Uniform semi-Latin squares and their Schur-optimality

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Abstract

Let n and k be integers, with $n > 1$ and $k > 0$. An $(n \times n)/k$ *semi-Latin square* S is an $n \times n$ array, whose entries are k-subsets of an nk-set, the set of *symbols* of S, such that each symbol of S is in exactly one entry in each row and exactly one entry in each column of S. Semi-Latin squares form an interesting class of combinatorial objects which are useful in the design of comparative experiments. We say that an $(n \times n)/k$ semi-Latin square S is *uniform* if there is a constant μ such that any two entries of S, not in the same row or column, intersect in exactly μ symbols (in which case $k = \mu(n-1)$). We prove that a uniform $(n \times n)/k$ semi-Latin square is Schur-optimal in the class of $(n \times n)/k$ semi-Latin squares, and so is optimal (for use as an experimental design) with respect to a very wide range of statistical optimality criteria. We give a simple construction to make an $(n \times n)/k$ semi-Latin square S from a transitive permutation group G of degree n and order nk, and show how certain properties of S can be determined from permutation group properties of G. If G is 2-transitive then S is uniform, and this provides us with Schur-optimal semi-Latin squares for many values of n and k for which optimal $(n \times n)/k$ semi-Latin squares were previously unknown for any optimality criterion. The existence of a uniform $(n \times n)/((n-1)\mu)$ semi-Latin square for all integers $\mu > 0$ is shown to be equivalent to the existence of $n-1$ mutually orthogonal Latin squares (MOLS) of order n. Although there are not even two MOLS of order 6, we construct uniform, and hence Schur-optimal, $(6 \times 6)/(5\mu)$ semi-Latin squares for all integers $\mu > 1$.

1 Introduction

Let n and k be integers, with $n > 1$ and $k > 0$. An $(n \times n)/k$ *semi-Latin square* S is an $n \times n$ array, whose entries are k-subsets of an nk-set, the set of *symbols* of S, such that each symbol of S is in exactly one entry in each row and exactly one entry in each column of S. The entry in row i and column j is called the (i,j)*-entry* of S and is denoted by $S(i,j)$. We consider two $(n \times n)/k$ semi-Latin squares to be *isomorphic* if one can be obtained from the other by applying an *isomorphism*, which is a sequence of one or more of: a row permutation, a column permutation, transposing, and renaming symbols. An *automorphism* of S is an isomorphism mapping S to itself. By identifying a 1-subset of symbols with the symbol it contains, we consider an $(n \times n)/1$ semi-Latin square to be the same thing as a Latin square of order n.

For example, here are two nonisomorphic $(3 \times 3)/2$ semi-Latin squares, both having symbol-set $\{1, \ldots, 6\}$:

$$X := \begin{array}{|c|c|c|} \hline 1\,4 & 2\,5 & 3\,6 \\ \hline 3\,6 & 1\,4 & 2\,5 \\ \hline 2\,5 & 3\,6 & 1\,4 \\ \hline \end{array}\,, \qquad Y := \begin{array}{|c|c|c|} \hline 1\,4 & 2\,5 & 3\,6 \\ \hline 3\,5 & 1\,6 & 2\,4 \\ \hline 2\,6 & 3\,4 & 1\,5 \\ \hline \end{array}\,. \tag{1}$$

Observe that symbols 2 and 5 occur together in the three entries $X(1,2)$, $X(2,3)$ and $X(3,1)$ of X, but no pair of distinct symbols occur together in more than one entry of Y.

Semi-Latin squares form an interesting class of combinatorial objects which are used in the design of comparative experiments (see [18, 1, 2, 20, 5]). Moreover, the duals of $(n \times n)/k$ semi-Latin squares are certain factorial designs, and optimal $(n \times n)/k$ semi-Latin squares dualize to optimal factorial designs of this type, with respect to a wide range of statistical optimality criteria (see [5]). However, until now, optimal $(n \times n)/k$ semi-Latin squares were only known (for certain optimality criteria) when there are k mutually orthogonal Latin squares (MOLS) of order n [11], when there are $n-1$ MOLS of order n and k is a multiple of $n-1$ [2], when $n=3$ [2], when $n=k=4$ [12], and for "regular-graph" $(6\times 6)/2$ [8] and $(6\times 6)/3$ [21, 5] semi-Latin squares.

In this paper, we introduce the concept of a uniform semi-Latin square. An $(n\times n)/k$ semi-Latin square S is *uniform* if there is a constant $\mu=\mu(S)$ such that any two entries of S, not in the same row or column, intersect in exactly μ symbols. For example, the semi-Latin square Y in (1) is uniform, with $\mu(Y)=1$. We prove that a uniform $(n\times n)/k$ semi-Latin square is Schur-optimal (defined in Section 2) in the class of $(n\times n)/k$ semi-Latin squares, and so, in particular, is Φ_p-optimal, for all $p\in(0,\infty)$, as well as A-, D-, and E-optimal (see [15, 6]).

We shall give a simple construction to make an $(n\times n)/k$ semi-Latin square S from a transitive permutation group G of degree n and order nk, and show how certain properties of S can be determined from permutation group properties of G. If G is 2-transitive then S is uniform, and this provides us with Schur-optimal semi-Latin squares for many values of n and k for which optimal $(n\times n)/k$ semi-Latin squares were previously unknown for any optimality criterion.

