Generalized wreath products of association schemes

R. A. Bailey
School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road, London E1 4NS, UK

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Address for correspondence: R. A. Bailey
School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road, London E1 4NS,
U.K.

email: r.a.bailey@qmul.ac.uk

Abstract

Given a finite partially ordered set X, and, for each x in X, an association scheme on a set Ω_x, we show how to define an association scheme on $\prod_x \Omega_x$ in a way that specializes to iterated crossing and nesting when X is series-parallel. The character table of the new association scheme is found.

1 Background

An association scheme of rank r on a finite set Ω is a partition of $\Omega \times \Omega$ into r subsets whose adjacency matrices A_0, A_1, …, A_{r-1} in $\mathbb{R}^{\Omega\times\Omega}$ satisfy:

(1) $A_0 = I_\Omega$, the identity matrix on Ω;

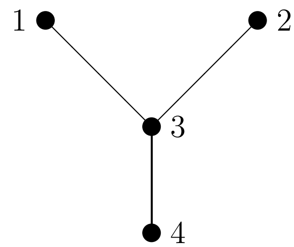
(2) for $i = 0, \ldots, r-1$, $A_i^\top = A_i$, where $A^\top$ denotes the transpose of the matrix A;

(3) for $i, j = 0, \ldots, r-1$, A_iA_j is a linear combination of $A_0, \ldots, A_{r-1}$.

Note that $\sum_{i=0}^{r-1} A_i = J_\Omega$, the all-1 matrix over Ω, and that each A_iJ_Ω is a scalar multiple of J_Ω. See [3].

There are two methods of combining two association schemes, called crossing and nesting in the statistical literature. For $t = 1, 2$, let $\mathcal{Q}_t$ be an association scheme on Ω_t whose adjacency matrices are A_{ti} for $i = 0, \ldots, r_t - 1$. Crossing $\mathcal{Q}_1$ and $\mathcal{Q}_2$ produces the association scheme $\mathcal{Q}_1 \times \mathcal{Q}_2$ on $\Omega_1 \times \Omega_2$ whose adjacency matrices are $A_{1i} \otimes A_{2j}$ for $0 \leq i \leq r_1 - 1$ and $0 \leq j \leq r_2 - 1$; this is called the direct product. Nesting $\mathcal{Q}_2$ within $\mathcal{Q}_1$ produces the association scheme $\mathcal{Q}_1/\mathcal{Q}_2$, also called the wreath product, whose adjacency matrices are $A_{1i} \otimes J_{\Omega_2}$, for $1 \leq i \leq r_1 - 1$, and $I_{\Omega_1} \otimes A_{2j}$, for $0 \leq j \leq r_2 - 1$.

The crossing operator is commutative and associative (up to isomorphism), and essentially corresponds to the 2-element antichain. By contrast, the nesting operator is not commutative but is associative: the foregoing construction corresponds to the 2-element chain in which $2 < 1$. Iterated crossing and nesting can lead to an association scheme such as $(\mathcal{Q}_1 \times \mathcal{Q}_2)/\mathcal{Q}_3/\mathcal{Q}_4$, which corresponds to the following partially ordered set.



Only series-parallel posets arise in this way. The goal of this paper is to give a poset operator for combining association schemes indexed by an arbitrary finite poset. The operator should specialize to iterated crossing and nesting if the poset is a series-parallel one.

The equivalent problem has already been solved for transitive permutation groups. If G is a transitive group of permutations of Ω then the orbits of G on $\Omega \times \Omega$ form a homogeneous coherent configuration [5], which is a generalization of association scheme in which (2) is weakened to

(2)′ for $i = 1, \ldots, r-1$, there is an index i^* such that $A_i^\top = A_{i^*}$.

For $t = 1, 2$, let G_t be a transitive group of permutations of Ω_t, with corresponding homogeneous coherent configuration $\mathcal{Q}_t$. The partition of $(\Omega_1 \times \Omega_2) \times (\Omega_1 \times \Omega_2)$ into the orbits of the permutation direct product $G_1 \times G_2$ is precisely $\mathcal{Q}_1 \times \mathcal{Q}_2$, while the partition of $(\Omega_1 \times \Omega_2) \times (\Omega_1 \times \Omega_2)$ into the orbits of the permutation wreath product $G_2 \operatorname{wr} G_1$ is precisely $\mathcal{Q}_1 / \mathcal{Q}_2$.

