

Cube designs

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Abstract

A cube design of order v , or a $\text{CUBE}(v)$, is a decomposition of all cyclicly oriented quadruples of a v -set into oriented cubes. A $\text{CUBE}(v)$ design is unoriented if its cubes can be paired so that the cubes in each pair are related by reflection through the center. A cube design is degenerate if it has repeated points on one of its cubes, otherwise it is nondegenerate.

We show that a nondegenerate $\text{CUBE}(v)$ design exists for all integers $v \geq 8$, and that an unoriented nondegenerate $\text{CUBE}(v)$ design exists if and only if $v \geq 8$ and $v \equiv 0, 1, 2$ or $3 \pmod{8}$.

A degenerate example of a CUBE(v) design is also given for each integer $v \geq 4$.

1 Introduction

Given a positive integer v and an unlimited supply of squares, white on one side and green on the other, consider making all possible labelled green squares from v labels, $\{0, 1, \dots, v-1\}$, with distinct labels written at the corners of a green side. Squares that can be obtained from each other by rotation are considered the same. Since we can tell the sides of the squares by their colour, the labelled squares are *oriented faces*. There are $v(v-1)(v-2)(v-3)$ ways to thusly label a square, starting at a given corner on a green side and moving counterclockwise, so (dividing by four for the rotations) the total number of such faces is $N = v(v-1)(v-2)(v-3)/4$.

Our problem is to assemble these N faces into $N/6$ cubes so that

- (a) each face is used exactly once,
- (b) the green side of each face is on the outside of a cube, and
- (c) at each corner of the cube, all of the labels agree.

Such a design is an *oriented cube design of order v* , denoted CUBE(v).

If we began instead with a supply of clear plastic squares and allow ourselves to flip over a face as well, then there are half as many available *unoriented faces*, $N' = v(v-1)(v-2)(v-3)/8$. If they can be assembled into $N'/6$ cubes according to (a) and (c) above, then we will have an *unoriented cube design of order v* , denoted uCUBE(v). We will show that certain trivial necessary conditions for such a construction to succeed are always sufficient.

Formally, let V be a v -set whose elements are called *labels* or *points*. Where a, b, c, d are distinct labels, an *oriented face*, f , is defined by

$$f = [a, b, c, d] = \{(a, b, c, d), (b, c, d, a), (c, d, a, b), (d, a, b, c)\},$$

so that $[a, b, c, d] = [b, c, d, a]$, and an *unoriented face*, f' , is defined by

$$f' = \langle a, b, c, d \rangle = [a, b, c, d] \cup [d, c, b, a],$$

so there is the additional symmetry $\langle a, b, c, d \rangle = \langle d, c, b, a \rangle$.

An *oriented (labeled) cube C* , is defined by

$$\begin{aligned} C &= N[a, b, c, d, e, f]S \\ &= \{[N, a, b, c], [N, c, d, e], [N, e, f, a], [S, f, e, d], [S, d, c, b], [S, b, a, f]\}, \end{aligned}$$

where N, a, b, c, d, e, f, S are labels in V . If two labels of C are the same then C is **degenerate**, otherwise it is **nondegenerate**. For example, $C = 0[1, 2, 3, 4, 5, 6]0$ is a degenerate cube. Note that even if a cube is degenerate, the labels on each of its faces must be distinct.

Our notation for a cube is called **equatorial notation**. Intuitively, the corner of C labeled N points upwards and the opposite corner of the cube, labeled S , points downwards. We call N the North pole and S the South pole. The six other vertices form a 6-cycle that we read counterclockwise when looking down from the North pole in the order a, b, c, d, e, f .

Note the following symmetries:

$$\begin{aligned} N[a, b, c, d, e, f]S &= N[c, d, e, f, a, b]S \quad (120^\circ \text{ rotation about the } NS\text{-axis}), \\ N[a, b, c, d, e, f]S &= S[f, e, d, c, b, a]N \quad (\text{exchanging poles}), \\ N[a, b, c, d, e, f]S &= a[b, c, N, e, f, S]d \quad (\text{moving } a \text{ to the North pole}). \end{aligned}$$

Of the $8!$ orderings of N, a, b, c, d, e, f, S there are 24 that represent the same cube as $N[a, b, c, d, e, f]S$.

The set \mathcal{F} of all oriented faces having points in V has cardinality $|\mathcal{F}| = v(v-1)(v-2)(v-3)/4$. An **oriented cube design of order v** , or a $\text{CUBE}(v)$, is a pair (V, \mathcal{C}) where \mathcal{C} is a set of $|\mathcal{F}|/6$ cubes such that each face in \mathcal{F} is contained in a unique cube of \mathcal{C} . If \mathcal{C} consists of a set of cubes with points in V such that no face with points in V is in two (or more) cubes of \mathcal{C} then (V, \mathcal{C}) is a **partial cube design of order v** .

