More on block intersection polynomials and new applications to graphs and block designs

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Abstract

The concept of intersection numbers of order r for t-designs is generalized to graphs and to block designs which are not necessarily t-designs. These intersection numbers satisfy certain integer linear equations involving binomial coefficients, and information on the non-negative integer solutions to these equations can be obtained using the block intersection polynomials introduced by P.J. Cameron and the present author. The theory of block intersection polynomials is extended, and new applications of these polynomials to the studies of graphs and block designs are obtained. In particular, we obtain a new method of bounding the size of a clique in an edge-regular graph with given parameters, which can improve on the Hoffman bound when applicable, and a new method for studying the possibility of a graph with given vertex-degree sequence being an induced subgraph of a strongly regular graph with given parameters.

1 Introduction

Consider the linear equations:

$$\sum_{i=0}^{s} \binom{i}{j} n_i = b_j \quad (j = 0, \ldots, t) \tag{1}$$

where s, t and the b_j are given non-negative integers, with $s \geq t$, and suppose we are interested in solution vectors $[n_0, \ldots, n_s]$ of non-negative integers. Problems of this form arise in the study of block designs, especially the study of t-designs (see [13, 8, 15, 1, 2, 5]). In [5], Cameron and the present author introduce the block intersection polynomial, and show how this polynomial gives useful information about the solutions to this problem when t is even and specified non-negative integers $m_0, \ldots, m_s$ are given for which $m_i \leq n_i$ must hold. They show how to use block intersection polynomials to obtain a bound on the multiplicity of a block in a t-(v, k, λ) design, and when t is even, a sometimes better bound for a resolvable t-(v, k, λ) design.

In this paper, we generalize the concept of intersection numbers of order r for t-designs, introduced by Mendelsohn [13] and studied further by van Trung, Wu and Mesner [15], to graphs and to block designs which are not necessarily t-designs. We show that these intersection numbers (defined in Section 2) satisfy equations of the form (1), and so may be studied using block intersection polynomials, or by methods of linear or integer programming. We then continue the study of block intersection polynomials, and obtain new applications of these polynomials to the studies of edge-regular graphs, amply regular graphs, and block designs. For example, we prove the following:

Theorem 1.1 *Let Γ be an edge-regular graph with parameters (v, k, λ), let r be a positive integer, and suppose that Γ contains a clique S of size $s \geq 2$. Let*

$$B(x) := x(x+1)\binom{v-s}{r} - 2xs\binom{k-s+1}{r} + s(s-1)\binom{\lambda-s+2}{r}. \tag{2}$$

Then $B(m) \geq 0$ for every integer m. Moreover, $B(m) = 0$ for some integer m if and only if for every r-set R of vertices, such that $R \cap S = \emptyset$, we have $|\cap_{\rho \in R} \Gamma(\rho) \cap S| \in \{m, m+1\}$, in which case m occurs exactly $B(m+1)/2$ times.

We apply this theorem with $r = 1$ to obtain what appears to be a good bound on the size of a clique in any edge-regular graph Γ with parameters (v, k, λ). We have that if Γ has a clique of size s then

$$m(m+1)(v-s) - 2ms(k-s+1) + s(s-1)(\lambda - s + 2) \geq 0 \qquad (3)$$

for every integer m. Then our clique bound is one less than the least integer $s \geq 2$ for which (3) does not hold for some integer m. (Such an s always exists, for considered as a polynomial in s, the left-hand side of (3) has leading term $-s^3$. For a fixed $s < v$, the left-hand side of (3) is a quadratic in m, and so it is easy to determine whether it is negative for some integer m.) Our bound is sometimes better than what the Hoffman bound [4, Proposition 1.3.2] gives when Γ is a strongly regular graph with parameters (v, k, λ, μ).

Our methods may also be applied to the study of induced subgraphs of amply regular graphs. We shall prove:

Theorem 1.2 *Let Γ be an amply regular graph with parameters (v, k, λ, μ), and suppose Δ is an induced subgraph of Γ, where Δ has $s \geq 2$ vertices and vertex-degree sequence $[d_1, \dots, d_s]$. Further suppose that Δ is connected with diameter at most 2 if Γ is not strongly regular. Let*

$$B(x) := x(x+1)(v-s) - 2xsk + (2x + \lambda - \mu + 1)\sum_{i=1}^{s} d_i + s(s-1)\mu - \sum_{i=1}^{s} d_i^2.$$

Then $B(m) \geq 0$ for every integer m. Moreover, $B(m) = 0$ for some integer m if and only if each vertex not in Δ is adjacent to exactly m or $m+1$ vertices of Δ, in which case exactly $B(m+1)/2$ vertices not in Δ are adjacent to just m vertices of Δ.

