On generalised t-designs and their parameters

Leonard H. Soicher
School of Mathematical Sciences
Queen Mary University of London
Mile End Road, London E1 4NS, UK
L.H.Soicher@qmul.ac.uk

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Abstract

Recently, P.J. Cameron studied a class of block designs which generalises the classes of t-designs, α-resolved 2-designs, orthogonal arrays, and other classes of combinatorial designs. In fact, Cameron's generalisation of t-designs (when there are no repeated blocks) is a special case of the "poset t-designs" in product association schemes studied ten years earlier by W.J. Martin, who further studied the special case of "mixed block designs". In this paper, we study Cameron's generalisation of t-designs from the point of view of classical t-design theory, in particular investigating the parameters of these generalised t-designs. We show that the t-design constants λ_i (the number of blocks containing an i-subset of the points, where $i \leq t$) and λ_i^j (the number of blocks containing an i-subset I of the points and disjoint from a j-subset J of the points, where $I \cap J = \emptyset$ and $i + j \leq t$) have very natural counterparts for generalised t-designs. Our main result places strong restrictions on the block structure of Cameron's t-$(\mathbf{v}, \mathbf{k}, \lambda)$ designs, an important subclass of generalised t-designs. We also generalise N.S. Mendelsohn's concept of "intersection numbers of order r" for t-designs, and show that analogous equations to those of Mendelsohn hold for generalised t-designs.

[Keywords: block design, t-design, generalised t-design, mixed block design, orthogonal array, intersection numbers]

1 Introduction

Recently, P.J. Cameron [1] studied a class of block designs which generalises the classes of t-designs, α-resolved 2-designs, orthogonal arrays, and other classes of combinatorial designs. His main interest was the possibility of using Markov chain methods to generate such designs with block size 3 and given parameters (such as 1-factorisations of complete graphs and Steiner triple systems) uniformly at random. In fact, Cameron's generalisation of t-designs (when there are no repeated blocks) is a special case of the "poset $\mathcal{T}$-designs" in product association schemes (in particular the "poset t-designs" in products of Johnson schemes) studied ten years earlier by W.J. Martin [6]. In [6], Martin applies the theory of Delsarte $\mathcal{T}$-designs to determine lower bounds on the number of blocks, and results on block intersections, for many classes of designs at once, implicitly including generalised t-designs. Martin [5] further studied in depth the special case of "mixed block designs". See also [7].

In this paper, we study Cameron's generalisation of t-designs from the point of view of classical t-design theory, in particular investigating the parameters of these generalised t-designs. We start by presenting some notation and definitions used in this paper, including defining exactly what we mean by a generalised t-design, as variants of this concept are given in Cameron's article [1].

Let V be a finite set and $\mathbf{V} = (V_1, \ldots, V_m)$ an ordered partition of V. For S a subset of V, we define the $\mathbf{V}$*-height* of S, denoted $[S]_{\mathbf{V}}$, to be the m-vector of non-negative integers

$$(|S \cap V_1|, \ldots, |S \cap V_m|).$$

When the ordered partition $\mathbf{V}$ is clear from the context, we may just say *height* for $\mathbf{V}$-height, and denote the height of S by $[S]$. (Note that if $\mathbf{V} = (V)$ then the $\mathbf{V}$-height of a subset S of V is simply a vector with a single coordinate giving the size of S.)

Let $\mathbf{r} = (r_1, \ldots, r_\ell)$, $\mathbf{s} = (s_1, \ldots, s_m)$ be vectors of integers. We write $\mathbf{r} \le \mathbf{s}$ to mean that $\ell = m$ and $r_i \le s_i$ for $i = 1, \ldots, m$. Now further suppose that $\ell = m$ and $s_1, \ldots, s_m$ are non-negative. We denote $\sum_{i=1}^m s_i$ by $|\mathbf{s}|$, and define

$$\binom{\mathbf{s}}{\mathbf{r}} := \prod_{i=1}^{m} \binom{s_i}{r_i},$$

with the convention that $\binom{s_i}{r_i} = 0$ if $r_i < 0$ or $r_i > s_i$. Thus, if V is a finite set, $\mathbf{V}$ an ordered partition of V, and $S \subseteq V$, with $\mathbf{s} = [S]_{\mathbf{V}}$, then $|\mathbf{s}| = |S|$, and $\binom{\mathbf{s}}{\mathbf{r}}$ is the number of subsets R of S having $[R]_{\mathbf{V}} = \mathbf{r}$.