The existence of a uniform $(n\times n)/((n-1)\mu)$ semi-Latin square for all integers $\mu>0$ is shown to be equivalent to the existence of $n-1$ MOLS of order n. Although there are not even two MOLS of order 6, we construct uniform, and hence Schur-optimal, $(6\times 6)/(5\mu)$ semi-Latin squares for all integers $\mu>1$.

The reader who is unfamiliar with statistical design theory and the theory of optimal designs should consult the excellent survey article [6], which was written for combinatorialists. Other useful references for these topics include [19, 3, 4, 10]. An excellent reference for permutation groups is [9].

2 Block designs and Schur-optimality

In this Section, we collect definitions we will need for block designs and Schur-optimality.

A *block design* is an ordered pair $(V,\mathcal{B})$, such that V is a finite non-empty set of *points*, and $\mathcal{B}$ is a (disjoint from V) finite non-empty collection (or multiset) of non-empty subsets of V called *blocks*, such that every point is in at least one block. Thus, all our block designs are "binary" in that no block can have a repeated point, but we certainly allow repeated blocks, and repeated blocks are counted in any count of blocks. A 1-(v,k,r) *design* is a block design having exactly v points, with each block having size k and with each point in exactly r blocks.

If we ignore the row and column structure of an $(n \times n)/k$ semi-Latin square S, we obtain its *underlying block design* (or *quotient block design* [2]), denoted $\Delta(S)$, the block design whose points are the symbols of S and whose block multiset is $[S(i,j) : 1 \le i,j \le n]$. Note that $\Delta(S)$ is a 1-(nk,k,n) design.

Let Δ be a block design having v points and b blocks. The *point graph* of Δ is the graph whose vertices are the points of Δ, and with $\{\alpha,\beta\}$ an edge precisely when points α and β are distinct and both in some block of Δ. We say that Δ is *connected* if its point graph is connected, and that a semi-Latin square is *connected* if its underlying block design is connected. Thus, for the examples in (1), we see that X is not connected and Y is connected. The *incidence matrix* of Δ is the $v \times b$ matrix whose rows are indexed by the points of Δ and columns by the blocks of Δ, with the (α,B)-entry being 1 if the point α is in the block B, and 0 otherwise. The *dual* of Δ is obtained by interchanging the roles of points and blocks, and is defined to be the block design whose incidence matrix is the transpose of that of Δ. Note that the dual of a 1-(v,k,r) design is a 1-$(vr/k,r,k)$ design. The *concurrence matrix* of Δ is the $v \times v$ matrix whose rows and columns are indexed by the points, and whose (α,β)-entry is the number of blocks containing both α and β. Note that if N is the incidence matrix of Δ, then its concurrence matrix is NN^{T}, and the concurrence matrix of the dual of Δ is $N^{\mathrm{T}}N$ (where N^{T} denotes the transpose of N).

Now suppose Δ is a 1-(v,k,r) design with incidence matrix N. The *information matrix* of Δ is

$$C(\Delta) := rI_v - k^{-1}NN^{\mathrm{T}}.$$

The eigenvalues of this information matrix are all real and lie in the interval $[0,r]$. At least one eigenvalue is zero: an associated eigenvector is the all-1 vector. The remaining eigenvalues are all non-zero if and only if Δ is connected. (See, for example, [6].) Let $\delta_0 \le \delta_1 \le \cdots \le \delta_{v-1}$ be the eigenvalues of $C(\Delta)$. We say that Δ is *Schur-optimal* in a class $\mathcal{C}$ of 1-(v,k,r) designs containing Δ if Δ is connected and for each design $\Gamma \in \mathcal{C}$, with information matrix $C(\Gamma)$ having eigenvalues $\gamma_0 \le \gamma_1 \le \cdots \le \gamma_{v-1}$, we have:

$$\sum_{i=0}^{\ell} \delta_i \ge \sum_{i=0}^{\ell} \gamma_i, \quad \text{for } \ell = 0, 1, \ldots, v-1.$$

A Schur-optimal design need not exist within a given class $\mathcal{C}$, but when it does, that design is optimal in $\mathcal{C}$ with respect to a very wide range of statistical optimality criteria, including being Φ_p-optimal, for all $p \in (0, \infty)$, and also A- D- and E-optimal. This was proved in [15]; see also [6, 19] for definitions of these optimality criteria and more on this result.

Following the analysis in [2], we consider an $(n \times n)/k$ semi-Latin square to be *optimal* with respect to a given optimality criterion if and only if its underlying block design is optimal with respect to that criterion in the class of underlying block designs of $(n \times n)/k$ semi-Latin squares. In particular, an $(n \times n)/k$ semi-Latin square is *Schur-optimal* if its underlying block design is Schur-optimal in the class of underlying block designs of $(n \times n)/k$ semi-Latin squares.

3 Uniform semi-Latin squares

Recall that a semi-Latin square S is *uniform* if there is a constant $\mu = \mu(S)$ such that any two entries of S, not in the same row or column, intersect in exactly μ symbols.