We close the introduction by stating a result on t-designs which we shall prove using block intersection polynomials in Section 5.

Theorem 1.3 *Let t be an even positive integer, let D be a t-(v, k, λ) design, and for S a block of D, define $I(D, S)$ to be the set of all i for which some block of D, other than S, meets S in exactly i points. Now suppose that for some block S of D, $I(D, S)$ is contained in a set of t consecutive integers. Then for every block T in every t-(v, k, λ) design E, the number of blocks of E meeting T in exactly i points $(0 \leq i \leq k)$ is the same as the number of blocks of D meeting S in exactly i points.*

2 Intersection numbers of order r for graphs and block designs

Throughout this paper, all graphs are finite and undirected, with no loops or multiple edges. For Γ a graph, $V(\Gamma)$ denotes its vertex-set, and for $\alpha \in V(\Gamma)$, $\Gamma_i(\alpha)$ denotes the set of vertices in Γ at distance i from α, with $\Gamma(\alpha) := \Gamma_1(\alpha)$, the set of vertices *adjacent* to α. A graph Γ is *edge-regular* with *parameters* (v, k, λ) if Γ has exactly v vertices, is regular of degree k, and every pair of adjacent vertices have exactly λ common neighbours. A graph Γ is *amply regular* with *parameters* (v, k, λ, μ) if it is edge-regular with parameters (v, k, λ) and every pair of vertices at distance 2 have exactly μ common neighbours. A graph Γ is *strongly regular* with *parameters* (v, k, λ, μ) if it is edge-regular with parameters (v, k, λ) and every pair of distinct nonadjacent vertices have exactly μ common neighbours (so in particular, every strongly regular graph is amply regular). A *clique* in a graph is a set of pairwise adjacent vertices.

Let Γ be a graph, and let $S, Q \subseteq V(\Gamma)$. Then for r a positive integer, the i-th *intersection number of order* r, denoted $n_i^{(r)}(\Gamma, S, Q)$, is defined to be the number of r-subsets R of Q such that

$$\left| \bigcap_{\rho \in R} \Gamma(\rho) \bigcap S \right| = i.$$

In particular, $n_i^{(1)}(\Gamma, S, Q)$ is the number of vertices $\alpha \in Q$ such that $|\Gamma(\alpha) \cap S| = i$.

For T any subset of $V(\Gamma)$, let $\lambda_T(\Gamma, Q)$ be the number of vertices in Q adjacent to each vertex in T; that is

$$\lambda_T(\Gamma, Q) = |\{\alpha \in Q : T \subseteq \Gamma(\alpha)\}|.$$

Now, for r a positive integer and $0 \le j \le s$, where $s := |S|$, define

$$\lambda_j^{(r)}(\Gamma, S, Q) := \binom{s}{j}^{-1} \sum_{T \subseteq S, |T| = j} \binom{\lambda_T(\Gamma, Q)}{r}. \tag{4}$$

For example, suppose that Γ is a strongly regular graph with parameters (v, k, λ, μ), and Δ is a subgraph of Γ, induced on a vertex-subset S of size $s \ge 2$. Let Q be the set of vertices of Γ not in S. Then $\lambda_{\{\}}(\Gamma, Q) = v - s$; if $\sigma \in V(\Delta)$ then $\lambda_{\{\sigma\}}(\Gamma, Q) = k - |\Delta(\sigma)|$; if $\{\rho, \sigma\}$ is an edge of

Δ then $\lambda_{\{\rho,\sigma\}}(\Gamma, Q) = \lambda - |\Delta(\sigma) \cap \Delta(\rho)|$; if $\{\rho,\sigma\}$ is a non-edge of Δ then $\lambda_{\{\rho,\sigma\}}(\Gamma, Q) = \mu - |\Delta(\sigma) \cap \Delta(\rho)|$. Thus, we can determine $\lambda_T(\Gamma, Q)$ for all $T \subseteq S$ with $|T| \leq 2$, and hence determine $\lambda_j^{(r)}(\Gamma, S, Q)$ for all $r > 0$ and $j = 0, 1, 2$, using only the parameters of the strongly regular graph Γ and the isomorphism class of the induced subgraph Δ.