The generalized wreath product of transitive permutation groups indexed by a finite poset was given in [2]. In that paper, the main work was in finding the correct definition of the action of the generalized wreath product, finding the correct form for the orbits and proving that they were indeed orbits. There was no need to prove an analogue of condition (3), because that condition is automatically satisfied for adjacency matrices of orbits of a permutation group. In this paper we "find" the correct adjacency matrices for the generalized wreath product of association schemes by simply mimicking the result for permutation groups: this is done in Section 2. Now the proof of condition (3) does require some work, and we put part of this into some technical lemmas before the main theorem.

Paper [2] concludes by giving the irreducible subspaces of the permutation representation of the generalized wreath product in terms of those of the components. The analogue for an association scheme consists of the minimal idempotents and the character table. This is dealt with in Section 3. Again, the correct form is obtained by mimicking the result from permutation groups. Having guessed eigenspaces correctly, it is a straightforward matter to demonstrate that they are indeed eigenspaces, finding the eigenvalues in the process.

2 Constructing the association scheme

For the remainder of this paper, X is a finite poset. For x in X, $\mathcal{Q}_x$ is an association scheme on a set Ω_x which is finite of cardinality at least two; the rank of $\mathcal{Q}_x$ is r_x and the adjacency matrices of $\mathcal{Q}_x$ are A_{xi} for i in an index set $\mathcal{K}_x$ of cardinality r_x. We abbreviate I_{Ω_x} and J_{Ω_x} to I_x and J_x, and choose the labelling of $\mathcal{K}_x$ so that $A_{x0} = I_x$. Put $\Omega = \prod_{x\in X} \Omega_x$.

A subset Y of X is called an *antichain* if, whenever $x < y$, then not both of x and y are in Y. For each antichain Y define

$$\mathrm{Up}(Y) = \{x \in X : \exists y \in Y,\ y < x\}\,,$$

$$\mathrm{Down}(Y) = \{x \in X : \exists y \in Y,\ x < y\}\,.$$

Also, let $A(Y)$ be the set of $\{0,1\}$ matrices in $\mathbb{R}^{\Omega\times\Omega}$ of the following form

$$\bigotimes_{x\notin Y\cup\mathrm{Down}(Y)} I_x \otimes \bigotimes_{y\in Y} A_{yi_y} \otimes \bigotimes_{z\in\mathrm{Down}(Y)} J_z,$$

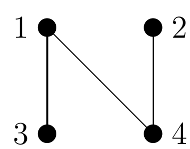
where, for each y in Y, $i_y \in \mathcal{K}_y \setminus \{0\}$. Thus

$$|A(Y)| = \prod_{y\in Y}(r_y - 1).$$

Put $A = \bigcup_{\text{antichains } Y} A(Y)$, and let $\mathcal{A}$ be the span of A over $\mathbb{R}$.

This collection of matrices is more natural than it appears at first sight. For each x, the x-th component is either an adjacency matrix of $\mathcal{Q}_x$ or is J_x. The whole idea of the partial order in the wreath product is that, for α, β in Ω, if $z < y$ then we are not interested in the relationship between α_z and β_z unless $\alpha_y = \beta_y$: thus the z-component must be J_z unless the y-component is I_y. The matrices in A are precisely those that satisfy this condition.

Example The smallest poset which is not series-parallel is the following poset N.



Its antichains are $\emptyset$, $\{1\}$, $\{2\}$, $\{1,2\}$, $\{3\}$, $\{2,3\}$, $\{4\}$ and $\{3,4\}$. Table 1 shows $\mathrm{Down}(Y)$ and $A(Y)$ for each of these antichains Y.

