Constructing t-designs from t-wise balanced designs

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Abstract

We give a construction to obtain a t-design from a t-wise balanced design. More precisely, given a positive integer k and a t-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design $\mathcal{D}$, with with all block-sizes k_i occurring in $\mathcal{D}$ and $1 \le t \le k \le k_1 < k_2 < \cdots < k_s$, the construction produces a t-$(v, k, n\lambda)$ design $\mathcal{D}^*$, with $n = \mathrm{lcm}(\binom{k_1-t}{k-t}, \ldots, \binom{k_s-t}{k-t})$. We prove that $\mathrm{Aut}(\mathcal{D})$ is a subgroup of $\mathrm{Aut}(\mathcal{D}^*)$, with equality when both $\lambda = 1$ and $t < k$. We employ our construction in another construction, which, given a t-(v, k, λ) design with $1 \le t < k < v$, and a point of this design, yields a t-$(v-1, k-1, (k-t)\lambda)$ design. Many of the t-designs coming from our constructions appear to be new.

1 Introduction

For t a positive integer, a *t-wise balanced design* $\mathcal{D}$ is an ordered pair $(X, \mathcal{B})$, where X is a finite non-empty set (of *points*) and $\mathcal{B}$ is a finite non-empty multiset of subsets of X (called *blocks*), such that every t-subset of X is contained in a constant number $\lambda > 0$ of blocks. If $v = |X|$ and K is the set of sizes of the blocks, then we call $\mathcal{D}$ a t-(v, K, λ) *design*. If all blocks of $\mathcal{D}$ have the same size k (i.e. $K = \{k\}$), then $\mathcal{D}$ is called a t-*design* or a t-(v, k, λ) *design*.

In this note we give a construction (the $*$-construction) to obtain a t-design from a t-wise balanced design. More precisely, given a positive integer k and a t-$(v, \{k_1, k_2, \dots, k_s\}, \lambda)$ design $\mathcal{D}$, with with all block-sizes k_i occurring in $\mathcal{D}$ and $1 \le t \le k \le k_1 < k_2 < \cdots < k_s$, the $*$-construction produces a t-$(v, k, n\lambda)$ design $\mathcal{D}^*$, with $n = \mathrm{lcm}(\binom{k_1-t}{k-t}, \dots, \binom{k_s-t}{k-t})$. We prove that $\mathrm{Aut}(\mathcal{D})$ is a subgroup of $\mathrm{Aut}(\mathcal{D}^*)$, with equality when both $\lambda = 1$ and $t < k$. We employ the $*$-construction in another construction (the #-construction), which, given a t-(v, k, λ) design with $1 \le t < k < v$, and a point of this design, yields a t-$(v-1, k-1, (k-t)\lambda)$ design. Many of the t-designs coming from our constructions appear to be new, and although they usually have repeated blocks, they often, via their constructions, have quite large automorphism groups.

2 The $*$-construction

The input to the $*$-construction consists of positive integers t and k, and a t-$(v, \{k_1, k_2, \dots, k_s\}, \lambda)$ design $\mathcal{D}$, with all block-sizes k_i occurring in $\mathcal{D}$ and $1 \le t \le k \le k_1 < k_2 < \cdots < k_s$. Now for $i = 1, 2, \dots, s$ define

$$n_i = \binom{k_i - t}{k - t}, \quad n = \mathrm{lcm}(n_1, n_2, \dots, n_s), \quad m_i = \frac{n}{n_i}. \tag{1}$$

The output of the $*$-construction is a block design $\mathcal{D}^*$, which we prove below to be a t-$(v, k, n\lambda)$ design.

The point-set of $\mathcal{D}^*$ is that of $\mathcal{D}$, and to construct the block-multiset $\mathcal{B}^*$ of $\mathcal{D}^*$ we proceed as follows:

- start by setting $\mathcal{B}^*$ to be the empty multiset;
- for each $i = 1, 2, \dots, s$ and for each block $B \in \mathcal{B}$ of size k_i (including repeats) do:
  - insert m_i copies of every k-subset of B into $\mathcal{B}^*$.