The following result shows how knowledge of certain $\lambda_j^{(r)}(\Gamma, S, Q)$ gives us integer linear equations of the form (1) which must be satisfied by the non-negative integers $n_i^{(r)}(\Gamma, S, Q)$, $i = 0, \ldots, |S|$.

Theorem 2.1 *Let Γ be a graph, let $S, Q \subseteq V(\Gamma)$, with $s := |S|$, and let r be a positive integer. Then if $0 \leq j \leq s$, we have*

$$\sum_{i=0}^{s} \binom{i}{j} n_i^{(r)}(\Gamma, S, Q) = \binom{s}{j} \lambda_j^{(r)}(\Gamma, S, Q). \tag{5}$$

Proof. We let $0 \leq j \leq s$, and count in two ways the number N_j of ordered pairs (R, T) such that R is an r-subset of Q, and T is a j-subset of both S and $\cap_{\rho \in R} \Gamma(\rho)$.

1. Each j-subset T of S contributes exactly $\binom{\lambda_T(\Gamma,Q)}{r}$ pairs of the form $(*, T)$ to N_j, and so
$$N_j = \sum_{T \subseteq S, |T|=j} \binom{\lambda_T(\Gamma, Q)}{r} = \binom{s}{j} \lambda_j^{(r)}(\Gamma, S, Q).$$

2. Each r-subset R of Q contributes exactly
$$\binom{\left|\bigcap_{\rho \in R} \Gamma(\rho) \cap S\right|}{j}$$
pairs of the form $(R, *)$ to N_j, and so
$$N_j = \sum_{R \subseteq Q, |R|=r} \binom{\left|\bigcap_{\rho \in R} \Gamma(\rho) \cap S\right|}{j} = \sum_{i=0}^{s} \binom{i}{j} n_i^{(r)}(\Gamma, S, Q).$$

Hence the result. ∎

We are often interested in the case where Γ is the incidence graph of a block design. A *block design* is an ordered pair $(V, \mathcal{B})$, such that V is a finite non-empty set, whose elements are called *points*, and $\mathcal{B}$ is a finite non-empty multiset of subsets of V called *blocks*. For t a non-negative integer and v, k, λ positive integers with $t \leq k \leq v$, a t-(v,k,λ) *design* (or simply a t-*design*) is a block design with exactly v points, such that each block has size k and each t-subset of the point-set is contained in exactly λ blocks. The *incidence graph* of a block design D is the graph whose vertices are the points and blocks of D (including repeated blocks), with $\{\alpha, \beta\}$ an edge precisely when one of α and β is a point and the other is a block containing that point.

Let D be a block design, and S a subset of the point-set of D. Then for r a positive integer, the i-th *intersection number of order* r, denoted $n_i^{(r)}(D,S)$, is defined to be the number of collections of r blocks of D whose intersection intersects S in exactly i points. In particular, $n_i^{(1)}(D,S)$ is the number of blocks of D (including repeats) intersecting S in exactly i points. Note that if Γ is the incidence graph of D, and Q is the set of vertices of Γ corresponding to the blocks of D, then

$$n_i^{(r)}(D,S) = n_i^{(r)}(\Gamma, S, Q).$$

For T any subset of the point-set of D, let $\lambda_T(D)$ be the number of blocks of D (including repeats) containing T. Now, for r a positive integer and $0 \leq j \leq s$, where $s := |S|$, define

$$\lambda_j^{(r)}(D,S) := \binom{s}{j}^{-1} \sum_{T \subseteq S, |T|=j} \binom{\lambda_T(D)}{r}. \tag{6}$$

Note that if Γ is the incidence graph of D, and Q is the set of vertices of Γ corresponding to the blocks of D, then

$$\lambda_j^{(r)}(D,S) = \lambda_j^{(r)}(\Gamma, S, Q).$$

Note further that if D is a t-(v,k,λ) design and T is any j-subset of S with $j \leq t$, we have $\lambda_T(D) = c_j := \lambda\binom{v-j}{t-j}/\binom{k-j}{t-j}$, and so in this case, for every $j = 0, \ldots, t$ and every $r > 0$, we have $\lambda_j^{(r)}(D,S) = \binom{c_j}{r}$.