Lemma 3.1 *If S is a uniform $(n \times n)/k$ semi-Latin square then $\mu(S) = k/(n-1)$, and in particular, $n-1$ divides k.*

Proof. Let S be a uniform $(n \times n)/k$ semi-Latin square, and let $i, j \in \{1, \ldots, n\}$. We count in two ways the number of triples (i', j', α), such that $i', j' \in \{1, \ldots, n\}$, $i' \ne i$, $j' \ne j$, and $\alpha \in S(i,j) \cap S(i', j')$. We get that $(n-1)^2 \mu(S) = k(n-1)$, and the result follows. ∎

Let s be a positive integer. An *s-fold inflation* of an $(n \times n)/k$ semi-Latin square is obtained by replacing each symbol α in the semi-Latin square by s symbols $\sigma_{\alpha,1}, \ldots, \sigma_{\alpha,s}$, such that $\sigma_{\alpha,i} = \sigma_{\beta,j}$ if and only if $\alpha = \beta$ and $i = j$. The result is an $(n \times n)/(ks)$ semi-Latin square. For example, the square X

in (1) is a 2-fold inflation of

1	2	3
3	1	2
2	3	1

.

The *superposition* of an $(n \times n)/k$ semi-Latin square with an $(n \times n)/\ell$ semi-Latin square (with disjoint symbol sets) is obtained by superimposing the first square upon the second, giving an $(n \times n)/(k + \ell)$ semi-Latin square. For example, the square Y in (1) is the superposition of

1	2	3
3	1	2
2	3	1

and

4	5	6
5	6	4
6	4	5

.

Lemma 3.2 *If S is a uniform semi-Latin square then an s-fold inflation of S is also uniform, and if S and T are both $n \times n$ uniform semi-Latin squares (with disjoint symbol sets) then the superposition of S and T is also uniform.*

Proof. Straightforward. ■

Theorem 3.3 *An $(n \times n)/(n-1)$ semi-Latin square S is uniform if and only if S is a superposition of $n-1$ MOLS of order n.*

Proof. Suppose S is a uniform $(n \times n)/(n-1)$ semi-Latin square. By Lemma 3.1, $\mu(S) = 1$, so any two entries of S in different positions meet in 0 or 1 points, so every pair of distinct symbols of S occur together in at most one entry. Bailey [2, Theorem 6.4] shows that an $(n \times n)/(n-1)$ semi-Latin square with this property must be a superposition of $n-1$ MOLS of order n.

Conversely, suppose S is a superposition of $n-1$ MOLS of order n, and consider entries $S(i,j)$ and $S(i',j')$ of S, with $i \neq i'$ and $j \neq j'$. Now $|S(i,j) \cap S(i',j')| \leq 1$, for otherwise there would be two (or more) symbols from orthogonal Latin squares occurring together in more than one entry of S, and this cannot happen. Now each of the $n-1$ symbols in $S(i,j)$ must occur in row i', no two of these can occur together in any entry in this row, and none can occur in column j, so we must have $|S(i,j) \cap S(i',j')| = 1$. ■

Uniform semi-Latin squares can thus be seen as generalizing the concept of complete sets of MOLS (i.e. sets of $n-1$ MOLS of order n). Since the μ-fold inflation of a uniform semi-Latin square is uniform, we see that the existence of a uniform $(n \times n)/((n-1)\mu)$ semi-Latin square for all integers $\mu > 0$ is equivalent to the existence of a complete set of MOLS of order n, and such a set exists if n is a prime power. It is a major unsolved problem whether such a set exists for some n not a prime power, so when n is not a prime power the existence question for a uniform $(n \times n)/((n-1)\mu)$ semi-Latin square for a given μ can be very difficult indeed. We will show how to make a uniform $(n \times n)/k$ semi-Latin square given a 2-transitive permutation group of degree n and order nk. (An equivalent construction is given in [23], where the interest is in producing efficient partially balanced incomplete-block designs with respect to rectangular association schemes. Semi-Latin squares, their optimality, or that of their duals, are not considered in [23].) We will also construct uniform $(6 \times 6)/(5\mu)$ semi-Latin squares for all integers $\mu > 1$.

The statistical importance of uniform semi-Latin squares comes from the following result. (We exclude the case $n = 2$ since no $(2 \times 2)/k$ semi-Latin square is connected.)

Theorem 3.4 *Let $n > 2$ and let S be a uniform $(n \times n)/k$ semi-Latin square. Then S is Schur-optimal; that is, the underlying block design of S is Schur-optimal in the class of underlying block designs of $(n \times n)/k$ semi-Latin squares.*

