is a $3A$-generated subgroup of order 3 (the 3^2 possibility contains $3B$-elements).

Remarks The computations required to obtain the results of this section were surprisingly quick, and Reichard [13] has since used the methods of Mathon to verify these results. We plan to apply our methods to larger pseudogeometric graphs. Reichard has done further work to establish the non-existence of a McLaughlin geometry having certain line-sets through a point. These line-sets correspond to certain "point-spreads" of a $\mathrm{pg}(3,9,1)$; all the point-spreads of such a geometry have been determined independently by Reichard and Brouwer (see [2]).

5.1 Eliminating some possible groups of automorphisms

We make use of the following:

Lemma 5.1 *Let $\mathcal{S} = (P, \mathcal{L})$ be a pls(s,t) with point graph Γ, and let $H \leq \mathrm{Aut}\,(\mathcal{S})$. Then there is a set $\mathcal{C}$ of H-invariant $(s+1)$-cliques of Γ, such that each edge of Γ fixed by H is contained in an element of $\mathcal{C}$, and no two distinct elements of $\mathcal{C}$ meet in two or more vertices.*

Proof. Suppose H fixes the edge e of Γ. There is just one $L \in \mathcal{L}$ containing e, and if $\sigma \in H$ then $e = e^{\sigma} \subseteq L^{\sigma}$, and so $L^{\sigma} = L$. Thus, the set of H-invariant edges of Γ must be covered by a set $\mathcal{C}$ of H-invariant $(s+1)$-cliques of Γ, corresponding to the H-invariant lines of $\mathcal{S}$ containing these fixed edges, and so no two distinct elements of $\mathcal{C}$ meet in two or more vertices. ∎

We shall apply the above lemma to eliminate certain cyclic subgroups H of $McL\colon 2$ as possible groups of automorphisms of a McLaughlin geometry by showing in these cases the required set $\mathcal{C}$ of fixed 5-cliques does not exist.

The McLaughlin graph has just 15400 edges and 15400 5-cliques, and so it is easy to store these on a computer and to determine which are H-invariant for a given $H \leq McL\colon 2$. However, we may use GRAPE to directly compute the H-invariant cliques of a given size (including size 2), as described below.

Let Γ be a (finite, simple) graph, and let $H \leq \mathrm{Aut}\,(\Gamma)$. We define a vertex-weighted graph $\Delta = \Delta(\Gamma, H)$, called the *collapsed complete-orbits graph of* Γ *with respect to* H, as follows. We have that v is a vertex of Δ if and only if v is an H-orbit of vertices of Γ as well as a clique of Γ. Furthermore, if v is a vertex of Δ then its *weight* is the size of v. Vertices v and w are adjacent in Δ if and only if $v \neq w$ and $v \cup w$ is a clique of Γ. Now let N be a subgroup of $\mathrm{Aut}\,(\Gamma)$ such that N normalizes H. Then N permutes the H-orbits of vertices of Γ and preserves the property of being a clique of Γ of a given size. We thus see that N acts on Δ as a group of vertex-weight preserving automorphisms. To classify the H-invariant k-cliques of Γ up to the action of N, we may use GRAPE to first compute $\Delta = \Delta(\Gamma, H)$ as an N-graph (see the GRAPE documentation for `CollapsedCompleteOrbitsGraph`) and then use the GRAPE function `CompleteSubgraphsOfGivenSize` applied to Δ to classify the cliques of Δ with vertex-weight sum k (corresponding to the H-invariant k-cliques of Γ). We can, if we desire, then list all H-invariant k-cliques of Γ.

We now assume we are given a subgroup H of the group G of automorphisms of the McLaughlin graph M, together with the set $\mathcal{E}$ of the edges of M fixed by H and the set $\mathcal{C}$ of the 5-cliques of M fixed by H and containing at least one element of $\mathcal{E}$. We then form a vertex-weighted graph Λ, with vertex-set $\mathcal{C}$, the weight of a vertex C being the number of edges in $\mathcal{E}$ contained in C, and with two distinct vertices $C, D \in \mathcal{C}$ adjacent if and only if their intersection has size at most 1. We then use the GRAPE function `CompleteSubgraphsOfGivenSize` to determine whether or not Λ has a complete-subgraph with vertex-weight sum equal to $|\mathcal{E}|$, the number of edges fixed by H. If not, then Lemma 5.1 tells us that H cannot be a group of automorphisms of a McLaughlin geometry.

We apply this method to a set of representatives H of the conjugacy classes of the non-trivial cyclic subgroups of $G = \mathrm{Aut}\,(M)$, such that H has order 2 or odd order, and fixes at least one edge of M. These are the groups $\langle x \rangle$, where x is in one of the conjugacy classes $2A, 3A, 3B, 5B, 9AB, 2B$ of G. These groups have, respectively, $280, 10, 37, 5, 1, 66$ fixed edges and $56, 91, 10, 0, 1, 110$ fixed cliques, each containing at least one fixed edge. Applying the above method, we obtain the following:

Theorem 5.2 *The automorphism group of a McLaughlin geometry contains no element in any of the conjugacy classes* $2A, 3B, 5B, 2B$ *of* $McL\colon 2$.