We now have the following corollary of Theorem 2.1:

Corollary 2.2 *Let D be a block design, let S be a subset of the point-set of D, with $s := |S|$, and let r be a positive integer. Then if $0 \le j \le s$, we have*

$$\sum_{i=0}^{s} \binom{i}{j} n_i^{(r)}(D,S) = \binom{s}{j} \lambda_j^{(r)}(D,S). \tag{7}$$

We remark that this result is proved for the case when D is a t-design and S is a block in [13], in general for t-designs in [15], and for the case $r = 1$ for general block designs in [5] (from which we have adapted our proof of Theorem 2.1).

3 Block intersection polynomials

The preceding section shows that the intersection numbers of order r for graphs and for block designs satisfy integer linear equations of the form (1). These equations can be studied by methods of linear or integer programming, or by using the block intersection polynomials introduced in [5]. In particular, we remark that the `simplex` linear programming package in MAPLE [10] is useful in being able to determine whether given equations of the form (1) have some solution $[n_0, \dots, n_s]$ with n_i a rational number and $\ell_i \le n_i \le u_i$ for given integers ℓ_i and u_i ($i = 0, \dots, s$). In this paper, however, we concentrate on block intersection polynomials, and proceed with their definition.

For k a non-negative integer, define the polynomial

$$P(x,k) := x(x-1)\cdots(x-k+1),$$

and for real number sequences $[m_0, \dots, m_s]$, $[\lambda_0, \dots, \lambda_t]$, with $t \le s$, define the *block intersection polynomial*

$$B(x,[m_0,\dots,m_s],[\lambda_0,\dots,\lambda_t]) := \sum_{j=0}^{t} \binom{t}{j} P(-x,t-j)[P(s,j)\lambda_j - \sum_{i=j}^{s} P(i,j)m_i]. \tag{8}$$

The following theorem shows how we may use block intersection polynomials to obtain information about the n_i satisfying the linear equations (1). Without loss of generality (and to maintain compatibility with [5]), we write b_j in the form $\binom{s}{j}\lambda_j$. The assertions of this theorem, after the first, can be translated directly into results, formulae and bounds for the intersection numbers of order r of both graphs and block designs, using Theorem 2.1 for graphs and Corollary 2.2 for block designs. We shall perform some of these translations explicitly in Corollary 3.2. In the case of block designs, this corollary extends and generalizes Theorem 4.1 of [5].

Theorem 3.1 *Let s and t be non-negative integers, with $s \geq t$, let $n_0, \ldots, n_s$, $m_0, \ldots, m_s$, and $\lambda_0, \ldots, \lambda_t$ be real numbers, such that*

$$\sum_{i=0}^{s} \binom{i}{j} n_i = \binom{s}{j} \lambda_j \quad (j = 0, \ldots, t), \tag{9}$$

and let $B(x)$ be the block intersection polynomial $B(x, [m_0, \ldots, m_s], [\lambda_0, \ldots, \lambda_t])$, defined in (8).

Then the following hold:

1. *$B(x) = \sum_{i=0}^{s} P(i - x, t)(n_i - m_i)$.*
2. *If n_i and m_i are integers for all i then $B(x)$ is a polynomial over the integers, and if $n_i = m_i$ for all i then $B(x) \equiv 0$.*
3. *Suppose that $m_i \leq n_i$ for all i or $m_i \geq n_i$ for all i. Then if $m_i \neq n_i$ for some i, $B(x)$ is a non-zero polynomial of degree t.*
4. *For every $\ell = 0, \ldots, s$ and every integer m with $\ell \notin \{m, m+1, \ldots, m+t-1\}$, we have:*

$$n_\ell = m_\ell + \frac{B(m) - \sum_{i \notin \{\ell, m, m+1, \ldots, m+t-1\}} P(i - m, t)(n_i - m_i)}{P(\ell - m, t)}. \tag{10}$$