For x in X, let $\mathcal{A}_x$ be the Bose–Mesner algebra of $\mathcal{Q}_x$; that is, $\mathcal{A}_x$ is the span over $\mathbb{R}$ of the adjacency matrices of $\mathcal{Q}_x$. Define B in $\bigotimes_{x\in X}\mathcal{A}_x$ to be *nice* if, whenever $x < y$, either B_x is a scalar multiple of J_x or B_y is a scalar multiple of I_y. Let $\mathcal{B}$ be the set of nice matrices.

Lemma 1 *The sum of the matrices in A is J_Ω.*

Y	$\mathrm{Down}(Y)$	$A(Y)$
$\emptyset$	$\emptyset$	$\{I_1 \otimes I_2 \otimes I_3 \otimes I_4\}$
$\{1\}$	$\{3,4\}$	$\{A_{1i} \otimes I_2 \otimes J_3 \otimes J_4 : 1 \leq i \leq r_1 - 1\}$
$\{2\}$	$\{4\}$	$\{I_1 \otimes A_{2j} \otimes I_3 \otimes J_4 : 1 \leq j \leq r_2 - 1\}$
$\{1,2\}$	$\{3,4\}$	$\{A_{1i} \otimes A_{2j} \otimes J_3 \otimes J_4 : 1 \leq i \leq r_1 - 1,\ 1 \leq j \leq r_2 - 1\}$
$\{3\}$	$\emptyset$	$\{I_1 \otimes I_2 \otimes A_{3l} \otimes I_4 : 1 \leq l \leq r_3 - 1\}$
$\{2,3\}$	$\{4\}$	$\{I_1 \otimes A_{2j} \otimes A_{3l} \otimes J_4 : 1 \leq j \leq r_2 - 1,\ 1 \leq l \leq r_3 - 1\}$
$\{4\}$	$\emptyset$	$\{I_1 \otimes I_2 \otimes I_3 \otimes A_{4m} : 1 \leq m \leq r_4 - 1\}$
$\{3,4\}$	$\emptyset$	$\{I_1 \otimes I_2 \otimes A_{3l} \otimes A_{4m} : 1 \leq l \leq r_3 - 1,\ 1 \leq m \leq r_4 - 1\}$

Table 1: Set of adjacency matrices corresponding to each antichain in the poset N

Proof For each antichain Y, put

$$B_Y = \bigotimes_{x \notin Y \cup \mathrm{Down}(Y)} I_x \otimes \bigotimes_{y \in Y} (J_y - I_y) \otimes \bigotimes_{z \in \mathrm{Down}(Y)} J_z.$$

Then B_Y is the sum of the matrices in $A(Y)$. As shown in [1], the poset X defines a structure on Ω called a poset block structure, which in turn defines an association scheme on Ω whose adjacency matrices are the B_Y for antichains Y. Hence

$$\sum_{\text{antichains } Y} B_Y = J_\Omega. \quad \blacksquare$$

Lemma 2 *(a) For each antichain Y, every matrix in $A(Y)$ is in $\mathcal{B}$.*

(b) The set $\mathcal{B}$ is closed under matrix multiplication.

(c) If B is nice then $B \in \mathcal{A}$.

Proof (a) Suppose that $B \in A(Y)$ and $x < y$. If $B_y \neq I_y$ then $y \in Y \cup \mathrm{Down}(Y)$ so $x \in \mathrm{Down}(Y)$ and $B_x = J_x$.

(b) Suppose that B and C are in $\mathcal{B}$ and $x < y$. If either B_x or C_x is a scalar multiple of J_x then so is B_xC_x. Otherwise, both B_y and C_y are scalar multiples of I_y and then so is B_yC_y.

(c) If B is a nice matrix then B is a linear combination of nice matrices all of whose components are either adjacency matrices or J, so it suffices to consider a nice matrix B which is itself of this form. Put $M = \{x \in X : B_x \neq I_x,\ B_x \neq J_x\}$. Then M is an antichain and $B_x = J_x$ whenever $x \in \mathrm{Down}(M)$. Let $P = \{x \in X : x \notin \mathrm{Down}(M),\ B_x = J_x\}$. Put $\bar{B} = \bigotimes_{x \notin P} B_x$ and $J_P = \bigotimes_{x \in P} J_x$.