A *block design* is an ordered pair $(V, \mathcal{B})$, such that V is a finite non-empty set, whose elements are called *points*, and $\mathcal{B}$ is a finite non-empty collection (multiset) of subsets of V called *blocks*. For t a non-negative integer, a *t-(v,k,λ) design* (or simply a *t-design*) is a block design $(V, \mathcal{B})$ satisfying:

- V has size v;
- each block has the same size k, with $k > 0$ and $t \leq k$;
- each t-subset of V is contained in the same (positive) number λ of blocks.

For t a non-negative integer and V a finite non-empty set, a *t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design* (or simply a *generalised t-design*) with *point-set* V is an ordered pair $(\mathbf{V}, \mathcal{B})$, such that $\mathbf{V}$ is an ordered partition of V, $(V, \mathcal{B})$ is a block design, and the following properties hold:

- V has $\mathbf{V}$-height $\mathbf{v}$;
- each block has the same $\mathbf{V}$-height $\mathbf{k}$, with each entry in $\mathbf{k}$ positive, and $t \leq |\mathbf{k}|$;
- for every tuple $\mathbf{t}$ of non-negative integers satisfying $|\mathbf{t}| = t$ and $\mathbf{t} \leq \mathbf{k}$, each t-subset T of V having $[T]_{\mathbf{V}} = \mathbf{t}$ is contained in the same (positive) number $\lambda_{\mathbf{t}}$ of blocks.

We call $t, \mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}})$ the *parameters* of a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design D. The sequence $(\lambda_{\mathbf{t}})$, if explicitly given, is with respect to some fixed total ordering (say lexicographic) of those $\mathbf{t}$ with $|\mathbf{t}| = t$ and $\mathbf{t} \leq \mathbf{k}$. As we shall see, any one such $\lambda_{\mathbf{t}}$ determines the others. We denote by $V(D)$ the point-set of D.

Example 1 Let $D := (\mathbf{V}, \mathcal{B})$, where $\mathbf{V} := (\{1,2,3,4\}, \{5,6,7\})$ and

$$\mathcal{B} := [\{1,2,5\}, \{3,4,5\}, \{1,3,6\}, \{2,4,6\}, \{1,4,7\}, \{2,3,7\}].$$

Then $V(D) = \{1, \ldots, 7\}$, and D is a 2-$((4,3),(2,1),(\lambda_{(1,1)}, \lambda_{(2,0)}))$ design with $\lambda_{(1,1)} = \lambda_{(2,0)} = 1$.

Note that a t-$(\mathbf{v}, \mathbf{k}, (\lambda_\mathbf{t}))$ design with $\mathbf{k} = (k)$ is the same thing as a t-design with block size k, and a t-$(\mathbf{v}, (1, 1, \ldots, 1), (\lambda_\mathbf{t}))$ design is the same thing as an orthogonal array of strength t (possibly over variable-size alphabets). The t-$(\mathbf{v}, (k_1, k_2), (\lambda_\mathbf{t}))$ designs are the mixed block designs studied by Martin [5], where many constructions for these are given. Further examples and classes of generalised t-designs are given by Cameron [1].

Cameron [1] almost exclusively studies t-$(\mathbf{v}, \mathbf{k}, (\lambda_\mathbf{t}))$ designs with $\lambda_\mathbf{t} = \lambda$ (a constant) for all $\mathbf{t}$; these are called t-$(\mathbf{v}, \mathbf{k}, \lambda)$ *designs*. (He also requires $0 < t < |\mathbf{k}|$, which we do not.) Thus, the design D in Example 1 is a 2-$((4, 3), (2, 1), 1)$ design. The main result of this paper is that a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with block size k and $2 \le t \le k-2$ must have $\mathbf{k} \in \{\mathbf{v}, (k), (k-1, 1), (1, k-1), (1, 1, \ldots, 1)\}$.

In the final section of this paper, we generalise Mendelsohn's concept of "intersection numbers of order r" for t-designs, and show that analogous equations to those of Mendelsohn [8] hold for generalised t-designs, and then apply the generalised equations to study a particular example.