5. *Suppose that $m_i \leq n_i$ for all i or $m_i \geq n_i$ for all i, and that t is even. Then for every $\ell = 0, \ldots, s$ and every integer m with $\ell \notin \{m, m+1, \ldots, m+t-1\}$, we have:*
   - *$n_\ell \leq m_\ell + B(m)/P(\ell - m, t)$ if $m_i \leq n_i$ for all i,*
   - *$n_\ell \geq m_\ell + B(m)/P(\ell - m, t)$ if $m_i \geq n_i$ for all i, and*
   - *$n_\ell = m_\ell + B(m)/P(\ell - m, t)$ if and only if $m_i = n_i$ for all $i \notin \{\ell, m, m+1, \ldots, m+t-1\}$, in which case the sequence $[n_0, \ldots, n_s]$ is uniquely determined by $[m_0, \ldots, m_s]$ and $[\lambda_0, \ldots, \lambda_t]$.*
6. *Suppose that $m_i \leq n_i$ for all i or $m_i \geq n_i$ for all i, and that t is even. Then for every integer m, we have:*

- *$B(m) \geq 0$ if $m_i \leq n_i$ for all i,*
- *$B(m) \leq 0$ if $m_i \geq n_i$ for all i, and*
- *$B(m) = 0$ if and only if $m_i = n_i$ for all $i \notin \{m, m+1, \ldots, m+t-1\}$, in which case the sequence $[n_0, \ldots, n_s]$ is uniquely determined by $[m_0, \ldots, m_s]$ and $[\lambda_0, \ldots, \lambda_t]$.*

Proof.

1. This is Theorem 3.2 of [5].

2. These follow from part 1.

3. First, note that $P(i-x, t)$ is a polynomial (in x) of degree t, with leading coefficient $(-1)^t$. Then, applying part 1, we have that $B(x)$ is a polynomial of degree at most t, the coefficient of x^t being $(-1)^t \sum_{i=0}^{s}(n_i - m_i)$. This coefficient is non-zero if we have $m_i \leq n_i$ for all i or $m_i \geq n_i$ for all i, and also $m_i \neq n_i$ for some i.

4. This follows from part 1, and the fact that $P(i - m, t) = 0$ precisely when $i \in \{m, m+1, \ldots, m+t-1\}$.

5. Suppose m is an integer and $\ell \in \{0, \ldots, s\} \setminus \{m, m+1, \ldots, m+t-1\}$. Since t is even, $P(i - m, t)$ is a product of evenly many consecutive integers, and so is non-negative, with $P(i-m, t) > 0$ when $i \notin \{m, m+1, \ldots, m+t-1\}$. In particular, $P(\ell - m, t) > 0$.

   Now let $q := \sum_{i \notin \{\ell, m, m+1, \ldots, m+t-1\}} P(i - m, t)(n_i - m_i)$, so by part 4,

   $$n_\ell = m_\ell + (B(m) - q)/P(\ell - m, t).$$

   If $m_i \leq n_i$ for all i, we have $q \geq 0$, in which case $n_\ell \leq m_\ell + B(m)/P(\ell - m, t)$. If $m_i \geq n_i$ for all i, we have $q \leq 0$, in which case $n_\ell \geq m_\ell + B(m)/P(\ell - m, t)$. In either case, we have $n_\ell = m_\ell + B(m)/P(\ell - m, t)$ if and only if $q = 0$, which holds if and only if $m_i = n_i$ for all $i \notin \{\ell, m, m+1, \ldots, m+t-1\}$.

   Now suppose that $n_\ell = m_\ell + B(m)/P(\ell - m, t)$, and so, in addition, $m_i = n_i$ for all $i \notin \{\ell, m, m+1, \ldots, m+t-1\}$. We show how to (uniquely) determine the remaining n_i.

   Let $M := \{0, \ldots, s\} \cap \{m, m+1, \ldots, m+t-1\}$. Then M is a set of consecutive integers, say $M = \{a, a+1, \ldots, b\}$. Then $n_a, n_{a+1}, \ldots, n_b$ are uniquely determined by applying (10) successively with that formula's (ℓ, m) taking the values $(a, a+1), (a+1, a+2), \ldots, (b, b+1)$ in turn. Note that at each such application, all the required n_i in the right-hand side have been determined, we do not divide by zero, and that the block intersection polynomial $B(x)$ depends only on $[m_0, \ldots, m_s]$ and $[\lambda_0, \ldots, \lambda_t]$.