If $\alpha : x \mapsto x^\alpha$ is a permutation of V , and $C = N[a, b, c, d, e, f]S$ is a cube with points in V , then we define the image of C under α to be

$$C^\alpha = N^\alpha[a^\alpha, b^\alpha, c^\alpha, d^\alpha, e^\alpha, f^\alpha]S^\alpha.$$

An **automorphism** of a partial cube design (V, \mathcal{C}) is a permutation α of V such that $C^\alpha \in \mathcal{C}$ for all $C \in \mathcal{C}$, and the **automorphism group** of (V, \mathcal{C}) , denoted $\text{AUT}(V, \mathcal{C})$, is the group of all automorphisms of (V, \mathcal{C}) .

For example, if $V = \{0, 1, 2, 3\}$ and $\mathcal{C} = \{0[1, 2, 3, 1, 2, 3]0\}$ then (V, \mathcal{C}) is a degenerate $\text{CUBE}(4)$ design, and the automorphism group is $\text{AUT}(V, \mathcal{C}) = \langle (1, 2, 3), (0, 1, 2, 3) \rangle$ and has order 24.

An **unoriented (labeled) cube** is defined similarly:

$$\begin{aligned} C &= N\langle a, b, c, d, e, f \rangle S \\ &= \{ \langle N, a, b, c \rangle, \langle N, c, d, e \rangle, \langle N, e, f, a \rangle, \langle S, f, e, d \rangle, \langle S, d, c, b \rangle, \langle S, b, a, f \rangle \} \end{aligned}$$

with the additional symmetry $N\langle a, b, c, d, e, f \rangle S = S\langle d, e, f, a, b, c \rangle N$, where labels are exchanged along great diagonals. An **unoriented cube design of order** v , denoted $u\text{CUBE}(v)$, is a pair (V, \mathcal{C}) where \mathcal{C} is a set of unoriented cubes with points in V such that each unoriented face with points in V is a face of exactly one unoriented cube in \mathcal{C} . Degenerate and nondegenerate unoriented cube designs, partial unoriented cube designs and their automorphisms are defined in a similar way to oriented ones. In particular, in a nondegenerate unoriented cube design, each (unoriented) cube has distinct labels, and if α is a permutation of the point set V and $C = N\langle a, b, c, d, e, f \rangle S$ is an unoriented cube with points in V , then $C^\alpha = N^\alpha\langle a^\alpha, b^\alpha, c^\alpha, d^\alpha, e^\alpha, f^\alpha \rangle S^\alpha$.

The **inversion map** I exchanges labels along great diagonals of a cube, that is

$$I(N[a, b, c, d, e, f]S) = S[d, e, f, a, b, c]N.$$

Now a $u\text{CUBE}(v)$ design is equivalent to a $\text{CUBE}(v)$ design whose cubes can be paired by the inversion mapping (except that we do not consider $a\langle b, c, d, b, c, d \rangle a$ to be a valid unoriented cube, since its six faces are not distinct).

The cube is one of the five Platonic solids, so the reader may be wondering about what is known of designs based on the other Platonic solids.

For the tetrahedron the corresponding oriented design is called a tetrahedral quadruple system; its existence and some embedding results are given in [4]. The unoriented version of this design is just the Steiner quadruple system $S(3, 4, v)$, which has been well studied, see [1].

For the octahedron the unoriented case was resolved by Hanani [3], and for the oriented case and for the embedding problem it was resolved by the first and third authors [5].

The existence of icosahedral and dodecahedral designs is still open.

2 Candelabra cube designs

Let $S, G_0, G_1, \dots, G_{t-1}$ be pairwise disjoint subsets of $V = S \cup G_0 \cup G_1 \cup \dots \cup G_{t-1}$. The set S is called the **stem** and each G_i is called a **group**.

If $f = [a, b, c, d]$ is a face with $\{a, b, c, d\} \subseteq V$ then the **width** of f is the number of groups G_i such that $\{a, b, c, d\} \cap G_i \neq \emptyset$. Thus a face has width 0 if and only if all of its points lie in S , and a width 4 face has no points in S and no two of its points in the same group.

If $W = \{w_1, w_2, \dots, w_r\}$ is a set of widths and (V, \mathcal{C}) is a partial cube design whose cubes cover precisely those faces in V with widths in W , then (V, \mathcal{C}) is a **candelabra cube design covering widths** w_1, w_2, \dots, w_r . It is denoted as a W -CAND_CUBE($g_0 g_1 \cdots g_{t-1} : s$), where $g_i = |G_i|$ for each i and $s = |S|$. If the faces and cubes are unoriented we use the notation W -uCAND_CUBE($g_0 g_1 \cdots g_{t-1} : s$).

An **automorphism** of a candelabra cube design (V, \mathcal{C}) with groups G_0, G_1, \dots, G_{t-1} is an automorphism α of (V, \mathcal{C}) that has the additional property that the stem and the set of groups are invariant under α , that is, $S^\alpha = S$ and for each i , $G_i^\alpha = G_j$ for some j . The **automorphism group** of a candelabra cube design is the group of all its automorphisms.