2 Generalising the constants λ_i and λ_i^j of a t-design

In a t-(v, k, λ) design, when $i \le t$, the number of blocks containing a given i-subset of the points is a constant $\lambda_i := \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$, not depending on the choice of i-subset. Here we prove an entirely analogous result for generalised t-designs, generalising Lemma 1.1 of [5], and providing useful necessary conditions on the parameters of a generalised t-design.

Proposition 2.1 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, (\lambda_\mathbf{t}))$ design and $I \subseteq V(D)$, with $|I| \le t$. Then the number of blocks λ_I containing I is a constant $\lambda_\mathbf{i}$, depending only on the parameters of D and the $\mathbf{V}$-height $\mathbf{i}$ of I. Moreover,*

$$\lambda_\mathbf{i} = \lambda_\mathbf{t}\binom{\mathbf{v}-\mathbf{i}}{\mathbf{t}-\mathbf{i}}\Big/\binom{\mathbf{k}-\mathbf{i}}{\mathbf{t}-\mathbf{i}},$$

for all tuples $\mathbf{t}$ of non-negative integers with $|\mathbf{t}| = t$ and $\mathbf{i} \le \mathbf{t} \le \mathbf{k}$.

Proof. If $\mathbf{i} \not\le \mathbf{k}$, then $\lambda_I = \lambda_\mathbf{i} := 0$. Now let $\mathbf{t}$ be a tuple of non-negative integers, such that $|\mathbf{t}| = t$ and $\mathbf{i} \le \mathbf{t} \le \mathbf{k}$. We count in two ways the number

N of ordered pairs (T, B), such that B is a block of D, $I \subseteq T \subseteq B$ and $[T]_{\mathbf{V}} = \mathbf{t}$. Each such T contributes exactly $\lambda_{\mathbf{t}}$ pairs, so $N = \binom{\mathbf{v}-\mathbf{i}}{\mathbf{t}-\mathbf{i}}\lambda_{\mathbf{t}}$. On the other hand, each block B containing I contributes exactly $\binom{\mathbf{k}-\mathbf{i}}{\mathbf{t}-\mathbf{i}}$ pairs, so $N = \lambda_I\binom{\mathbf{k}-\mathbf{i}}{\mathbf{t}-\mathbf{i}}$. Thus $\lambda_I = \lambda_{\mathbf{i}} := \lambda_{\mathbf{t}}\binom{\mathbf{v}-\mathbf{i}}{\mathbf{t}-\mathbf{i}}/\binom{\mathbf{k}-\mathbf{i}}{\mathbf{t}-\mathbf{i}}$. ∎

Corollary 2.2 *A generalised t-design is a generalised s-design for each $s = 0, \dots, t$.*

Corollary 2.3 *Let $b = \lambda_{(0,\dots,0)}$ be the number of blocks of D. Then*

$$\lambda_{\mathbf{t}}\binom{\mathbf{v}}{\mathbf{t}} = b\binom{\mathbf{k}}{\mathbf{t}},$$

for all tuples $\mathbf{t}$ of non-negative integers with $|\mathbf{t}| = t$ and $\mathbf{t} \leq \mathbf{k}$. In particular, the sequence $(\lambda_{\mathbf{t}})$ is determined by $t, \mathbf{v}, \mathbf{k}$ and b.

Example 2 Let H be the cyclic group of order 10 generated by

$$(1, 2, 3, 4, 5)(7, 8, 9, 10, 11, 12, 13, 14, 15, 16),$$

and let $\mathcal{B}$ be the union of the H-orbits of the sets $\{1, 2, 3, 9, 11, 14, 16\}$, $\{1, 2, 4, 7, 10, 11, 13\}$, $\{1, 2, 6, 7, 8, 9, 11\}$ and $\{1, 3, 6, 10, 11, 15, 16\}$. The reader can verify that

$$E := ((\{1, \dots, 6\}, \{7, \dots, 16\}), \mathcal{B})$$

is a 2-$((6, 10), (3, 4), (\lambda_{(0,2)} = 4, \lambda_{(1,1)} = 6, \lambda_{(2,0)} = 6))$ design. (The design E was found using the function `BlockDesigns` in the DESIGN package [10] for GAP [3]. Indeed, using this function it is easy to classify, up to isomorphism, the 36 H-invariant generalised 2-designs with the same parameters as E.) Now any generalised 2-design with the same parameters as E is also a 1-$((6, 10), (3, 4), (\lambda_{(0,1)} = 12, \lambda_{(1,0)} = 15))$ design, and also a 0-$((6, 10), (3, 4), 30)$ design.