For each antichain Q in P, let $\mathrm{Down}_P(Q) = \mathrm{Down}(Q) \cap P$ and define $A_P(Q)$ analogously to $A(Q)$ but with indices restricted to P.

If $x \in Q$ and $y \in M$ then $x \not< y$, because $x \notin \mathrm{Down}(M)$, and $y \not< x$, because $B_x = J_x$ and B is nice. Therefore $M \cup Q$ is an antichain. If $x \in \mathrm{Down}(Q)$ then $B_x = J_x$ so $x \in \mathrm{Down}(M) \cup P$, so $\mathrm{Down}(M \cup Q) = \mathrm{Down}(M) \cup \mathrm{Down}(Q) = \mathrm{Down}(M) \cup \mathrm{Down}_P(Q)$. This shows that if $C \in A_P(Q)$ then $\bar{B} \otimes C \in A(M \cup Q) \subseteq A$. Now, $B = \bar{B} \otimes J_P$, and Lemma 1 for P shows that J_P is the sum of the matrices in

$$\bigcup_{\text{antichains } Q \subseteq P} A_P(Q),$$

so B is a sum of elements of A and is therefore in $\mathcal{A}$. ■

Theorem 3 *The matrices in A are the adjacency matrices of an association scheme on Ω.*

Proof Lemma 1 shows that the corresponding subsets form a partition of $\Omega \times \Omega$. The empty set is an antichain and $A(\emptyset) = \{I_\Omega\}$. Each component of every matrix in A is symmetric, so every matrix in A is symmetric. Parts (a) and (c) of Lemma 2 show that $\mathcal{A} = \mathcal{B}$, so part (b) of Lemma 2 shows that $\mathcal{A}$ is closed under matrix multiplication. ■

Definition Call the association scheme in Theorem 3 the *generalized wreath product* of the association schemes $\mathcal{Q}_x$ over the poset X.

Remark The same construction can be applied to homogeneous coherent configurations, in which case the generalized wreath product is also a homogeneous coherent configuration.

Theorem 4 *The generalized wreath product over the poset X is the same as the result of iterated crossing and nesting if X is a series-parallel poset.*

Proof The proof is by induction on the cardinality of X. If $X = \{x\}$ then the antichains are $\emptyset$ and X: $A(\emptyset) = \{I_x\}$ and $A(X) = \{A_{xi} : i \in \mathcal{K}_x \setminus \{0\}\}$, so the generalized wreath product is just $\mathcal{Q}_x$.

Now suppose that X is the disjoint union of non-empty series-parallel posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, for both of which the result is true. That is, for $t = 1, 2$, after crossing and nesting, the adjacency matrices of the association scheme $\mathcal{Q}_t$ on Ω_t have the form

$$\bigotimes_{x \notin Y_t \cup \mathrm{Down}_t(Y_t)} I_x \otimes \bigotimes_{y \in Y_t} A_{yi_y} \otimes \bigotimes_{z \in \mathrm{Down}_t(Y_t)} J_z$$

for some i_y in $\mathcal{K}_y \setminus \{0\}$, where Y_t is an antichain in X_t and Down_t is defined for X_t analogously to Down for X. Let $A_t(Y_t)$ be the set of such matrices for each fixed antichain Y_t in X_t.

Crossing corresponds to taking the cardinal sum of $(X_1, \leq_1)$ and $(X_2, \leq_2)$ to obtain the partial order on X defined by

$$x \leq y \quad \text{if} \quad \begin{cases} x \in X_1,\ y \in X_1 \text{ and } x \leq_1 y & \text{or} \\ x \in X_2,\ y \in X_2 \text{ and } x \leq_2 y. \end{cases}$$

Each antichain Y in X is the disjoint union of antichains Y_1 and Y_2 with $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$; moreover, $\mathrm{Down}(Y) = \mathrm{Down}_1(Y_1) \cup \mathrm{Down}_2(Y_2)$. Thus the tensor product of an element of $A_1(Y_1)$ with an element of $A_2(Y_2)$ gives an element of $A(Y)$, and all elements of $A(Y)$ are of this form; that is $A(Y) = \{C \otimes D : C \in A_1(Y_1),\ D \in A_2(Y_2)\}$. Thus $\mathcal{Q}_1 \times \mathcal{Q}_2$ is the generalized wreath product over $(X, \leq)$.