We believe that candelabra cube designs may be of interest in their own right. However, in this paper, we make use of specific small candelabra cube designs as ingredients to build larger cube designs, and we abbreviate the notation for the class of candelabra cube designs we shall employ. We write g^t for $g_0 g_1 \cdots g_{t-1}$ when $g = g_0 = g_1 = \cdots = g_{t-1}$. In this case and when there is a single width that is equal to the number of groups, we use the notation CAND_CUBE($g^t : s$) and uCAND_CUBE($g^t : s$), respectively, for $\{t\}$ -CAND_CUBE($g^t : s$) and $\{t\}$ -uCAND_CUBE($g^t : s$). In all our candelabra cube notations, we omit s from the notation to mean that $s = 0$, i.e., the stem is empty.

Note that both oriented and unoriented cube designs are special cases of candelabra cube designs. For example, a CUBE(v) design is the same thing as a $\{4\}$ -CAND_CUBE(1^v) and also the same as a CAND_CUBE(v^1).

3 Counting

Let V be a v -set, and let Q be a set of quadruples in V , that is, $Q \subseteq \binom{V}{4}$. Let

$$\mathcal{F} = \{[a, b, c, d] : \{a, b, c, d\} \in Q\}.$$

If (V, \mathcal{C}) is a partial cube design on V whose cubes cover precisely the faces in \mathcal{F} and no others, then (V, \mathcal{C}) will be said to be **based** on Q , and we shall also say that (V, \mathcal{C}) is **quadruple-based**.

Lemma 3.1 *If (V, \mathcal{C}) is a partial (oriented) cube design based on Q , then $|\mathcal{C}| = |Q|$.*

Proof: With notation as above, we have that $|\mathcal{F}| = 6|Q|$, and a single cube covers exactly six faces, so $|\mathcal{C}| = |Q|$. ■

Thus a $\text{CUBE}(v)$ design has $\binom{v}{4}$ cubes, which is also true for the degenerate cases with $4 \leq v \leq 7$. The table below gives cube counts for some designs we will need, the entries readily verified by quadruple-based counting:

Design	number of cubes
$\text{CUBE}(v)$	$\binom{v}{4}$
$u\text{CUBE}(v)$	$\binom{v}{4}/2$
$\{2\}\text{-CAND_CUBE}(g^t)$	$\binom{t}{2} \cdot \left[\binom{g}{2} \binom{g}{2} + 2 \binom{g}{3} g \right]$
$\{3\}\text{-CAND_CUBE}(g^t)$	$\binom{t}{3} \cdot 3 \binom{g}{2} g^2$
$\{4\}\text{-CAND_CUBE}(g^t)$	$\binom{t}{4} g^4$
$\text{CAND_CUBE}(g^2 : s)$	$\binom{s}{2} g^2 + 2 \binom{g}{2} g s + \binom{g}{2} \binom{g}{2} + 2 \binom{g}{3} g$
$\text{CAND_CUBE}(g^3 : s)$	$3 \binom{g}{2} g^2 + s g^3$

Note that a $\{2\}\text{-CAND_CUBE}(g^t)$ does not exist in general, but the entry can still be useful for counting. For example, the number of cubes in a $\{2, 4\}\text{-CAND_CUBE}(g^t)$ can be obtained by adding the counts for each width separately, since all the designs are quadruple-based.

If (U, \mathcal{D}) and (V, \mathcal{C}) are cube designs such that $U \subseteq V$ and $\mathcal{D} \subseteq \mathcal{C}$, then (U, \mathcal{D}) is a **subdesign** of (V, \mathcal{C}) . Given a v -set V let

$$\mathcal{C} = \left\{ a[b, c, d, b, c, d]a : \{a, b, c, d\} \in \binom{V}{4} \right\}.$$

Then (V, \mathcal{C}) is trivially a degenerate $\text{CUBE}(v)$ design with a subdesign (U, \mathcal{D}) on any subset U of V , where \mathcal{D} consists of those cubes originating from quadruples contained in U . However, we are interested in nondegenerate cube designs, for which the following holds.

Lemma 3.2 *If a nondegenerate $\text{CUBE}(v)$ design has a $\text{CUBE}(u)$ subdesign, then $v = u$ or $v \geq u + 5$.*

Proof: For a contradiction suppose that (V, \mathcal{C}) is a nondegenerate cube design with a subdesign (U, \mathcal{D}) and that $0 < |V - U| \leq 4$. Let $f = [x, y, z, w]$ be a face such that $V - U \subseteq \{x, y, z, w\}$, and let $C \in \mathcal{C}$ be the unique cube that contains the face f . Then the face f' of C that is opposite the face f

has all of its points in U , and it follows that $C \in \mathcal{D}$, a contradiction as C has points that are not in U . ■

4 The main result

Our main result is the following:

- Theorem 4.1** 1. A degenerate CUBE(v) design exists for each integer $v \geq 4$.
2. A nondegenerate CUBE(v) design exists for each integer $v \geq 8$.
3. A nondegenerate uCUBE(v) design exists if and only if v is an integer with $v \geq 8$ and $v \equiv 0, 1, 2$ or $3 \pmod{8}$.