In a t-(v, k, λ) design, when $i + j \leq t$, the number of blocks containing an i-subset I of the points and disjoint from a j-subset J of the points, with $I \cap J = \emptyset$, is a constant $\lambda_i^j := \lambda\binom{v-i-j}{k-i}/\binom{v-t}{k-t}$, not depending on the choice of i-subset or disjoint j-subset (see [9]). Here we prove an entirely analogous result for generalised t-designs. In addition, the proof will generalise the usual "intersection triangles" for t-designs and Steiner systems.

Proposition 2.4 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design. Let $I, J \subseteq V(D)$, $I \cap J = \emptyset$, and suppose that $|I \cup J| \leq t$, or D is a t-$(\mathbf{v}, \mathbf{k}, 1)$ design and $I \cup J$ is contained in some block of D. Then the number λ_I^J of blocks containing I and disjoint from J is a constant $\lambda_{\mathbf{i}}^{\mathbf{j}}$, depending only on the parameters of D and the $\mathbf{V}$-heights $\mathbf{i}$ and $\mathbf{j}$ of I and J respectively. Moreover, if $|\mathbf{i}+\mathbf{j}| \leq t$ and b is the number of blocks of D, then*

$$\lambda_{\mathbf{i}}^{\mathbf{j}} = b\binom{\mathbf{v}-\mathbf{i}-\mathbf{j}}{\mathbf{k}-\mathbf{i}} \Big/ \binom{\mathbf{v}}{\mathbf{k}} = \lambda_{\mathbf{t}}\binom{\mathbf{v}-\mathbf{i}-\mathbf{j}}{\mathbf{k}-\mathbf{i}} \Big/ \binom{\mathbf{v}-\mathbf{t}}{\mathbf{k}-\mathbf{t}},$$

for all tuples $\mathbf{t}$ of non-negative integers with $|\mathbf{t}| = t$ and $\mathbf{t} \leq \mathbf{k}$.

Proof. We shall use the following notation. For $\mathbf{u} = (u_1, \ldots, u_m)$ an m-tuple of integers, and $\ell \in \{1, \ldots, m\}$, let

$$\mathbf{u}(\ell+) := (u_1, \ldots, u_{\ell-1}, u_\ell + 1, u_{\ell+1}, \ldots, u_m),$$

$$\mathbf{u}(\ell-) := (u_1, \ldots, u_{\ell-1}, u_\ell - 1, u_{\ell+1}, \ldots, u_m).$$

We proceed by induction on $|\mathbf{j}|$, the size of J. If $|\mathbf{j}| = 0$ then $J = \emptyset$. We then have $\lambda_I^J = \lambda_{\mathbf{i}}$ if $|I| \leq t$, and otherwise, by assumption, I is contained in a unique block, so $\lambda_I^J = 1$.

Now suppose $|\mathbf{j}| > 0$, $\mathbf{j} = (j_1, \ldots, j_m)$, and $\mathbf{V} = (V_1, \ldots, V_m)$. Choose ℓ such that $j_\ell > 0$ and let $a \in J \cap V_\ell$. Then $a \notin I$, and

$$\lambda_I^J = \lambda_I^{J\setminus\{a\}} - \lambda_{I\cup\{a\}}^{J\setminus\{a\}}.$$

Thus, by induction, for every $\ell \in \{1, \ldots, m\}$ such that $j_\ell > 0$, we have:

$$\lambda_I^J = \lambda_{\mathbf{i}}^{\mathbf{j}} := \lambda_{\mathbf{i}}^{\mathbf{j}(\ell-)} - \lambda_{\mathbf{i}(\ell+)}^{\mathbf{j}(\ell-)}.$$

In particular, λ_I^J depends only on the parameters of D and $\mathbf{i}$ and $\mathbf{j}$.