Nesting corresponds to taking the ordinal sum of $(X_1, \leq_1)$ and $(X_2, \leq_2)$ to obtain the partial order $\sqsubseteq$ on X defined by

$$x \sqsubseteq y \quad \text{if} \quad \begin{cases} x \in X_1,\ y \in X_1 \text{ and } x \leq_1 y & \text{or} \\ x \in X_2,\ y \in X_2 \text{ and } x \leq_2 y & \text{or} \\ x \in X_2 \text{ and } y \in X_1. \end{cases}$$

If Y_1 is a nonempty antichain in X_1 then Y_1 is an antichain in X, and $\mathrm{Down}(Y_1) = \mathrm{Down}_1(Y_1) \cup X_2$. If $C \in A_1(Y_1)$ then the nesting construction gives the adjacency matrix $C \otimes J_{\Omega_2}$, which is in $A(Y_1)$: moreover, all elements of $A(Y_1)$ arise in this way. All other antichains Y_2 in X are antichains in X_2. For D in $A_2(Y_2)$ the nesting construction gives the adjacency matrix $I_{\Omega_1} \otimes D$, which is in $A(Y_2)$ because $X_1 \subseteq X \setminus Y_2 \setminus \mathrm{Down}_2(Y_2)$; moreover, all elements of $A(Y_2)$ arise in this way. Thus $\mathcal{Q}_1/\mathcal{Q}_2$ is the generalized wreath product over $(X, \sqsubseteq)$. ■

3 The character table

The Bose–Mesner algebra of an association scheme is commutative and so its matrices are simultaneously diagonalizable. Thus if $\mathcal{Q}$ is an association scheme of rank r on a set Ω then $\mathbb{R}^\Omega$ is the direct sum of mutually orthogonal spaces $W_0, \ldots, W_{r-1}$ which are contained in eigenspaces of every adjacency matrix: see [4, Chapter 17]. Statisticians call these subspaces *strata*; the orthogonal projectors onto them, which are themselves in the Bose–Mesner algebra, are variously called minimal idempotents or stratum projectors. The table of the eigenvalues is called the character table of the association scheme.

The strata and character table for crossed and nested association schemes were found in [6] in terms of the strata and character table of the components. In this section we do the same thing for generalized wreath products.

For x in X, suppose that the strata for $\mathcal{Q}_x$ are W_{xe} for e in an index set $\mathcal{E}_x$ of cardinality r_x, where the labelling is chosen so that W_{x0} is the one-dimensional space consisting of the constant vectors. Let T_x be the orthogonal projector onto W_{x0}, so that $T_x = n_x^{-1} J_x$ where $n_x = |\Omega_x|$. For e in $\mathcal{E}_x \setminus \{0\}$, let S_{xe} be the projector onto W_{xe}. Let $\lambda_x(i,e)$ be the eigenvalue of A_{xi} on W_{xe}, for i in $\mathcal{K}_x$ and e in $\mathcal{E}_x$. In particular, $\lambda_x(0,e) = 1$ for all e and $\lambda_x(i,0)$ is equal to the constant row sum k_{xi} of A_{xi}.

For each antichain Y of X, let $S(Y)$ consist of all matrices of the form

$$\bigotimes_{x\in \mathrm{Up}(Y)} I_x \otimes \bigotimes_{y\in Y} S_{ye_y} \otimes \bigotimes_{z\notin Y\cup \mathrm{Up}(Y)} T_z,$$

which is the orthogonal projector onto the space

$$\bigotimes_{x\in \mathrm{Up}(Y)} \mathbb{R}^{\Omega_x} \otimes \bigotimes_{y\in Y} W_{ye_y} \otimes \bigotimes_{z\notin Y\cup \mathrm{Up}(Y)} W_{z0};$$

we make the restriction that $e_y \in \mathcal{E}_y \setminus \{0\}$ for y in Y. We shall show that these spaces are the strata for the generalized wreath product. Put $S = \bigcup_{\text{antichains } Y} S(Y)$.