6. Suppose t is even and m is an integer. Then $P(i-m, t)$ is a product of evenly many consecutive integers, and so is non-negative. It then follows from part 1 that if $m_i \le n_i$ for all i we have $B(m) \ge 0$, and if $m_i \ge n_i$ for all i we have $B(m) \le 0$.

   Now suppose $m_i = n_i$ for all $i \notin \{m, m+1, \ldots, m+t-1\}$. Since $P(i-m, t) = 0$ if $i \in \{m, m+1, \ldots, m+t-1\}$, we conclude that $B(m) = \sum_{i=0}^{s} P(i-m, t)(n_i - m_i) = 0$.

   Conversely, suppose $B(m) = 0$. Since $P(i-m, t) > 0$ unless $i \in \{m, m+1, \ldots, m+t-1\}$, we conclude that since $B(m) = 0$ and the $n_i - m_i$ are all non-negative or all non-positive, then we must have $m_i = n_i$ for all $i \notin \{m, m+1, \ldots, m+t-1\}$. In this case, we can uniquely determine the remaining n_i as in part 5.

■

Corollary 3.2 *Let r be a positive integer, and let s and t be non-negative integers, with t even and $t \le s$, and suppose that either:*

1. *Γ is a graph, and $S, Q \subseteq V(\Gamma)$, with S of size s, $n_i := n_i^{(r)}(\Gamma, S, Q)$ $(i = 0, \ldots, s)$, and $\lambda_j := \lambda_j^{(r)}(\Gamma, S, Q)$ $(j = 0, \ldots, t)$; or*

2. *D is a block design, S is a subset of size s of the point-set of D, $n_i := n_i^{(r)}(D, S)$ $(i = 0, \ldots, s)$, and $\lambda_j := \lambda_j^{(r)}(D, S)$ $(j = 0, \ldots, t)$.*

In either case, let $m_0, \ldots, m_s$ be non-negative integers with either $m_i \le n_i$ for all i or $m_i \ge n_i$ for all i, and let

$$B(x) := B(x, [m_0, \ldots, m_s], [\lambda_0, \ldots, \lambda_t]).$$

Then $B(x) \equiv 0$ if and only if $m_i = n_i$ for all i; otherwise, $B(x)$ is a degree t polynomial with integer coefficients. Furthermore, $B(m) \ge 0$ for every integer m if $m_i \le n_i$ for all i, and $B(m) \le 0$ for every integer m if $m_i \ge n_i$

for all i. Moreover, $B(m) = 0$ for some integer m if and only if $m_i = n_i$ for all $i \notin \{m, m+1, \ldots, m+t-1\}$, in which case $[n_0, \ldots, n_s]$ is uniquely determined by $[m_0, \ldots, m_s]$ and $[\lambda_0, \ldots, \lambda_t]$.

Proof. Apply Theorem 3.1 to the integer linear equations described in Theorem 2.1 and Corollary 2.2, satisfied by the (non-negative) intersection numbers of order r of graphs and blocks designs, respectively. ∎

We remark that algorithms to construct and analyse block intersection polynomials are implemented in the freely available DESIGN package [14] for GAP [7]. For example, you can use GAP and DESIGN to construct a block intersection polynomial $B(x)$, determine the integer zeros of $B(x)$, and determine whether there is any integer m for which $B(m) < 0$. The methods used for this last test are described in [5].

4 Applications to edge-regular and amply regular graphs

Here we apply block intersection polynomials to the study of cliques in edge-regular graphs and to the study of induced subgraphs of amply regular graphs. In particular, we prove Theorems 1.1 and 1.2.

4.1 Cliques in edge-regular graphs

When studying edge-regular graphs Γ with parameters (v, k, λ), we may wish to consider the possibility of the existence of a clique S in Γ of some given size $s \geq 2$. Where r is a positive integer and $Q := V(\Gamma) \setminus S$, we may determine $\lambda_j^{(r)}(\Gamma, S, Q)$ for $j = 0, 1, 2$, and so can use block intersection polynomials (or linear or integer programming) to obtain information on the numbers $n_i^{(r)}(\Gamma, S, Q)$ $(i = 0, \ldots, s)$. Applying block intersection polynomials in the case $r = 1$ appears to provide a good bound on the size of cliques in an edge-regular graph with given parameters. We shall give an example of this, after proving Theorem 1.1.