Proof. Let Δ be the underlying block design of S, let N be the incidence matrix of Δ, and let $i, j, i', j' \in \{1, \ldots, n\}$, with $(i, j) \neq (i', j')$. If $i = i'$ or $j = j'$ then $|S(i, j) \cap S(i', j')| = 0$, and otherwise $|S(i, j) \cap S(i', j')| = \mu(S) = k/(n-1)$. Thus the dual Δ^* of Δ is a partially balanced incomplete-block design with respect to the L_2-type association scheme, so it is straightforward to work out the eigenvalues and their multiplicities for the concurrence matrix $N^{\mathrm{T}}N$ of Δ^* (see, for example, [24]). These eigenvalues are nk with multiplicity 1, $nk/(n-1)$ with multiplicity $(n-1)^2$, and 0 with multiplicity $2n-2$. The non-zero eigenvalues of $N^{\mathrm{T}}N$, as well as their multiplicities, are the same as for NN^{T}. It follows that the eigenvalues $\delta_0, \ldots, \delta_{nk-1}$ of the information matrix $C(\Delta) := nI_{nk} - k^{-1}NN^{\mathrm{T}}$ of Δ, in non-decreasing order, satisfy:

$$0 = \delta_0 < n - n/(n-1) = \delta_1 = \cdots = \delta_{(n-1)^2} < n = \delta_{(n-1)^2+1} = \cdots = \delta_{nk-1}.$$

(Note that, since S is uniform and $n > 2$, we have $nk - 1 \geq n(n-1) - 1 > (n-1)^2$.)

Now let R be any $(n \times n)/k$ semi-Latin square, let Γ^* be the dual block design of the underlying block design Γ of R, and let M be the incidence matrix of Γ. The rows and columns of the concurrence matrix $M^{\mathrm{T}}M$ of Γ^* are indexed by the n^2 entries of R, with the $(R(i,j), R(i',j'))$-entry of $M^{\mathrm{T}}M$ being $|R(i,j) \cap R(i',j')|$. Now consider a row $\mathbf{r}$ of $M^{\mathrm{T}}M$. If we just look at the positions in $\mathbf{r}$ indexed by the n entries in a given row (or column) of R, then the values in these positions sum to k. Thus the n^2-vector having $n-1$ in these positions and -1 elsewhere is in the null space of $M^{\mathrm{T}}M$. Such null vectors corresponding to the rows of R span an $(n-1)$-space (they sum to $\mathbf{0}$), and such null vectors correponding to the columns of R span another $n-1$ space, and these two spaces have trivial intersection. Thus the null space of $M^{\mathrm{T}}M$ has dimension at least $2n-2$, and so the rank of both $M^{\mathrm{T}}M$ and MM^{T} is at most $(n-1)^2+1$. It follows that the eigenvalues $\gamma_0, \ldots \gamma_{nk-1}$ of the information matrix $C(\Gamma) := nI_{nk} - k^{-1}MM^{\mathrm{T}}$ of Γ, in non-decreasing order, satisfy:

$$0 = \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{(n-1)^2} \leq n = \gamma_{(n-1)^2+1} = \cdots = \gamma_{nk-1}.$$

Now suppose that for some $\ell \in \{0, 1, \ldots, nk-1\}$ we have $\sum_{i=0}^{\ell} \delta_i < \sum_{i=0}^{\ell} \gamma_i$, and choose ℓ to be the least index with this property. Then $\delta_\ell < \gamma_\ell$ and $0 < \ell \leq (n-1)^2$. Moreover, for $j = \ell, \ldots, (n-1)^2$, the δ_j are constant and the γ_j are non-decreasing, and for $j = (n-1)^2+1, \ldots, nk-1$, $\delta_j = \gamma_j = n$, and so $\sum_{i=0}^{nk-1} \delta_i < \sum_{i=0}^{nk-1} \gamma_i$. But this contradicts the fact that the sums of the eigenvalues of $C(\Delta)$ and $C(\Gamma)$ are the same (both information matrices have trace $n^2(k-1)$). We conclude that Δ is Schur-optimal in the class of underlying block designs of $(n \times n)/k$ semi-Latin squares, and we are done. (We note that a similar, and simpler, argument shows that Δ^* is Schur-optimal in the class of duals of underlying block designs of $(n \times n)/k$ semi-Latin squares.) ∎

Remark 3.5 The proof of Theorem 3.4 could be shortened, but made less self-contained and explicit, as follows. After determining the eigenvalues and their multiplicities for $C(\Delta)$, we may observe that S is "maximally balanced" in the sense of [1, Section 4]. The result then follows from [7, Theorem 3.3].

Remark 3.6 Theorem 3.4 generalizes Theorem 5.4 of [2], where it is shown that if S is the superposition of $n-1$ MOLS of order n, or the s-fold inflation of such a superposition, then S is A-, D-, and E-optimal among semi-Latin squares of the same size as S. Bailey remarks in [5] that this extends to Φ_p-optimality for all $p \in (0, \infty)$, which is also covered by our result.

4 Semi-Latin squares from transitive permutation groups

We now present a simple construction to obtain a semi-Latin square from a transitive permutation group. The construction applied to a 2-transitive group yields a uniform semi-Latin square. First, we give some definitions.

A *permutation group* G on a finite set Ω of *points* is a subgroup of the group of all permutations of Ω. If $|\Omega| = n$ then we say that G has *degree* n. The *symmetric group of degree* n, denoted S_n, is the group of all permutations of $\{1, \ldots, n\}$. A permutation group G on Ω is *transitive* if for every $i, j \in \Omega$ there is a $g \in G$ with $i^g = j$ (our permutations act on the right), and G is 2-*transitive* if for every $i, i', j, j' \in \Omega$ with $i \neq i'$ and $j \neq j'$, there is a $g \in G$ with $i^g = j$ and $i'^g = j'$. A permutation group is *regular* if it is transitive and no non-identity element fixes a point. Note that a regular permutation group of degree n has order n. A *Frobenius group* is a transitive permutation group such that each non-identity element fixes at most one point.