Clearly, $\mathcal{D}^*$ depends on the choice of k as well as on $\mathcal{D}$. Less obviously, since the t-wise balanced design $\mathcal{D}$ may be t'-wise balanced for some $t' \neq t$, $\mathcal{D}^*$ may depend on the choice of t. When we wish to make these dependencies explicit, we shall use the notation $\mathcal{D}^*(t, k)$ instead of $\mathcal{D}^*$.

Theorem 2.1 *Let k be a positive integer and let $\mathcal{D} = (X, \mathcal{B})$ be a t-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design, with all block-sizes k_i occurring in $\mathcal{D}$ and $1 \leq t \leq k \leq k_1 < k_2 < \cdots < k_s$. Then $\mathcal{D}^* = \mathcal{D}^*(t, k) = (X, \mathcal{B}^*)$ is a t-$(v, k, n\lambda)$ design, where $n = \mathrm{lcm}(n_1, n_2, \ldots, n_s)$ and $n_i = \binom{k_i - t}{k - t}$.*

Proof. Let T be any t-subset of X. Suppose that B is a block of $\mathcal{B}$ of size k_i containing T. Then the number of k-subsets of B which contain T is $n_i = \binom{k_i - t}{k - t}$. Each of these k-subsets is added to $\mathcal{B}^*$ exactly $m_i = n/n_i$ times. Hence B contributes exactly $n_i m_i = n$ blocks containing T to $\mathcal{B}^*$. Now T is contained in exactly λ blocks in $\mathcal{B}$, and so in exactly $n\lambda$ blocks in $\mathcal{B}^*$. ■

We have defined n to be $\mathrm{lcm}(n_1, n_2, \ldots, n_s)$. We could have chosen n to be any common multiple of $\{n_1, n_2, \ldots, n_s\}$, but, in order to keep $n\lambda$ as small as possible, we choose the least common multiple. We also remark that the $*$-construction works perfectly well when $s = 1$, that is, when $\mathcal{D}$ is a t-design.

Example 1 Let $\mathcal{D}$ be the 2-$(11, \{3, 5\}, 1)$ design with point-set $X = \{1, 2, \ldots, 9, T, E\}$ (here $T = 10$ and $E = 11$), and block-multiset $\mathcal{B} =$

$$[167, 18E, 19T, 268, 279, 2TE, 369, 37E, 38T, 46T, 478, 49E, 56E, 57T, 589, 12345]$$

(see [1, p.187]).

(a) Suppose $t = k = 2$. Here $k_1 = 3$, $k_2 = 5$, and each $n_i = n = m_i = 1$. So $\mathcal{D}^*(2, 2)$ is the 2-$(11, 2, 1)$ design consisting of all the 2-subsets of X.

(b) The case $t = 2$, $k = 3$ is more interesting. Here $k = k_1 = 3$, $k_2 = 5$, $n_1 = 1$, $n_2 = 3$, $n = 3$, $m_1 = 3$, and $m_2 = 1$. So $\mathcal{D}^* = \mathcal{D}^*(2, 3)$ is a 2-$(11, 3, 3)$ design, an $(11, 55, 15, 3, 3)$-BIBD. The block-multiset of $\mathcal{D}^*$ consists of three copies of each block of $\mathcal{D}$ of size 3, together with all the 3-subsets of $\{1, 2, 3, 4, 5\}$.

The $*$-construction was found as a result of looking for 2-designs with repeated blocks to help fill up Preece's catalogue [4]. Many new examples coming from this construction have since gone into the catalogue.