Now let b be the number of blocks of D, and let $\mathbf{i}, \mathbf{j}$ be tuples of non-negative integers, with $\mathbf{i}+\mathbf{j} \leq \mathbf{v}$ and $|\mathbf{i}+\mathbf{j}| \leq t$. To establish the stated formulae for the constant $\lambda_{\mathbf{i}}^{\mathbf{j}}$, we count in two ways the number N of ordered triples (I, J, B), such that $I, J \subseteq V(D)$, $I \cap J = \emptyset$, $[I]_{\mathbf{V}} = \mathbf{i}$, $[J]_{\mathbf{V}} = \mathbf{j}$, B is a block, $I \subseteq B$, and $J \cap B = \emptyset$. There are $\binom{\mathbf{v}}{\mathbf{i}}$ choices for I, for each I there are $\binom{\mathbf{v}-\mathbf{i}}{\mathbf{j}}$ choices for J, and for each I and J there are $\lambda_{\mathbf{i}}^{\mathbf{j}}$ choices for B, so $N = \binom{\mathbf{v}}{\mathbf{i}}\binom{\mathbf{v}-\mathbf{i}}{\mathbf{j}}\lambda_{\mathbf{i}}^{\mathbf{j}}$. For the second count, there are b choices for B, for each B

there are $\binom{\mathbf{k}}{\mathbf{i}}$ choices for I, and for each B and I there are $\binom{\mathbf{v}-\mathbf{k}}{\mathbf{j}}$ choices for J, so $N = b\binom{\mathbf{k}}{\mathbf{i}}\binom{\mathbf{v}-\mathbf{k}}{\mathbf{j}}$. Thus

$$\lambda_{\mathbf{i}}^{\mathbf{j}} = \frac{b\binom{\mathbf{k}}{\mathbf{i}}\binom{\mathbf{v}-\mathbf{k}}{\mathbf{j}}}{\binom{\mathbf{v}}{\mathbf{i}}\binom{\mathbf{v}-\mathbf{i}}{\mathbf{j}}} = b\binom{\mathbf{v}-\mathbf{i}-\mathbf{j}}{\mathbf{k}-\mathbf{i}}\Big/\binom{\mathbf{v}}{\mathbf{k}}.$$

Now, by Corollary 2.3, for all tuples $\mathbf{t}$ of non-negative integers with $|\mathbf{t}| = t$ and $\mathbf{t} \leq \mathbf{k}$, we have

$$\lambda_{\mathbf{i}}^{\mathbf{j}} = \frac{\lambda_{\mathbf{t}}\binom{\mathbf{v}}{\mathbf{t}}\binom{\mathbf{v}-\mathbf{i}-\mathbf{j}}{\mathbf{k}-\mathbf{i}}}{\binom{\mathbf{k}}{\mathbf{t}}\binom{\mathbf{v}}{\mathbf{k}}} = \lambda_{\mathbf{t}}\binom{\mathbf{v}-\mathbf{i}-\mathbf{j}}{\mathbf{k}-\mathbf{i}}\Big/\binom{\mathbf{v}-\mathbf{t}}{\mathbf{k}-\mathbf{t}}.$$

■

3 The block structure of t-$(\mathbf{v}, \mathbf{k}, \lambda)$ designs

In this section, we determine strong restrictions on the block structure of a generalised t-design with constant $\lambda_{\mathbf{t}}$.

Lemma 3.1 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $t > 0$, and let $I \subseteq V(D)$, with $|I| = t-1$ and $[I]_{\mathbf{V}} = (i_1, \ldots, i_m) \leq \mathbf{k} = (k_1, \ldots, k_m)$. Then, for each j with $k_j > i_j$, the number of blocks of D containing I is equal to*

$$\lambda(v_j - i_j)/(k_j - i_j).$$

Proof. This follows from Proposition 2.1. ■

Lemma 3.2 *Suppose $D = (\mathbf{V}, \mathcal{B})$ is a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $t > 0$, $\mathbf{v} = (v_1, \ldots, v_m)$, $\mathbf{k} = (k_1, \ldots, k_m)$, block size $k \geq t+1$, and with $k_i = v_i$ for some i. Then $\mathbf{k} = \mathbf{v}$.*