The restriction in the case of unoriented cube designs arises from the fact that a uCUBE(v) has $\binom{v}{4}/2$ cubes, which must be an integer.

We have shown how to construct a degenerate CUBE(v) design in the discussion preceding Lemma 3.2. Now we handle the nondegenerate case.

Theorem 4.2 Let $s \in \{0, 1, 2, 3\}$ and let $t \geq 2$ be an integer. Then a nondegenerate CUBE($4t + s$) exists if the following designs exist:

- (a) a nondegenerate CAND_CUBE(4^4),
- (b) a nondegenerate CAND_CUBE($4^3 : s$),
- (c) a nondegenerate CAND_CUBE($4^2 : s$),
- (d) a nondegenerate CUBE($8 + s$),
- (e) a nondegenerate CUBE($12 + s$).

Furthermore, a nondegenerate uCUBE($4t + s$) exists if t is even and unoriented versions exist of the designs in (a)–(d).

Proof: Let $V = \{\infty_0, \infty_1, \dots, \infty_{s-1}\} \cup (\{0, 1, 2, 3\} \times \{0, 1, \dots, t-1\})$. Let $S = \{\infty_0, \infty_1, \dots, \infty_{s-1}\}$ and $G_i = \{0, 1, 2, 3\} \times \{i\}$, $0 \leq i \leq t-1$.

Case 1. The number of groups, t , is even. In this case pair the groups as $(G_0, G_1), (G_2, G_3), \dots, (G_{t-2}, G_{t-1})$. Form the following sets of cubes:

- (a) Initialise \mathcal{C}_4 to be the empty set, and then for each set $\{G_i, G_j, G_k, G_l\}$ of four groups, add to \mathcal{C}_4 the cubes of a nondegenerate CAND_CUBE(4^4) design with groups G_i, G_j, G_k, G_l .
- (b) Initialise \mathcal{C}_3 to be the empty set, and then for each set $\{G_i, G_j, G_k\}$ of three groups, add to \mathcal{C}_3 the cubes of a nondegenerate CAND_CUBE($4^3 : s$) with groups G_i, G_j, G_k and stem S .
- (c) Initialise \mathcal{C}_2 to be the empty set, and then for each set $\{G_i, G_j\}$ of two groups that are NOT one of the paired groups, i.e., $\{i, j\} \neq \{2r, 2r+1\}$ for $r = 0, \dots, (t-2)/2$, add to \mathcal{C}_2 the set of cubes of a nondegenerate CAND_CUBE($4^2 : s$) with groups G_i, G_j and stem S .
- (d) Initialise $\mathcal{C}_{1,2}$ to be the empty set, and then for each set $\{G_{2r}, G_{2r+1}\}$ of two groups that are paired, add to $\mathcal{C}_{1,2}$ the cubes of a nondegenerate CUBE($s+8$) on the points $G_{2r} \cup G_{2r+1} \cup S$.

It is readily verified that $(V, \mathcal{C}_{1,2} \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4)$ is a nondegenerate CUBE($4t+s$) design.

Case 2. The number of groups, t , is odd. In this case pair the first $t-3$ groups as $(G_0, G_1), (G_2, G_3), \dots, (G_{t-5}, G_{t-4})$ and set aside the last three groups as a triple $(G_{t-3}, G_{t-2}, G_{t-1})$. Form the following sets of cubes:

- (a) Initialise \mathcal{C}_4 to be the empty set, and then for each set $\{G_i, G_j, G_k, G_l\}$ of four groups, add to \mathcal{C}_4 the cubes of a nondegenerate CAND_CUBE(4^4) design with groups G_i, G_j, G_k, G_l .
- (b) Initialise \mathcal{C}_3 to be the empty set, and then for each set $\{G_i, G_j, G_k\}$ of three groups, where $\{i, j, k\} \neq \{t-3, t-2, t-1\}$, add to \mathcal{C}_3 the cubes of a nondegenerate CAND_CUBE($4^3 : s$) with groups G_i, G_j, G_k and stem S .
- (c) Initialise \mathcal{C}_2 to be the empty set, and then for each set $\{G_i, G_j\}$ of two groups such that $\{i, j\} \neq \{2r, 2r+1\}$ for $r = 0, \dots, (t-5)/2$, and $\{i, j\} \not\subseteq \{t-3, t-2, t-1\}$, add to \mathcal{C}_2 the set of cubes of a nondegenerate CAND_CUBE($4^2 : s$) with groups G_i, G_j and stem S .
- (d) Initialise $\mathcal{C}_{1,2}$ to be the empty set, and then for each set $\{G_{2r}, G_{2r+1}\}$ of two groups that are paired, add to $\mathcal{C}_{1,2}$ the cubes of a nondegenerate CUBE($s+8$) on the points $G_{2r} \cup G_{2r+1} \cup S$.