Let n and k be integers, with $n > 1$ and $k > 0$, and let P be a set of nk permutations of $\{1, \ldots, n\}$, such that, for all $i, j \in \{1, \ldots, n\}$ there are exactly k elements of P mapping i to j. Then P determines a unique $(n \times n)/k$ semi-Latin square, denoted $\mathrm{SLS}(P)$, with symbol-set P, and whose (i, j)-entry consists precisely of those $p \in P$ with $i^p = j$.

Now let G be a transitive permutation group on $\{1, \ldots, n\}$, with $n > 1$. For all $i, j \in \{1, \ldots, n\}$, there are exactly $|G|/n$ elements of G mapping i to j (the elements mapping i to j are precisely those in $h^{-1}G_1hg$, where h is any element of G with $1^h = i$, G_1 is the stabilizer in G of 1, and g is any element of G with $i^g = j$). Thus, the set of elements of G define an $(n \times n)/k$ semi-Latin square $\mathrm{SLS}(G)$, with $k = |G|/n$. For example, $\mathrm{SLS}(S_3)$ is isomorphic to the square Y in (1).

Theorem 4.1 *Let G be a transitive permutation group on $\{1, \ldots, n\}$, with $n > 1$, and let $S := \mathrm{SLS}(G)$.*

1. *Let H be a transitive subgroup of G. Then S is the superposition of $|G|/|H|$ $(n \times n)/(|H|/n)$ semi-Latin squares, each isomorphic to* $\mathrm{SLS}(H)$*. In particular, if H is regular then S is a superposition of Latin squares, each isomorphic to* $\mathrm{SLS}(H)$*.*

2. *The group G contains a non-identity element with exactly f fixed points if and only if there are two distinct symbols of S which occur together in exactly f entries of S.*

3. *G is a Frobenius group if and only if S is a superposition of MOLS.*

4. *G is 2-transitive if and only if S is uniform.*

Proof.

1. Let i and j be elements of $\{1,\ldots,n\}$, $g \in G$ and $h \in H$. There are exactly $|H|/n$ elements of H mapping i to $j^{g^{-1}}$, and so there are exactly $|H|/n$ elements of the right coset Hg mapping i to j. We thus obtain an $(n \times n)/(|H|/n)$ semi-Latin square $\mathrm{SLS}(Hg)$, which can be formed from $\mathrm{SLS}(H)$ by first permuting its columns by g (so if $i^g = j$ then the current i-th column becomes the new j-th column), and then right multiplying each symbol by g. Thus, if $\{Hg_1,\ldots,Hg_m\}$ is the partition of G into the $m := |G|/|H|$ right cosets of H, then S is the superposition of $\mathrm{SLS}(Hg_1),\ldots,\mathrm{SLS}(Hg_m)$, and these semi-Latin squares are all isomorphic to $\mathrm{SLS}(H)$.

   (We remark that a similar argument works just as well for the left cosets of H in G, with $\mathrm{SLS}(gH)$ obtained from $\mathrm{SLS}(H)$ by permuting its rows by g^{-1} and then left multiplying each symbol by g.)

2. Suppose g is a non-identity element of G, and g has exactly f fixed points. Then g occurs together with the identity element of G in exactly f entries of S.

   Conversely, suppose g and h are distinct elements of G occurring together in exactly f entries of S. Then there are exactly f points $i \in \{1,\ldots,n\}$ with $i^g = i^h$, and so gh^{-1} is a non-identity element of G having exactly f fixed points.

3. Suppose G is a Frobenius group. By Frobenius' Theorem [9, Theorem 2.1], G has a regular (normal) subgroup, and so by part 1 above,

S is a superposition of Latin squares. Since G is a Frobenius group, only the identity element fixes more than one point, so by part 2, each pair of distinct symbols of S occur together in at most one entry of S. It follows that a superposition of Latin squares forming S must be a superposition of MOLS.

Conversely, if S is a superposition of MOLS, then each pair of distinct symbols of S occur together in at most one entry of S, and so by part 2, no non-identity element of G fixes more than one point, and so G is a Frobenius group.

4. Suppose G is 2-transitive. Then for every $i, i', j, j' \in \{1, \ldots, n\}$ with $i \neq i'$ and $j \neq j'$, there are precisely $\mu := |G|/(n(n-1))$ elements $g \in G$ with $i^g = j$ and $i'^g = j'$. Thus, $S(i, j)$ and $S(i', j')$ intersect in exactly these μ elements, and so S is uniform.

   Conversely, suppose S is uniform. Then if $i, i', j, j' \in \{1, \ldots, n\}$ with $i \neq i'$ and $j \neq j'$, then $S(i, j)$ and $S(i', j')$ intersect in $\mu := k/(n-1) > 0$ symbols (recall that $n > 1$, $k > 0$), so there is an element of G mapping i to j and i' to j'. Thus G is 2-transitive.