Theorem 5 *The elements of S are the stratum projectors for the generalized wreath product $\mathcal{Q}$. Moreover, if C is the adjacency matrix*

$$\bigotimes_{x\notin Z\cup \mathrm{Down}(Z)} I_x \otimes \bigotimes_{y\in Z} A_{yi_y} \otimes \bigotimes_{z\in \mathrm{Down}(Z)} J_z$$

in $A(Z)$ and W is the stratum

$$\bigotimes_{x\in \mathrm{Up}(Y)} \mathbb{R}^{\Omega_x} \otimes \bigotimes_{y\in Y} W_{ye_y} \otimes \bigotimes_{z\notin Y\cup \mathrm{Up}(Y)} W_{z0}$$

whose projector D is in $S(Y)$, then the eigenvalue of C on W is equal to 0 *if* $\mathrm{Down}(Z) \cap Y \neq \emptyset$ *and is equal to*

$$\prod_{x \in Z \setminus Y} k_{xi_x} \prod_{x \in Z \cap Y} \lambda_x(i_x, e_x) \prod_{x \in \mathrm{Down}(Z)} n_x$$

otherwise.

Proof For each antichain Y, put

$$E_Y = \bigotimes_{x \in \mathrm{Up}(Y)} I_x \otimes \bigotimes_{y \in Y} (I_y - T_y) \otimes \bigotimes_{z \notin Y \cup \mathrm{Up}(Y)} T_z.$$

The projectors in $S(Y)$ are non-zero and pairwise orthogonal, because they have orthogonal components for at least one index y in Y. Their sum is E_Y. The matrices E_Y, over all antichains Y, are the stratum projectors for the poset block structure on Ω defined by X, so they are pairwise orthogonal and sum to I_Ω.

The size of S is

$$\sum_{\text{antichains } Y} \prod_{y \in Y} (r_y - 1),$$

which is equal to the rank of $\mathcal{Q}$. Thus it suffices to show that each putative stratum is contained in an eigenspace of every adjacency matrix, in other words that CD is a scalar multiple of D for all C in $A(Z)$ and all D in $S(Y)$, for all antichains Y and Z. This demonstration also gives the eigenvalues.

First suppose that $\mathrm{Down}(Z) \cap Y \neq \emptyset$. Now $J_x S_{xe_x} = 0$ for all x in $\mathrm{Down}(Z) \cap Y$, so $CD = 0$.

Secondly suppose that $\mathrm{Down}(Z) \cap Y = \emptyset$. Then there are no pairs y, z with y in Y, z in Z and $y < z$. Hence $Z \cap \mathrm{Up}(Y) = \mathrm{Down}(Z) \cap \mathrm{Up}(Y) = \emptyset$. If $x \notin Z \cup \mathrm{Down}(Z)$ then $C_x D_x = I_x D_x = D_x$. If $x \in Z$ then D_x is a stratum projector for $\mathcal{Q}_x$ so $C_x D_x = \lambda_x(i_x, e_x) D_x$ if $x \in Y$ and $C_x D_x = k_{xi_x} D_x$ if $x \notin Y \cup \mathrm{Up}(Y)$. If $x \in \mathrm{Down}(Z)$ then $x \notin Y \cup \mathrm{Up}(Y)$ so $C_x D_x = J_x T_x = n_x T_x = n_x D_x$. ∎

Example Tables 2–3 show part of the character table for the generalized wreath product of association schemes $\mathcal{Q}_1, \ldots, \mathcal{Q}_4$ over the poset N. There is one row for each antichain Z, showing one adjacency matrix in $A(Z)$. Here $i \in \mathcal{K}_1 \setminus \{0\}$, $j \in \mathcal{K}_2 \setminus \{0\}$, $l \in \mathcal{K}_3 \setminus \{0\}$ and $m \in \mathcal{K}_4 \setminus \{0\}$. There is one column for each antichain Y, showing one stratum projector in $S(Y)$. Here $e \in \mathcal{E}_1 \setminus \{0\}$, $f \in \mathcal{E}_2 \setminus \{0\}$, $g \in \mathcal{E}_3 \setminus \{0\}$, and $h \in \mathcal{E}_4 \setminus \{0\}$. The entries in the body of the table are the relevant eigenvalues.