- (e) Let $\mathcal{C}_{1,2,3}$ be the set of cubes of a nondegenerate CUBE($s + 12$) on the points $G_{t-3} \cup G_{t-2} \cup G_{t-1} \cup S$.

It is readily verified that $(V, \mathcal{C}_{1,2,3} \cup \mathcal{C}_{1,2} \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4)$ is a nondegenerate CUBE($4t + s$) design.

Finally, when $t = 2k$ is even, only the nondegenerate designs in (a)-(d) are needed, and these have an even number of cubes, so if unoriented versions of these exist, then a nondegenerate u CUBE($8k + s$) exists. ■

In order to prove the main theorem it suffices to produce the ingredients required by Theorem 4.2. These small designs are given in the next section. The main theorem is thus proved.

5 Small nondegenerate cube designs

In what follows we represent 10, 11, 12, ... by a, b, c, \dots , and denote the cyclic group of order n by C_n . For most of the designs, (generators for) a permutation group H acting on the point set is given, such that H is a subgroup of the automorphism group of the given design. If H is not given then all cubes in the design are listed. Otherwise, only H -orbit representatives of the cubes of the design are given, where the orbit of a cube C under H is $C^H = \{C^\alpha : \alpha \in H\}$. The group H was used as an assumed group of automorphisms when searching for the design, as described in the next section.

Design: u CUBE(8) on points $\{0, 1, \dots, 7\}$.

Cubes: $0\langle 123654 \rangle 7, 0\langle 132467 \rangle 5, 0\langle 154376 \rangle 2, 0\langle 213475 \rangle 6, 0\langle 247356 \rangle 1, 0\langle 256347 \rangle 1,$
 $0\langle 264157 \rangle 3, 0\langle 346257 \rangle 1, 0\langle 356247 \rangle 1, 0\langle 357246 \rangle 1, 0\langle 527163 \rangle 4, 0\langle 124563 \rangle 7,$
 $0\langle 125673 \rangle 4, 0\langle 126743 \rangle 5, 0\langle 127453 \rangle 6, 0\langle 142635 \rangle 7, 0\langle 143527 \rangle 6, 0\langle 146275 \rangle 3,$
 $0\langle 147365 \rangle 2, 0\langle 152736 \rangle 4, 0\langle 162437 \rangle 5, 0\langle 164257 \rangle 3, 0\langle 165347 \rangle 2, 0\langle 214673 \rangle 5,$
 $0\langle 215743 \rangle 6, 0\langle 216453 \rangle 7, 0\langle 217563 \rangle 4, 0\langle 245176 \rangle 3, 0\langle 254167 \rangle 3, 0\langle 314572 \rangle 6,$
 $0\langle 315642 \rangle 7, 0\langle 316752 \rangle 4, 0\langle 317462 \rangle 5, 0\langle 345167 \rangle 2, 0\langle 354176 \rangle 2.$

Design: u CUBE(9) on points $\{0, 1, \dots, 8\}$.

$H = \langle (012345678) \rangle \cong C_9.$

H -orbit representatives: $3\langle 462581 \rangle 0, 1\langle 562734 \rangle 0, 5\langle 817324 \rangle 0, 6\langle 351472 \rangle 0,$
 $5\langle 783241 \rangle 0, 5\langle 374126 \rangle 0, 3\langle 518642 \rangle 0.$

Design: $uCUBE(10)$ on points $\{0, 1, \dots, 9\}$.

$H = \langle (02468)(13579) \rangle \cong C_5$.

H -orbit representatives: $0\langle 942861 \rangle 7$, $0\langle 935427 \rangle 8$, $7\langle 961402 \rangle 3$, $1\langle 769083 \rangle 5$,
 $6\langle 718495 \rangle 0$, $3\langle 195607 \rangle 2$, $2\langle 107543 \rangle 9$, $8\langle 605972 \rangle 1$, $9\langle 560741 \rangle 8$, $7\langle 483625 \rangle 0$,
 $6\langle 721854 \rangle 3$, $1\langle 879350 \rangle 6$, $8\langle 073254 \rangle 6$, $4\langle 680379 \rangle 2$, $1\langle 027458 \rangle 9$, $4\langle 301267 \rangle 9$,
 $3\langle 842019 \rangle 7$, $9\langle 517436 \rangle 0$, $2\langle 631809 \rangle 5$, $7\langle 018452 \rangle 9$, $3\langle 526470 \rangle 8$.

Design: $uCUBE(11)$ on points $\{0, 1, \dots, 10\}$.

$H = \langle (0123456789a) \rangle \cong C_{11}$.