■

Using the Classification of Finite Simple Groups, all the finite 2-transitive permutation groups have been classified (see [9, Section 4.8]), and tables of these groups are given in Sections 7.3 and 7.4 of [9]. Each 2-transitive group G gives rise to a uniform semi-Latin square SLS(G), certain properties of which can be deduced from properties of G. For example, consideration of the groups $PGL_2(q)$ and $PSL_2(q)$, of degree $q+1$, where q is a prime power, yields the following result.

Theorem 4.2 *Let q be a prime power. Then there exists a uniform, and hence Schur-optimal, $((q+1) \times (q+1))/(q(q-1))$ semi-Latin square S which is the superposition of isomorphic Latin squares and in which every pair of distinct symbols occur together in at most two entries. Moreover, if q is odd then S is also the superposition of two isomorphic uniform $((q+1) \times (q+1))/(q(q-1)/2)$ semi-Latin squares.*

Proof. The proof is an application of Theorem 4.1.

Let $G := PGL_2(q)$ in its natural 2-transitive action of degree $q+1$ (coming from the the action of $GL_2(q)$ on the 1-spaces of $GF(q)^2$), and let $S :=$

SLS(G). Then $|G| = (q+1)q(q-1)$, and so S is a uniform $((q+1) \times (q+1))/(q(q-1))$ semi-Latin square. The only element of G fixing three (or more) points is the identity (in fact, when $q > 2$, G is a "sharply 3-transitive group" (see [9])). Thus every pair of distinct symbols of S occur together in at most two entries. Moreover, G has a regular cyclic subgroup [17, Theorem 27.6], generated by a so-called Singer cycle, and so, by part 1 of Theorem 4.1, S is the superposition of isomorphic Latin squares.

If q is odd then G has a 2-transitive subgroup $PSL_2(q)$ of index 2, and so S is also the superposition of two isomorphic uniform $((q+1)\times(q+1))/(q(q-1)/2)$ semi-Latin squares. ∎

4.1 More on SLS(G)

In this subsection, we record further results of interest on the semi-Latin squares of the form SLS(G), where G is a transitive, but not necessarily 2-transitive, permutation group. The final section does not depend on these results.

We start by defining certain operations which may be applied (on the right) to any semi-Latin square of the form SLS(P), where P is a set of permutations of $\{1, \ldots, n\}$. It is easy to see that all these operations are isomorphisms.

- Where $g \in S_n$, the operation ρ_g permutes the rows according to g (so that, if $i^g = j$, then the current row i becomes the new row j) and then left multiplies each symbol by g^{-1}.
- Where $g \in S_n$, the operation γ_g permutes the columns according to g (so that, if $i^g = j$, then the current column i becomes the new column j) and then right multiplies each symbol by g.
- The operation τ transposes the square and then inverts each symbol.

Note that, for all $g, h \in S_n$, the operations ρ_g and γ_h commute, τ^2 is the identity, and $\tau\rho_g\gamma_h = \rho_h\gamma_g\tau$. Moreover, if P is a group and $g \in P$, then ρ_g, γ_g and τ are all automorphisms of SLS(P).

Theorem 4.3 *Let G be a transitive permutation group on $\{1, \ldots, n\}$, and let $S :=$ SLS(G). Then S is connected if and only if G has no normal subgroup N satisfying $G_1 \leq N \neq G$.*

Proof. Let Γ be the point graph of the underlying block design of S.

We first suppose that S is not connected, so Γ is not connected, and let N be the set of vertices of the connected component of Γ containing the identity element 1_G of G. (Recall that the vertices of Γ are the symbols of S, which are the elements of G.) Now 1_G is in the $(1,1)$-entry of S, together with all the other elements of G_1, the stabilizer in G of 1, and so G_1 is a subset of N, which is not equal to G. We shall show that N is a subgroup of G and is normal in G.

Let $x \in N$. Then, since γ_x is an automorphism of S, we have that Nx is the vertex-set of some connected component of Γ. This component contains the vertex $1_G x = x \in N$, so this component must be the one with vertex-set N. We conclude that $Nx = N$ for all $x \in N$, and so N is a subgroup of G. Now let $g \in G$. Then $\rho_g \gamma_g$ is an automorphism of S and so $g^{-1}Ng$ is the vertex-set of the connected component of Γ containing $g^{-1}1_G g = 1_G \in N$, so this component must be the one with vertex-set N. Thus $g^{-1}Ng = N$ for all $g \in G$, and so N is normal in G.

Conversely, suppose that N is a normal subgroup of G, with $G_1 \leq N \neq G$. For each $i = 1, \ldots, n$, the stabilizer G_i of i is conjugate in G to G_1, and so each G_i is contained in N, and so no element of G not in N fixes a point. Thus, if $x \in N$ and $y \in G \setminus N$, then $g := xy^{-1} \notin N$, so g has no fixed points and so there is no edge joining x and y in Γ. Thus no element of N is joined by an edge to any element of $G \setminus N$, so Γ is not connected, and so S is not connected. ∎

We now determine the automorphism group of a semi-Latin square of the form $\mathrm{SLS}(G)$.