In particular, since $McL\colon 2$ has just two conjugacy classes of involutions, we have that the automorphism group of a McLaughlin geometry contains no element of even order.

The above method does not eliminate elements in classes $3A$ or $9AB$ (in both cases the required set of fixed cliques exists), but we can eliminate the possibility of an automorphism in class $9AB$ as follows. Let $H = \langle x \rangle$, where $x \in 9AB$. Then H fixes just one edge, $e = \{v, w\}$ say, of M. We then consider the 5-cliques of M intersecting e nontrivially, and determine that there is no H-invariant subset $\mathcal{D}$ of these cliques, such that the elements of $\mathcal{D}$ meet pairwise in at most one vertex, and the elements of $\mathcal{D}$ cover all edges of M incident to v or w (or both). This is done via further applications of the GRAPE functions `CollapsedCompleteOrbitsGraph` and `CompleteSubgraphsOfGivenSize`.

6 No McLaughlin geometry satisfies the Axiom of Pasch

A partial geometry satisfies the *Axiom of Pasch* if for any pair L_1, L_2 of distinct lines meeting in one point x, and any two further lines M_1, M_2 each meeting both L_1 and L_2 in one point other than x each, we have that M_1 and M_2 meet (see [5]). In this section, we show that there is no McLaughlin geometry satisfying the Axiom of Pasch.

As usual, let M be the McLaughlin graph, and $G = \mathrm{Aut}\,(M)$. By Theorem 3.1 we see that, up to the action of G, there is a unique pair X, Y of 5-cliques meeting in just one vertex, say x. Let x_1, x_2, x_3, x_4 be the remaining vertices of X. Now x_1 is adjacent to just two vertices of Y (since M is a pseudogeometric $(4, 27, 2)$-graph), one of which is x. Call the other y_1. As y_1 is adjacent to just two vertices of X, these must be x and x_1. Similarly, there are three further vertices y_i of Y $(i = 2, 3, 4)$, with x_i adjacent to x and y_i in Y, and y_i adjacent to x and x_i in X. Calculation (see [16]) shows that there is only one set $\{Z_1, Z_2, Z_3, Z_4\}$ of four 5-cliques of M with the properties that Z_i contains x_i and y_i and no other vertex of X or Y, and that, if $i \neq j$, then Z_i and Z_j meet in exactly one vertex. Hence, if X and Y are lines in a McLaughlin geometry (with point graph M) satisfying the Axiom of Pasch, then Z_1, Z_2, Z_3, Z_4 are lines as well. We call $\mathcal{P} := \{X, Y, Z_1, Z_2, Z_3, Z_4\}$ the *Pasch closure* of the pair $\{X, Y\}$ of 5-cliques of M (which meet in just one vertex). We remark that the $3A$-subgroups corresponding to the cliques in $\mathcal{P}$ generate a subgroup $2^{\cdot}A_8$ of G, and that the G-stabilizer of $\mathcal{P}$ is of shape $4.S_6$ (these facts are verified computationally in [16]).

Theorem 6.1 *No McLaughlin geometry satisfies the Axiom of Pasch.*

Proof. Let $\mathcal{M}$ be a McLaughlin geometry satisfying the the Axiom of Pasch and having point graph M. Without loss of generality, we assume the line set of $\mathcal{M}$ includes two given 5-cliques X and Y of M meeting in just one vertex, x say.

Let H be the stablilizer of $\{X, Y\}$ in $\mathrm{Aut}\,(M)$. Then H (of order 192) acts on the set $\mathcal{C}$ of 5-cliques containing x and meeting neither X nor Y in any other vertex. ($\mathcal{C}$ is the set of candidates for the lines through x in addition to X and Y.) Now H acting on $\mathcal{C}$ has orbit-lengths $96, 64, 48, 2$. Let C_1, C_2, C_3, C_4 be representatives of the orbits with these respective lengths. For each $i = 1, 2, 3$ we set $\mathcal{L}_i = \{X, Y, C_i\}$, and then, we add to each $\mathcal{L}_i$ the elements of the Pasch closures of $\{X, Y\}$, of $\{X, C_i\}$, and of $\{Y, C_i\}$, so that each $\mathcal{L}_i$ has size 15. It turns out that $\mathcal{L}_3$ has 6 pairs of 5-cliques meeting in two points, and so no cliques in the third H-orbit can be lines of $\mathcal{M}$. For $i = 1, 2$ we attempt to further enlarge $\mathcal{L}_i$ by adding to it the elements of the Pasch closures of every distinct pair of intersecting cliques in the original $\mathcal{L}_i$ of size 15. After doing this, $|\mathcal{L}_1| = |\mathcal{L}_2| = 63$, but $\mathcal{L}_1$ has 15 pairs of cliques meeting in 2 points and $\mathcal{L}_2$ has 12 such pairs. We conclude that no cliques in the first or second orbit can be lines of $\mathcal{M}$. This leaves just the two cliques in the fourth orbit as possible lines through x together with X and Y, but there should be 28 lines through a point of $\mathcal{M}$. We conclude that $\mathcal{M}$ does not exist. ■

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