H -orbit representatives: $4\langle 1a5397 \rangle 0$, $4\langle 21368a \rangle 0$, $2\langle 849a15 \rangle 0$, $9\langle 57a138 \rangle 0$,
 $6\langle 795a14 \rangle 0$, $3\langle 748956 \rangle 0$, $2\langle 96783a \rangle 0$, $1\langle 8742a6 \rangle 0$, $4\langle 72a581 \rangle 0$, $6\langle 3519a2 \rangle 0$,
 $3\langle 815794 \rangle 0$, $8\langle 267a54 \rangle 0$, $4\langle 931275 \rangle 0$, $2\langle 415963 \rangle 0$, $1\langle 429653 \rangle 0$.

Design: $CUBE(12)$ on points $\{0, 1, \dots, 11\}$.

$H = \langle (0123456789a), (12485a9736), (0b)(1a)(25)(37)(48)(69) \rangle \cong PGL(2, 11)$.

H -orbit representatives: $1\langle 457ba9 \rangle 8$, $0\langle 153ab9 \rangle 8$.

Design: $CUBE(13)$ on points $\{0, 1, \dots, 12\}$.

$H = \langle (123456789a), (012935a8476) \rangle$, $|H| = 110$.

H -orbit representatives: $4\langle 758a9c \rangle b$, $0\langle 1ab925 \rangle c$, $0\langle 43598c \rangle a$, $3\langle 498a65 \rangle b$, $2\langle 5c68b9 \rangle a$,
 $2\langle 378a45 \rangle 9$, $1\langle 265a89 \rangle 7$, $0\langle 398672 \rangle a$.

Design: $CUBE(14)$ on points $\{0, 1, \dots, 13\}$.

$H = \langle (2ca864)(3db975), (018)(279)(34b)(5cd) \rangle \cong PSL(2, 13)$.

H -orbit representatives: $0\langle 174cba \rangle d$, $4\langle 576acd \rangle b$, $5\langle 7bca86 \rangle d$.

Design: $CUBE(15)$ on points $\{0, 1, \dots, 14\}$.

$H = \langle (034)(179)(2be)(5ca)(68d), (084dc15)(2e367ba) \rangle \cong Alt(7)$.

H -orbit representatives: $5\langle 7dcba6 \rangle e$, $7\langle 8adce9 \rangle b$, $7\langle 8d9aec \rangle b$.

Design: $uCAND_CUBE(4^2)$ on points $\{0, 1, \dots, 7\}$ with groups $\{0, 1, 2, 3\}$,
 $\{4, 5, 6, 7\}$.

Cubes: $3\langle 425067 \rangle 1$, $0\langle 725134 \rangle 6$, $2\langle 347065 \rangle 1$, $7\langle 140352 \rangle 6$, $4\langle 062135 \rangle 7$, $3\langle 546021 \rangle 7$,
 $5\langle 403672 \rangle 1$, $4\langle 063152 \rangle 7$, $2\langle 413065 \rangle 7$, $0\langle 265374 \rangle 1$, $5\langle 741362 \rangle 0$, $3\langle 716524 \rangle 0$,
 $2\langle 157346 \rangle 0$, $3\langle 765420 \rangle 1$, $2\langle 763410 \rangle 5$, $0\langle 462157 \rangle 3$, $2\langle 140573 \rangle 6$, $3\langle 015276 \rangle 4$,
 $1\langle 542367 \rangle 0$, $3\langle 762510 \rangle 4$, $5\langle 034617 \rangle 2$, $1\langle 347506 \rangle 2$, $4\langle 013576 \rangle 2$, $0\langle 372465 \rangle 1$,
 $7\langle 412630 \rangle 5$, $2\langle 165437 \rangle 0$, $2\langle 043756 \rangle 1$, $6\langle 231470 \rangle 5$, $1\langle 673502 \rangle 4$, $5\langle 246710 \rangle 3$,
 $0\langle 341725 \rangle 6$, $7\langle 532401 \rangle 6$, $1\langle 327460 \rangle 5$, $3\langle 706145 \rangle 2$.

Design: $uCAND_CUBE(4^2 : 1)$ on points $\{0, 1, \dots, 8\}$ with groups $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$ and stem $\{8\}$.

$H = \langle (132)(576), (04)(15)(26)(37), (03)(12)(47)(56), (02)(13)(46)(57) \rangle$, $|H| = 24$.

H -orbit representatives: $0\langle 216374 \rangle 8$, $0\langle 174658 \rangle 3$, $0\langle 163527 \rangle 4$, $0\langle 264157 \rangle 3$, $0\langle 246175 \rangle 3$.

Design: $uCAND_CUBE(4^2 : 2)$ on points $\{0, 1, \dots, 9\}$ with groups $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$ and stem $\{8, 9\}$.

$H = \langle (0426)(1537), (03)(12)(47)(56) \rangle \cong C_4 \times C_2$.

H -orbit representatives: $1\langle 487659 \rangle 3$, $0\langle 526374 \rangle 8$, $1\langle 478569 \rangle 3$, $1\langle 469857 \rangle 3$, $0\langle 469158 \rangle 3$, $0\langle 146397 \rangle 5$, $1\langle 459768 \rangle 3$, $0\langle 245387 \rangle 9$, $0\langle 136527 \rangle 4$, $0\langle 126437 \rangle 5$, $0\langle 647985 \rangle 3$, $0\langle 365147 \rangle 2$, $0\langle 354167 \rangle 2$, $0\langle 145276 \rangle 3$, $0\langle 154267 \rangle 3$, $0\langle 257346 \rangle 1$.