Proof. Without loss of generality, suppose $k_1 = v_1$, and to obtain a contradiction, assume $\mathbf{k} \neq \mathbf{v}$. Then $m > 1$, and we can suppose, without loss of generality, that $k_2 < v_2$. Now take $I \subseteq V$ with $|I| = t-1$ and $[I]_{\mathbf{V}} = (i_1, \ldots, i_m) \leq \mathbf{k}$, such that $k_1 - i_1 \geq 1$ and $k_2 - i_2 \geq 1$ (this is possible since $t > 0$ and $k \geq t + 1$). Now, applying Lemma 3.1, we get that the number of blocks containing I is equal to

$$\lambda(v_1 - i_1)/(k_1 - i_1) = \lambda,$$

and is also equal to

$$\lambda(v_2 - i_2)/(k_2 - i_2) > \lambda,$$

a contradiction. ∎

Theorem 3.3 *Suppose D is a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $t \geq 2$ and block size $k \geq t + 2$. Then $\mathbf{k} \in \{\mathbf{v}, (k), (k-1, 1), (1, k-1), (1, 1, \ldots, 1)\}$.*

Proof. Suppose $D = (\mathbf{V}, \mathcal{B})$, $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{k} = (k_1, \ldots, k_m)$. If $m = 1$ then $\mathbf{k} = (k)$, and there is nothing to prove. We shall consider the cases $m = 2$ and $m \geq 3$ separately.

Suppose $m = 2$. If $k_1 = 1$ or $k_2 = 1$ there is nothing to prove, and so we assume that $k_1, k_2 \geq 2$. Now take $U \subseteq V(D)$ with $|U| = t - 2$ and $[U]_{\mathbf{V}} = (u_1, u_2)$, such that $k_1 - u_1 \geq 2$ and $k_2 - u_2 \geq 2$ (this is possible since $t \geq 2$ and $k \geq t + 2$). Let $\mathbf{V} = (V_1, V_2)$, let $a_i \in V_i \setminus U$ and let n_i be the number of blocks containing $U \cup \{a_i\}$ $(i = 1, 2)$. Now, applying Lemma 3.1, we have:

$$n_1 = \lambda(v_1 - (u_1 + 1))/(k_1 - (u_1 + 1)) = \lambda(v_2 - u_2)/(k_2 - u_2),$$

$$n_2 = \lambda(v_1 - u_1)/(k_1 - u_1) = \lambda(v_2 - (u_2 + 1))/(k_2 - (u_2 + 1)).$$

Thus

$$(v_1 - u_1 - 1)(k_2 - u_2) = (v_2 - u_2)(k_1 - u_1 - 1), \qquad (1)$$

$$(v_1 - u_1)(k_2 - u_2 - 1) = (v_2 - u_2 - 1)(k_1 - u_1). \qquad (2)$$

Subtracting (2) from (1), we obtain

$$-k_2 + u_2 + v_1 - u_1 = -v_2 + u_2 + k_1 - u_1,$$

and so $v_1 + v_2 = k_1 + k_2$, and since $0 \leq k_i \leq v_i$, we must have $k_1 = v_1$ and $k_2 = v_2$; that is, $\mathbf{k} = \mathbf{v}$.

Suppose now $m \geq 3$ and $\mathbf{k} \neq (1, 1, \ldots, 1)$. Without loss of generality, $k_1 \geq 2$. Now take $U \subseteq V$ with $|U| = t - 2$ and $[U]_{\mathbf{V}} = (u_1, \ldots, u_m) \leq \mathbf{k}$, such that $k_1 - u_1 \geq 2$, $k_2 - u_2 \geq 1$ and $k_3 - u_3 \geq 1$ (this is possible since $t \geq 2$ and

$k \geq t+2$). Let $\mathbf{V} = (V_1, \ldots, V_m)$, let $a_i \in V_i \setminus U$ and let n_i be the number of blocks containing $U \cup \{a_i\}$ ($i = 1, 2, 3$). Now, applying Lemma 3.1, we have:

$$n_1 = \lambda(v_1 - (u_1 + 1))/(k_1 - (u_1 + 1)) = \lambda(v_2 - u_2)/(k_2 - u_2),$$

$$n_3 = \lambda(v_1 - u_1)/(k_1 - u_1) = \lambda(v_2 - u_2)/(k_2 - u_2).$$

Thus $n_1 = n_3$, and so

$$(v_1 - u_1 - 1)(k_1 - u_1) = (v_1 - u_1)(k_1 - u_1 - 1).$$

From this we obtain $-k_1 + u_1 = -v_1 + u_1$, and so $v_1 = k_1$. By Lemma 3.2, we have $\mathbf{k} = \mathbf{v}$, and the proof is complete. ∎