Theorem 4.4 *Let G be a transitive permutation group on $\{1, \ldots, n\}$, and let $S := \mathrm{SLS}(G)$. Then the automorphism group of S is of shape*

$$(G \times G).((N_{S_n}(G)/G) \times C_2),$$

where $N_{S_n}(G)$ is the normalizer in S_n of G, and C_2 is the cyclic group of order 2. This automorphism group acts transitively on the symbols of S, on the Cartesian product of the rows and columns of S, and on the union of the rows and columns of S.

Proof. Let A be the group of all automorphisms of S. Since no two distinct symbols of S (i.e. distinct permutations in G) occupy exactly the same set of positions in S, we see that an automorphism of S is uniquely determined by its action on the rows and columns of S, and so A is a subgroup of the group $(R \times C)\langle\tau\rangle$, where $R := \{\rho_g : g \in S_n\}$ and $C := \{\gamma_g : g \in S_n\}$.

We first note that $\tau \in A$, and consider $B := A \cap (R \times C)$. Let $\rho_x\gamma_y \in R \times C$. Then $\rho_x\gamma_y \in A$ if and only if $x^{-1}Gy = G$, in which case $x^{-1}1_G y = g$, for some $g \in G$, and we have $y = xg$. Thus $\rho_x\gamma_y \in A$ implies that for some $g \in G$, $x^{-1}hxg \in G$ for all $h \in G$, and so $x \in N_{S_n}(G)$. Thus B is contained in the group H generated by

$$\{\rho_x\gamma_x : x \in N_{S_n}(G)\} \cup \{\rho_{1_G}\gamma_g : g \in G\}.$$

But for each generator $\rho_a\gamma_b$ of H, we have $a^{-1}Gb = G$, so $B = H$. Thus $A = B\langle\tau\rangle = H\langle\tau\rangle$, which, in ATLAS notation [13], is of shape $(G \times G).((N_{S_n}(G)/G) \times C_2)$.

We complete the proof by showing how A acts transitively on various sets. Let $g, h \in G$ be symbols of S. Then the automorphism $\gamma_{g^{-1}h}$ maps g to h. Let $i, j, i', j' \in \{1, \ldots, n\}$. Since G is transitive on $\{1, \ldots, n\}$, there are elements $g, h \in G$ with $i^g = i'$ and $j^h = j'$. Thus, the automorphism $\rho_g\gamma_h$ maps row i and column j respectively to row i' and column j'. In particular, A can map any row to any row and any column to any column, and since τ interchanges the rows and columns, we have that A acts transitively on the union of the rows and columns of S. ∎

5 Uniform $(6 \times 6)/(5\mu)$ semi-Latin squares for all $\mu > 1$

In this Section, we provide a constructive proof of the following:

Theorem 5.1 *There exist uniform, and hence Schur-optimal,* $(6 \times 6)/(5\mu)$ *semi-Latin squares for all integers* $\mu > 1$.

Proof. If μ is even, then we take the $\mu/2$-fold inflation of the uniform $(6 \times 6)/10$ semi-Latin square SLS($PSL_2(5)$).

If $\mu = 3$, then we take the semi-Latin square T, whose columns are listed below:

1	7	13	19	25	31	37	43	49	55	61	67	73	79	85
2	10	15	23	30	34	39	45	53	56	65	72	78	80	88
3	8	17	20	28	32	40	47	54	60	63	69	77	82	90
4	11	14	24	29	33	38	48	50	57	64	70	75	84	89
5	9	16	21	27	36	42	44	52	59	66	68	76	83	86
6	12	18	22	26	35	41	46	51	58	62	71	74	81	87

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2	8	14	20	26	32	38	44	50	56	62	68	74	80	86
1	7	13	24	29	35	42	46	52	60	63	69	76	81	89
4	9	18	23	27	36	37	43	53	58	65	70	73	84	87
5	12	15	19	28	31	40	45	51	59	66	71	78	82	85
6	10	17	22	30	33	41	47	54	57	61	67	75	79	88
3	11	16	21	25	34	39	48	49	55	64	72	77	83	90

,

3	9	15	21	27	33	39	45	51	57	63	69	75	81	87
4	12	17	19	25	31	41	44	54	58	64	68	77	84	86
1	7	13	22	30	34	38	48	50	59	66	71	74	83	88
6	8	16	23	26	32	42	46	53	55	65	67	76	79	90
2	11	18	24	28	35	40	43	49	56	62	72	73	82	89
5	10	14	20	29	36	37	47	52	60	61	70	78	80	85

,

4	10	16	22	28	34	40	46	52	58	64	70	76	82	88
5	8	18	20	26	33	37	43	49	57	66	71	75	83	90
6	11	15	24	29	35	39	44	51	55	61	68	78	79	86
1	7	13	21	27	36	41	47	54	56	62	72	77	80	87
3	12	14	23	25	31	38	48	53	60	65	69	74	81	85
2	9	17	19	30	32	42	45	50	59	63	67	73	84	89

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5	11	17	23	29	35	41	47	53	59	65	71	77	83	89
6	9	14	21	28	36	40	48	50	55	61	67	74	82	87
2	12	16	19	25	33	42	46	52	57	62	72	75	80	85
3	10	18	22	30	34	37	44	49	60	63	68	73	81	86
1	7	13	20	26	32	39	45	51	58	64	70	78	84	90
4	8	15	24	27	31	38	43	54	56	66	69	76	79	88