3 The #-construction

Let $\mathcal{T} = (X, \mathcal{B})$ be a t-(v, k, λ) design with $1 \le t < k < v$, and let $x \in X$. We employ the $*$-construction in a new construction (the #-construction) which produces a t-$(v-1, k-1, (k-t)\lambda)$ design when given input $\mathcal{T}$ and x. The #-construction proceeds as follows:

Let $X' = X \setminus \{x\}$, and let $\mathcal{B}'$ be the multiset consisting of all $B \setminus \{x\}$ with $B \in \mathcal{B}$ (counting repeats). Denote the resulting block design $(X', \mathcal{B}')$ by $\mathcal{T} \setminus x$, which is a t-$(v-1, \{k-1, k\}, \lambda)$ design (whose isomorphism class may depend on the choice of x). Next, apply the $*$-construction with input t, $k-1$ and $\mathcal{T} \setminus x$ to obtain $(\mathcal{T} \setminus x)^*(t, k-1)$, a t-$(v-1, k-1, (k-t)\lambda)$ design. We denote this output of the #-construction by $\mathcal{T}^{\#}(t, x)$.

Example 2 Start with the large Witt design $\mathcal{W}$, the unique (up to isomorphism) 5-$(24, 8, 1)$ design; see [3, Chapter 8], where $\mathcal{W}$ is called the Mathieu design $\mathcal{M}_{24}$. Now $\mathcal{W}$ is also a 4-$(24, 8, 5)$ design, a 3-$(24, 8, 21)$ design, and a 2-$(24, 8, 77)$ design. Let x be a point of $\mathcal{W}$ (it matters not which one, since the automorphism group M_{24} of $\mathcal{W}$ acts transitively (in fact 5-transitively) on the point-set of $\mathcal{W}$). Then $\mathcal{W}^{\#}(5, x)$ is a 5-$(23, 7, 3)$ design, $\mathcal{W}^{\#}(4, x)$ is a 4-$(23, 7, 20)$ design, $\mathcal{W}^{\#}(3, x)$ is a 3-$(23, 7, 105)$ design, and $\mathcal{W}^{\#}(2, x)$ is a 2-$(23, 7, 462)$ design.

Example 3 Start with a projective plane $\mathcal{P} = (X, \mathcal{B})$ of order $m \ge 2$, a 2-$(m^2+m+1, m+1, 1)$ design. Now, given any $x \in X$, construct $\mathcal{P}^{\#}(2, x)$, which is a 2-$(m^2+m, m, m-1)$ design.

4 Automorphism groups

The automorphism group of a t-wise balanced design $\mathcal{D} = (X, \mathcal{B})$, denoted $\mathrm{Aut}(\mathcal{D})$, is the group consisting of all the permutations of X which leave the block-multiset $\mathcal{B}$ invariant. We now investigate the relationship of the automorphism groups of $\mathcal{D}$ and $\mathcal{D}^*(t, k)$. For a block $B \in \mathcal{B}$, we let $\mathrm{mult}(B)$ denote its multiplicity in $\mathcal{B}$.

Theorem 4.1 *Let k be a positive integer, let $\mathcal{D} = (X, \mathcal{B})$ be a t-$(v, \{k_1, k_2, \ldots, k_s\}, \lambda)$ design, with all block-sizes k_i occurring in $\mathcal{D}$ and $1 \le t \le k \le k_1 < k_2 < \cdots < k_s$, and let $\mathcal{D}^* = \mathcal{D}^*(t, k) = (X, \mathcal{B}^*)$ be the t-design obtained from the $*$-construction. Then*

(i) $\mathrm{Aut}(\mathcal{D}) \subseteq \mathrm{Aut}(\mathcal{D}^*)$*;*
(ii) *if* $\lambda = 1$ *and* $t < k$*, then* $\mathrm{Aut}(\mathcal{D}) = \mathrm{Aut}(\mathcal{D}^*)$*.*

Proof. (i) Let $\alpha \in \mathrm{Aut}(\mathcal{D})$. Let B^* be an arbitrary block in $\mathcal{B}^*$, hence there is a block $B \in \mathcal{B}$ which contains B^* as a k-subset. Suppose that $\alpha(B) = C$ for some block $C \in \mathcal{B}$, and that $\alpha(B^*) = C^*$. Then clearly C^* is a k-subset of C, a block of $\mathcal{B}$, hence $C^* \in \mathcal{B}^*$. Now we must show that $\mathrm{mult}(C^*) = \mathrm{mult}(B^*)$ (in $\mathcal{B}^*$) to conclude that $\alpha \in \mathrm{Aut}(\mathcal{D}^*)$.