We remark that the block multiset of a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $\mathbf{k} = \mathbf{v}$ consists of the complete point-set repeated λ times. A t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design with $\mathbf{k} = (k-1, 1)$ (or $(1, k-1)$) and $k > t > 0$ is more interesting. In such a design, the point set is the disjoint union of a v_1-set V_1 and a v_2-set V_2, a block consists of a $(k-1)$-subset of V_1, together with a "label" from V_2. Restricting the blocks to V_1, we get a t-$(v_1, k-1, \lambda)$ design, and the labels from V_2 define a partition of this t-design into $(t-1)$-$(v_1, k-1, \lambda)$ designs.

We additionally remark that the preceding theorem says nothing about the interesting case of t-$(\mathbf{v}, \mathbf{k}, \lambda)$ designs with block size $k = t+1$. Many interesting examples of such designs, with small t and k, are studied in [1].

4 Intersection numbers of order r for generalised t-designs

Mendelsohn [8] introduced the concept of intersection numbers of order r for a t-design, with respect to a block of that design, and showed that these intersection numbers satisfy a certain system of integer linear equations. The concept of intersection numbers of order r, and Mendelsohn's equations, have since been generalised and applied to block designs which are not necessarily t-designs and to graphs (see [12, 2, 11]).

In this section we provide another generalisation of intersection numbers of order r and the equations they satisfy, this time in a way appropriate for generalised t-designs.

Let $D = (\mathbf{V}, \mathcal{B})$ be a generalised t-design, with $\mathcal{B} = [B_1, \dots, B_b]$, and let $S \subseteq V(D)$ with $|S| \geq t$ and $\mathbf{s} := [S]_{\mathbf{V}}$. Further, let $\mathbf{i}$ be a tuple of non-negative integers with $\mathbf{i} \leq \mathbf{s}$. Then for r a positive integer, the $\mathbf{i}$-th *intersection number of order* r, with respect to S, denoted $n_{\mathbf{i}}^{(r)}(S)$ (or $n_{\mathbf{i}}^{(r)}(D, S)$), is defined to be the number of r-subsets R of $\{1, \dots, b\}$, such that $\cap_{i \in R} B_i \cap S$ has $\mathbf{V}$-height $\mathbf{i}$. In particular, $n_{\mathbf{i}}^{(1)}(S)$ is the number of blocks B of D (counting repeats), such that $[B \cap S]_{\mathbf{V}} = \mathbf{i}$.

Theorem 4.1 *Let $D = (\mathbf{V}, [B_1, \dots, B_b])$ be a generalised t-design, and let all heights be with respect to $\mathbf{V}$. Let $S \subseteq V(D)$ with $s := |S| \geq t$ and let r be a positive integer. Then for each tuple $\mathbf{j}$ of non-negative integers such that $|\mathbf{j}| \leq t$ and $\mathbf{j} \leq \mathbf{s} := [S]$, we have:*

$$\sum_{\mathbf{j} \leq \mathbf{i} \leq \mathbf{s}} \binom{\mathbf{i}}{\mathbf{j}} n_{\mathbf{i}}^{(r)}(S) = \binom{\mathbf{s}}{\mathbf{j}} \binom{\lambda_{\mathbf{j}}}{r}$$

(where $\lambda_{\mathbf{j}}$ is the number of blocks containing a point-subset of height $\mathbf{j}$).

Proof. Let $\mathbf{j}$ be a tuple of non-negative integers with $|\mathbf{j}| \leq t$ and $\mathbf{j} \leq \mathbf{s}$, and count in two ways the number $N_{\mathbf{j}}$ of ordered pairs (R, J), such that R is an r-subset of $\{1, \dots, b\}$ and J is a subset, of height $\mathbf{j}$, of both S and $\cap_{i \in R} B_i$.