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6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
3	11	16	22	27	32	38	47	51	59	62	70	73	79	85
5	10	14	21	26	31	41	45	49	56	64	67	76	81	89
2	9	17	20	25	35	39	43	52	58	61	69	74	83	88
4	8	15	19	29	34	37	46	50	55	63	71	77	80	87
1	7	13	23	28	33	40	44	53	57	65	68	75	82	86

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(The semi-Latin square T was discovered and studied using the DESIGN package [22] for GAP [14]. Up to isomorphism, T is the unique uniform $(6\times 6)/15$ semi-Latin square having a group of automorphisms of order 25. In fact, the image of the (full) automorphism group of T acting on the rows and columns of T has order 200. In addition, T is the superposition of 15 Latin squares, which have respective symbol-sets $\{1,\ldots,6\},\{7,\ldots,12\},\ldots,\{85,\ldots,90\}$.)

Finally, if μ is odd and $\mu > 3$, then we take the superposition of T with the $(\mu-3)/2$-fold inflation of SLS($PSL_2(5)$). ∎

Thus, when n is a prime power or $n = 6$, we know precisely the values of μ for which there exists a uniform $(n\times n)/(\mu(n-1))$ semi-Latin square, but we do not know exactly which values of μ have this property for any other $n > 1$. The first unsettled case is $n = 10$. There is no projective plane of order 10 [16], so there do not exist nine MOLS of order 10, and so a uniform $(10 \times 10)/9$ semi-Latin square does not exist. On the other hand, SLS($PSL_2(9)$) and inflations of this square yield uniform $(10 \times 10)/(9\mu)$ semi-Latin squares for $\mu = 4, 8, 12, 16, \ldots$.

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References

[1] R.A. Bailey, Semi-Latin squares, *J. Statist. Plann. Inference* **18** (1988), 299–312.

[2] R.A. Bailey, Efficient semi-Latin squares, *Statist. Sinica* **2** (1992), 413–437.

[3] R.A. Bailey, *Association Schemes. Designed Experiments, Algebra and Combinatorics*, Cambridge University Press, Cambridge, 2004.

[4] R.A. Bailey, *Design of Comparative Experiments*, Cambridge University Press, Cambridge, 2008.

[5] R.A. Bailey, Symmetric factorial designs in blocks, *J. Stat. Theory Pract.*, to appear.

[6] R.A. Bailey and P.J. Cameron, Combinatorics of optimal designs, in *Surveys in Combinatorics 2009*, S. Huczynska et al. (eds), Cambridge University Press, Cambridge, 2009, pp. 19–73.

[7] R.A. Bailey, H. Monod and J.P. Morgan, Construction and optimality of affine-resolvable designs, *Biometrika* **82** (1995), 187–200.

[8] R.A. Bailey and G. Royle, Optimal semi-Latin squares with side six and block size two, *Proc. Roy. Soc. London Ser. A* **453** (1997), 1903–1914.

[9] P.J. Cameron, *Permutation Groups*, Cambridge University Press, Cambridge, 1999.

[10] P.J. Cameron, P. Dobcsányi, J.P. Morgan and L.H. Soicher, The External Representation of Block Designs, `http://designtheory.org/library/extrep/extrep-1.1-html/`

[11] C.-S. Cheng and R.A. Bailey, Optimality of some two-associate-class partially balanced incomplete-block designs, *Ann. Statist.* **19** (1991), 1667–1671.

[12] P.E. Chigbu, Semi-Latin Squares: Methods for Enumeration and Comparison, PhD thesis, University of London, 1996.

[13] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[14] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.4.12, 2008, `http://www.gap-system.org`

[15] A. Giovagnoli and H.P. Wynn, Optimum continuous block designs, *Proc. Roy. Soc. London Ser. A* **377** (1981), 405–416.

[16] C.W.H. Lam, L. Thiel and S. Swiercz, The non-existence of finite projective planes of order 10, *Canad. J. Math.* **41** (1989), 1117–1123.

[17] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.

[18] D.A. Preece and G.H. Freeman, Semi-Latin squares and related designs, *J. Roy. Statist. Soc. Ser. B* **45** (1983), 267–277.

[19] K.R. Shah and B.K. Sinha, *Theory of Optimal Designs*, Lecture Notes in Statistics **54**, Springer, Berlin-Heidelberg-New York, 1989.

[20] L.H. Soicher, On the structure and classification of SOMAs: generalizations of mutually orthogonal Latin squares, *Electron. J. Combin.* **6** (1999), R32, 15 pp. printed version: *J. Combin.* **6** (1999), 427–441.

[21] L.H. Soicher, SOMA Update, `http://www.maths.qmul.ac.uk/~leonard/soma/`

[22] L.H. Soicher, The DESIGN package for GAP, Version 1.4, 2009, `http://designtheory.org/software/gap_design/`

[23] C.-Y. Suen and I.M. Chakravarti, Efficient two-factor balanced designs, *J. Roy. Statist. Soc. Ser. B* **48** (1986), 107–114.

[24] M.N. Vartak, The non-existence of certain PBIB designs, *Ann. Math. Stat.* **30** (1959), 1051–1062.