Fix i. Let $B_1, B_2, \ldots, B_d$ be the *distinct* blocks of $\mathcal{B}$ of size k_i which contain B^*, and let $C_1, C_2, \ldots, C_e$ be the distinct blocks of $\mathcal{B}$ of size k_i which contain C^*. Now, because $\alpha \in \mathrm{Aut}(\mathcal{D})$, we must have $d = e$ and for every j with $1 \leq j \leq d$ there must exist a unique j' with $1 \leq j' \leq d$ for which $\alpha(B_j) = C_{j'}$. Hence $\mathrm{mult}(B_j) = \mathrm{mult}(C_{j'})$ since α preserves block multiplicities.

Now let f_i be the number of blocks (counting multiplicities) of $\mathcal{B}$ of size k_i which contain B^*, and let g_i be the number of blocks (counting multiplicities) of $\mathcal{B}$ of size k_i which contain C^*. Then $g_i = \sum_{j'=1}^{d} \mathrm{mult}(C_{j'}) = \sum_{j=1}^{d} \mathrm{mult}(B_j) = f_i$, and so, in $\mathcal{B}^*$, we have $\mathrm{mult}(C^*) = \sum_{i=1}^{s} g_i m_i = \sum_{i=1}^{s} f_i m_i = \mathrm{mult}(B^*)$ (m_i defined in (1)), as required. Hence $\alpha \in \mathrm{Aut}(\mathcal{D}^*)$.

(ii) We first note that, because $\lambda = 1$, then $\mathrm{mult}(B) = 1$ for every block $B \in \mathcal{B}$. Secondly, if R^* is an arbitrary block in $\mathcal{B}^*$ then, again because $\lambda = 1$, there is a unique block $R \in \mathcal{B}$, with $R^* \subseteq R$.

Now let $\gamma \in \mathrm{Aut}(\mathcal{D}^*)$. We must show that, for every block $B \in \mathcal{B}$, we have $\gamma(B) \in \mathcal{B}$. Then, from above, $\mathrm{mult}(\gamma(B)) = 1 = \mathrm{mult}(B)$, so $\gamma \in \mathrm{Aut}(\mathcal{D})$. This will show that $\mathrm{Aut}(\mathcal{D}^*) \subseteq \mathrm{Aut}(\mathcal{D})$; part (i) then gives the result.

Fix i. Let B be an arbitrary block of $\mathcal{B}$ of size k_i, and let B^* be an arbitrary k-subset of B, and let $\gamma(B^*) = C^*$. Now, because $\gamma \in \mathrm{Aut}(\mathcal{D}^*)$, then $C^* \in \mathcal{B}^*$. So, from above, there is a unique block $C \in \mathcal{B}$, with $C^* \subseteq C$. We will show that $\gamma(B) = C$.