	$T_1 \otimes T_2 \otimes T_3 \otimes T_4$	$S_{1e} \otimes T_2 \otimes T_3 \otimes T_4$	$T_1 \otimes S_{2f} \otimes T_3 \otimes T_4$	$S_{1e} \otimes S_{2f} \otimes T_3 \otimes T_4$
$I_1 \otimes I_2 \otimes I_3 \otimes I_4$	1	1	1	1
$A_{1i} \otimes I_2 \otimes J_3 \otimes J_4$	$k_{1i}n_3n_4$	$\lambda_1(i,e)n_3n_4$	$k_{1i}n_3n_4$	$\lambda_1(i,e)n_3n_4$
$I_1 \otimes A_{2j} \otimes I_3 \otimes J_4$	$k_{2j}n_4$	$k_{2j}n_4$	$\lambda_2(j,f)n_4$	$\lambda_2(j,f)n_4$
$A_{1i} \otimes A_{2j} \otimes J_3 \otimes J_4$	$k_{1i}k_{2j}n_3n_4$	$\lambda_1(i,e)k_{2j}n_3n_4$	$k_{1i}\lambda_2(j,f)n_3n_4$	$\lambda_1(i,e)\lambda_2(j,f)n_3n_4$
$I_1 \otimes I_2 \otimes A_{3l} \otimes I_4$	k_{3l}	k_{3l}	k_{3l}	k_{3l}
$I_1 \otimes A_{2j} \otimes A_{3l} \otimes J_4$	$k_{2j}k_{3l}n_4$	$k_{2j}k_{3l}n_4$	$\lambda_2(j,f)k_{3l}n_4$	$\lambda_2(j,f)k_{3l}n_4$
$I_1 \otimes I_2 \otimes I_3 \otimes A_{4m}$	k_{4m}	k_{4m}	k_{4m}	k_{4m}
$I_1 \otimes I_2 \otimes A_{3l} \otimes A_{4m}$	$k_{3l}k_{4m}$	$k_{3l}k_{4m}$	$k_{3l}k_{4m}$	$k_{3l}k_{4m}$

Table 2: Character table for the generalized wreath product over the N poset: one associate class per antichain and strata for four antichains

	$I_1 \otimes T_2 \otimes S_{3g} \otimes T_4$	$I_1 \otimes S_{2f} \otimes S_{3g} \otimes T_4$	$I_1 \otimes I_2 \otimes T_3 \otimes S_{4h}$	$I_1 \otimes I_2 \otimes S_{3g} \otimes S_{4h}$
$I_1 \otimes I_2 \otimes I_3 \otimes I_4$	1	1	1	1
$A_{1i} \otimes I_2 \otimes J_3 \otimes J_4$	0	0	0	0
$I_1 \otimes A_{2j} \otimes I_3 \otimes J_4$	$k_{2j} n_4$	$\lambda_2(j,f) n_4$	0	0
$A_{1i} \otimes A_{2j} \otimes J_3 \otimes J_4$	0	0	0	0
$I_1 \otimes I_2 \otimes A_{3l} \otimes I_4$	$\lambda_3(l,g)$	$\lambda_3(l,g)$	k_{3l}	$\lambda_3(k,g)$
$I_1 \otimes A_{2j} \otimes A_{3l} \otimes J_4$	$k_{2j}\lambda_3(l,g) n_4$	$\lambda_2(j,f)\lambda_3(l,g) n_4$	0	0
$I_1 \otimes I_2 \otimes I_3 \otimes A_{4m}$	k_{4m}	k_{4m}	$\lambda_4(m,h)$	$\lambda_4(m,h)$
$I_1 \otimes I_2 \otimes A_{3l} \otimes A_{4m}$	$\lambda_3(l,g) k_{4m}$	$\lambda_3(l,g) k_{4m}$	$k_{3l}\lambda_4(m,h)$	$\lambda_3(k,g)\lambda_4(m,h)$

Table 3: Character table for the generalized wreath product over the N poset: one associate class per antichain and strata for the four other antichains

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