Design: $uCAND_CUBE(4^2 : 3)$ on points $\{0, 1, \dots, 10\}$ with groups $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$ and stem $\{8, 9, 10\}$.

$H = \langle (89a), (03)(12), (47)(56), (02)(13)(46)(57) \rangle \cong C_3 \times C_2 \times C_2 \times C_2$.

H -orbit representatives: $2\langle 49568a \rangle 3$, $0\langle 152637 \rangle 4$, $0\langle 469578 \rangle 3$, $0\langle 127453 \rangle 6$, $0\langle 279548 \rangle 3$, $1\langle 25a348 \rangle 9$, $0\langle 264157 \rangle 3$, $0\langle 254176 \rangle 3$, $0\langle 256147 \rangle 3$, $0\langle 154376 \rangle 2$, $0\langle 164357 \rangle 2$, $0\langle 156347 \rangle 2$, $0\langle 179348 \rangle 2$, $0\langle 145367 \rangle 2$, $0\langle 354276 \rangle 1$, $0\langle 364257 \rangle 1$, $0\langle 356247 \rangle 1$.

Design: $uCAND_CUBE(4^3)$ on points $\{0, 1, \dots, 11\}$ with groups $\{0, 3, 6, 9\}$, $\{1, 4, 7, 10\}$, $\{2, 5, 8, 11\}$.

$H = \langle (0369)(147a)(258b), (012)(345)(678)(9ab) \rangle \cong C_{12}$.

H -orbit representatives: $5\langle 83706a \rangle b$, $0\langle 214867 \rangle 9$, $3\langle 8a0576 \rangle 9$, $a\langle 269531 \rangle 4$, $9\langle 824b61 \rangle 3$, $4\langle ba9816 \rangle 0$, $a\langle 708495 \rangle 1$, $1\langle 94b078 \rangle 6$, $5\langle 276813 \rangle 4$, $3\langle 85412a \rangle 9$, $6\langle a54281 \rangle 0$, $1\langle 506892 \rangle a$.

Design: $uCAND_CUBE(4^3 : 1)$ on points $\{0, 1, \dots, 12\}$ with groups $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$, $\{8, 9, 10, 11\}$ and stem $\{12\}$.

$H = \langle (486a597b), (03)(12), (45)(67), (02)(13) \rangle$, $|H| = 64$.

H -orbit representatives: $2\langle 4b679c \rangle 3$, $2\langle 4a958b \rangle 3$, $2\langle 49b58a \rangle 3$, $0\langle 379168 \rangle 2$, $0\langle 148259 \rangle 3$, $0\langle 279368 \rangle 1$.

Design: $uCAND_CUBE(4^3 : 2)$ on points $\{0, 1, \dots, 13\}$ with groups $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$, $\{8, 9, 10, 11\}$ and stem $\{12, 13\}$.

$H = \langle (048)(159)(26a)(37b), (03)(12), (47)(56), (8b)(9a), (8a)(9b), (46)(57), (02)(13), (cd), (48)(59)(6a)(7b) \rangle$, $|H| = 768$.

H -orbit representatives: $2\langle 4bd5ac \rangle 3$, $2\langle 4a859b \rangle 3$, $2\langle 4985ba \rangle 3$, $2\langle 49a58b \rangle 3$.

Design: $uCAND_CUBE(4^3 : 3)$ on points $\{0, 1, \dots, 14\}$ with groups $\{0, 1, 2, 3\}$, $\{4, 5, 6, 7\}$, $\{8, 9, 10, 11\}$ and stem $\{12, 13, 14\}$.

$H = \langle (cde), (23)(67)(8b)(9a)(de), (23)(46)(57), (03)(12), (45)(67), (8a)(9b), (02)(13) \rangle$, $|H| = 192$.

H -orbit representatives: $2\langle 4bd58c \rangle 3$, $2\langle 4a958b \rangle 3$, $2\langle 49b58a \rangle 3$, $2\langle 4985ab \rangle 3$, $2\langle 47a568 \rangle 3$, $2\langle 47b659 \rangle 3$, $2\langle 45a678 \rangle 3$, $0\langle 37a168 \rangle 2$, $0\langle 15a348 \rangle 2$, $0\langle 27a368 \rangle 1$.

Design: $uCAND_CUBE(2^4)$ on points $\{0, 1, \dots, 7\}$ with groups $\{0, 4\}$, $\{1, 5\}$, $\{2, 6\}$, $\{3, 7\}$.

$H = \langle (04), (15), (26), (37) \rangle \cong C_2 \times C_2 \times C_2 \times C_2$.

H -orbit representative: $0\langle 123567 \rangle 4$.