First we show that $\gamma(B) \subseteq C$. Suppose that $\gamma(B) \nsubseteq C$, then there is an element $x \in B \backslash B^*$ with $\gamma(x) \notin C$. Let D be a $(k-1)$-subset of $B^* \subseteq B$, then $D^* = \{x\} \cup D$ is a k-subset of $B \in \mathcal{B}$, so $D^* \in \mathcal{B}^*$. Hence $E^* = \gamma(D^*) \in \mathcal{B}^*$, and there is a block $E \in \mathcal{B}$ with $E^* \subseteq E$. Now $E \neq C$ because $\gamma(x) \in E$ but $\gamma(x) \notin C$. Hence E and C are distinct blocks of $\mathcal{B}$. However, $\gamma(D) \subseteq E$, and $D \subseteq B^*$ so $\gamma(D) \subseteq \gamma(B^*) = C^* \subseteq C$. Now $t < k$ so $t \leq k - 1 = |\gamma(D)|$. Now let T be any t-subset of $\gamma(D)$, then the distinct blocks E and C both contain T, a contradiction since $\lambda = 1$. Hence $\gamma(B) \subseteq C$.

To show that $C \subseteq \gamma(B)$ we show that $\gamma^{-1}(C) \subseteq B$ by noting that $\gamma^{-1}(C^*) = B^*$, and so the proof follows as above. Hence $\gamma(B) = C$ and, since i was arbitrary, the result is proved. ∎

Example 4 We take $\mathcal{D}$ to be the 2-$(11, \{3, 5\}, 1)$ design of Example 1. Then $|\mathrm{Aut}(\mathcal{D})| = 120$; indeed $\mathrm{Aut}(\mathcal{D})$ is isomorphic to $\mathrm{Sym}(5)$, and acts naturally as this group on the subset $\{1, 2, 3, 4, 5\}$ of the point-set (checked using GAP [2] and its DESIGN package [5]).

(a) $\mathcal{D}^*(2, 2)$ is the complete 2-$(11, 2, 1)$ design. Hence $\mathrm{Aut}(\mathcal{D}) \subseteq \mathrm{Aut}(\mathcal{D}^*(2, 2)) = \mathrm{Sym}(11)$, illustrating Theorem 4.1(i), and also showing that if $\lambda = 1$ and $t = k$ then $\mathrm{Aut}(\mathcal{D}) \neq \mathrm{Aut}(\mathcal{D}^*(t, k))$ is possible (see Theorem 4.1(ii)).

(b) $\mathcal{D}^* = \mathcal{D}^*(2, 3)$ is a 2-$(11, 3, 3)$ design with $|\mathrm{Aut}(\mathcal{D}^*)| = 120$ (double checked with the DESIGN package). This illustrates Theorem 4.1(ii).

Example 5 This example shows that if $\lambda > 1$ then $\mathrm{Aut}(\mathcal{D}) \neq \mathrm{Aut}(\mathcal{D}^*(t, k))$ is possible, even when $t < k$. We apply the #-construction to the projective plane $\mathcal{P}$ of order 4, to obtain a 2-$(20, 4, 3)$ design $\mathcal{P}^\# = \mathcal{P}^\#(2, x) = (X, \mathcal{B})$, which has a point-transitive automorphism group of order 5760. Then, we take $x \in X$ and obtain a 2-$(19, \{3, 4\}, 3)$ design $\mathcal{D} = \mathcal{P}^\# \setminus x$ (using the notation of Section 3). (The choice of x does not affect the isomorphism class of $\mathcal{D}$ since $\mathcal{P}^\#$ is point-transitive). Finally, construct a 2-$(19, 3, 6)$ design $\mathcal{D}^* = \mathcal{D}^*(2, 3)$. It turns out that $|\mathrm{Aut}(\mathcal{D})| = 288$, but $|\mathrm{Aut}(\mathcal{D}^*)| = 576$. The construction of these designs and the determination of their automorphism groups was done using the DESIGN package.

Example 6 The DESIGN package shows that, up to isomorphism, there are exactly four 2-$(11, \{4, 5\}, 2)$ designs (not counting the unique 2-$(11, 5, 2)$ design). These designs $\mathcal{D}$ have automorphism groups of orders 6, 8, 12, and 120, as do the corresponding $\mathcal{D}^*(2, 4)$, which are (believed to be new) 2-$(11, 4, 6)$ designs. Note that these examples show that the converse of Theorem 4.1(ii) does not hold.

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