Now each subset J of S with $[J] = \mathbf{j}$ contributes exactly $\binom{\lambda_{\mathbf{j}}}{r}$ pairs of the form $(*, J)$ to $N_{\mathbf{j}}$, and so

$$N_{\mathbf{j}} = \sum_{J \subseteq S, [J] = \mathbf{j}} \binom{\lambda_{\mathbf{j}}}{r} = \binom{\mathbf{s}}{\mathbf{j}} \binom{\lambda_{\mathbf{j}}}{r}.$$

On the other hand, each r-subset R of $\{1, \dots, b\}$ contributes exactly

$$\binom{[\bigcap_{i \in R} B_i \cap S]}{\mathbf{j}}$$

pairs of the form $(R, *)$ to $N_{\mathbf{j}}$, and so

$$N_{\mathbf{j}} = \sum_{R \subseteq \{1, \dots, b\}, |R| = r} \binom{[\bigcap_{i \in R} B_i \cap S]}{\mathbf{j}} = \sum_{\mathbf{j} \leq \mathbf{i} \leq \mathbf{s}} \binom{\mathbf{i}}{\mathbf{j}} n_{\mathbf{i}}^{(r)}(S).$$

Hence the result. ∎

We remark that this result is proved for the case when D is a t-design and S is a block in [8], and in general for t-designs in [12]. A different generalisation to graphs and general block designs is given in [11] (from which we have adapted our proof of Theorem 4.1).

Example 3 Let $D = (\mathbf{V}, \mathcal{B})$ be any 2-$((6,10),(3,4),(\lambda_{(0,2)} = 4, \lambda_{(1,1)} = 6, \lambda_{(2,0)} = 6))$ design. We apply our generalisation of Mendelsohn's equations in the case $r = 1$ to obtain some information about the $\mathbf{V}$-heights of intersections of blocks of D.

Let B be a block of D, $\mathbf{k} := [B]_{\mathbf{V}} = (3,4)$, and let $n_{\mathbf{i}} := n_{\mathbf{i}}^{(1)}(B)$ for the (twenty) non-negative integer tuples $\mathbf{i} \leq \mathbf{k}$. Then the following six linear equations hold:

$$\sum_{\mathbf{j} \leq \mathbf{i} \leq \mathbf{k}} \binom{\mathbf{i}}{\mathbf{j}} n_{\mathbf{i}} = \binom{\mathbf{k}}{\mathbf{j}} \lambda_{\mathbf{j}} \quad \text{for} \quad \mathbf{j} \in \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)\}, \tag{3}$$

together with the inequalities:

$$n_{\mathbf{i}} \geq 0 \ \text{ for } \ \mathbf{i} < \mathbf{k}, \quad \text{and} \quad n_{\mathbf{k}} \geq 1. \tag{4}$$

Now the system consisting of all the linear constraints in (3) and (4) can be studied using exact linear or integer programming methods, which in particular can be used to obtain upper and lower bounds on each $n_{\mathbf{i}}$. Here we make use of the exact linear programming package `simplex` in the computer algebra system `Maple` [4].

For example, given the linear constraints in (3) and (4), the function `maximize` of the `simplex` package tells us that $n_{(3,4)} \leq 9/5$ in any solution to these constraints with rational $n_{\mathbf{i}}$, and so $n_{(3,4)} \leq 1$ in any solution with integer $n_{\mathbf{i}}$, and so D cannot have repeated blocks. Similarly, the `minimize` function of `simplex` tells us that $n_{(1,2)} \geq 2$ in any solution with rational $n_{\mathbf{i}}$, and so each block B of D must intersect at least two blocks in a subset of $\mathbf{V}$-height $(1,2)$. Furthermore, after adding to (3) and (4) the two constraints $n_{(1,4)} \geq 1$ and $n_{(2,3)} \geq 1$, the `simplex` package function `feasible` informs us that there are no solutions with rational $n_{\mathbf{i}}$ (and hence none with integer $n_{\mathbf{i}}$). This tells us that in D, there can be no blocks B, X, Y, with $B \cap X$ having $\mathbf{V}$-height $(1,4)$ and $B \cap Y$ having $\mathbf{V}$-height $(2,3)$.

Finally, we remark that, as well as our generalisation of Mendelsohn's equations, the block intersection polynomial techniques of Cameron and Soicher [2, 11] can also be used to study generalised t-designs with given parameters.

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