Design: $uCAND_CUBE(4^4)$ on points $\{a, b, c, d, e, f, g, h\} \times \{0, 1\}$ with groups $G_1 = \{a, b\} \times \{0, 1\}$, $G_2 = \{c, d\} \times \{0, 1\}$, $G_3 = \{e, f\} \times \{0, 1\}$, $G_4 = \{g, h\} \times \{0, 1\}$.

Let $G'_1 = \{a, b\}$, $G'_2 = \{c, d\}$, $G'_3 = \{e, f\}$, $G'_4 = \{g, h\}$. For each quadruple $q = \{x_1, x_2, x_3, x_4\}$ such that $|q \cap G'_i| = 1$ for each i , construct a nondegenerate $uCAND_CUBE(2^4)$ on points $q \times \{0, 1\}$ with groups $\{x_i\} \times \{0, 1\}$, $1 \leq i \leq 4$. This yields a set of $2^4 \cdot 8 = 4^4/2$ unoriented cubes that covers each unoriented face of width 4 relative to the groups G_i , hence the result is a nondegenerate $uCAND_CUBE(4^4)$.

6 The computation of cube designs

Small nondegenerate cube designs were found by hill climbing and by clique search. Most of the designs were found using a proposed subgroup H of their automorphism group, so that if a cube C is in the design, then so are all cubes in the H -orbit of C .

The hill climbs were programmed in the C language and the chosen group H was usually cyclic, but this was adequate for finding only about a third of the required designs. For example, a $uCUBE(8)$ (with H the trivial group) was first found by a hill climb with parameters `steps = 15000` and `stumble = 2`. To begin with, 35 random nondegenerate unoriented cubes were generated and a count was made of the total number of unoriented faces that were covered by these 35 cubes. A step consisted of picking a cube randomly and swapping two of its labels. If the swap did not decrease the count of

covered faces then it was kept, otherwise the cube was left alone. If 15000 such steps were taken and no design was found yet, then two “stumbles” were taken, that is, a cube was selected at random and a random change was made to it, and this was done two times and the count of faces updated. The process of taking steps was then repeated. No restarting was performed, only taking steps and stumbling until a design was found or the program was terminated because it was taking too long. With these parameters a $u\text{CUBE}(8)$ was found in 548740 total steps. Setting `steps = 14000` and `stumble = 2` required 10667619 total steps, while setting `steps = 10000` and `stumble = 1` required 10251849 total steps.

No systematic attempt was made to try to find ideal settings for the basic parameters, though some designs required more than a billion steps to be found this way, such as the bicyclic $u\text{CUBE}(10)$ given above, which required about 1.4 billion total steps.

The $\text{CUBE}(v)$ designs with $v = 12, 13, 14, 15$, and the $u\text{CAND_CUBE}(4^2 : s)$ and $u\text{CAND_CUBE}(4^3 : s)$ designs with $s = 1, 2, 3$, were found using the computational algebra and discrete mathematics system `GAP` [2], together with its `GRAPE` package [6] for computing with graphs with groups acting on them. The sophisticated permutation group machinery in `GAP` allowed us to generate and try out many possibilities for a subgroup H of the automorphism group of a required design, and the flexible and powerful clique finding machinery in `GRAPE` was used to find an H -invariant design (for a suitable H) when one existed. Possible groups H acting on $\text{CUBE}(v)$ designs were obtained from the `GAP` library of primitive groups, usually from those of degree v . For a candelabra cube design, possible groups of automorphisms were found amongst conjugacy class representatives H of the subgroups of the group G consisting of all the permutations of the points which fix both the set of groups and the stem (setwise). These representative subgroups were computed via the `GAP` function `ConjugacyClassesSubgroups`.

Now given a possible group H of automorphisms, we proceed as follows. For a cube design we compute the H -orbits of all nondegenerate cubes with labels in the point set. For a (unoriented) candelabra cube design, however, we only consider those orbits of (unoriented) nondegenerate cubes having each face of the required width. We then only keep the cube orbits containing no pair of distinct cubes with a common face. Call the resulting set of orbits Ω . We construct a vertex-weighted graph Γ with vertex set Ω , the weight of a vertex being its size (as an orbit), and we join two distinct vertices in Γ by an edge precisely when the union of these orbits contains no pair of distinct

cubes with a common face. Then the union of the vertices in a clique (set of pairwise adjacent vertices) of Γ gives the set of cubes of an H -invariant nondegenerate partial cube design, and such a clique gives a design of the required type if and only if its vertex weights sum to the number of cubes in such a design. For example, for a CUBE(15) the number of cubes is 1365, and for a u CAND_CUBE($4^3 : 3$) this number is 240. To find such a vertex-weighted clique, or to determine that none such exist, we used the GRAPE function `CompleteSubgraphsOfGivenSize`.

Each small design given in the previous section was double-checked to ensure it has the required properties. Electronic files of these designs are available from the authors upon request, together with log-files of the GAP/GRAPE computations.

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