Proof. (of Theorem 1.1) Let Γ be an edge-regular graph with parameters (v, k, λ), let r be a positive integer, and suppose that Γ contains a clique

S of size $s \geq 2$. Let $Q := V(\Gamma) \setminus S$, and for $j = 0, 1, 2$, let $\lambda_j := \lambda_j^{(r)}(\Gamma, S, Q)$. We have

$$\begin{aligned}
\lambda_0 &= \binom{|Q|}{r} = \binom{v-s}{r}, \\
\lambda_1 &= \binom{k-s+1}{r}, \\
\lambda_2 &= \binom{\lambda-s+2}{r}.
\end{aligned}$$

Let

$$B(x) := B(x, [0^{s+1}], [\lambda_0, \lambda_1, \lambda_2]).$$

Applying (8), we have

$$B(x) = x(x+1)\binom{v-s}{r} - 2xs\binom{k-s+1}{r} + s(s-1)\binom{\lambda-s+2}{r}.$$

Now applying Corollary 3.2, we have that $B(m) \geq 0$ for every integer m, and that $B(m) = 0$ for some integer m if and only if $n_i^{(r)}(\Gamma, S, Q) = 0$ for all $i \notin \{m, m+1\}$. In this case, we may apply formula (10), substituting the values $m, m+1$ respectively for that formula's ℓ, m, to determine that $n_m^{(r)}(\Gamma, S, Q) = 0 + B(m+1)/P(-1, 2) = B(m+1)/2$. ∎

Example 1 Consider the strongly regular graphs with parameters $(37, 18, 8, 9)$. The Paley graph Paley(37) is one such graph, and it happens to have a maximum clique of size 4. In [12], McKay and Spence describe a computer search which found 6760 pairwise non-isomorphic such graphs. A list of these graphs was downloaded from [11], and it turned out that the first graph on this list has a maximum clique of size 5. It is not known whether there are more strongly regular graphs with these parameters.

The complement of a strongly regular graph with parameters $(37, 18, 8, 9)$ (and such a graph) has least eigenvalue $\tau \approx -3.541$, and so the Hoffman bound gives an upper bound of $6 = \lfloor 37/(1 - 18/\tau) \rfloor$ on the size of a clique.

Now let Γ be any edge-regular graph with parameters $(37, 18, 8)$, and suppose that Γ has a clique S of size 6. Let $Q := V(\Gamma) \setminus S$, and for $j = 0, 1, 2$, let $\lambda_j := \lambda_j^{(1)}(\Gamma, S, Q)$. We have $\lambda_0 = 31$, $\lambda_1 = 13$, and $\lambda_2 = 4$. We calculate

$$B(x) := B(x, [0^7], [31, 13, 4]) = 31x^2 - 125x + 120,$$

and find that $B(2) = -6$. Hence Γ can have no clique of size 6.

4.2 Induced subgraphs of amply regular graphs

When studying strongly regular graphs Γ with parameters (v, k, λ, μ), we may wish to consider the possibility of a given graph Δ being an induced subgraph of such a Γ. As noted in Section 2, where $S := V(\Delta)$ and $Q := V(\Gamma)\backslash S$, using only the parameters of Γ and the isomorphism type of Δ, we can compute $\lambda_j^{(r)}(\Gamma, S, Q)$ for all $r > 0$ and $j = 0, 1, 2$, and so can use block intersection polynomials (or linear or integer programming) to obtain information on the numbers $n_i^{(r)}(\Gamma, S, Q)$ $(i = 0, \ldots, s)$, and may be able to prove that Γ can have no induced subgraph isomorphic to Δ. This may be useful to eliminate search subtrees when performing a computer backtrack search to find or classify strongly regular graphs with given parameters if we build up such graphs via induced subgraphs. We need only the vertex-degree sequence of Δ to compute $\lambda_j^{(1)}(\Gamma, S, Q)$ for $j = 0, 1, 2$, as shown in the proof below of Theorem 1.2.

Proof. (of Theorem 1.2) Let Γ be an amply regular graph with parameters (v, k, λ, μ), and suppose Δ is an induced subgraph of Γ, where Δ has $s \geq 2$ vertices and vertex-degree sequence $[d_1, \ldots, d_s]$. Further suppose that Δ is connected with diameter at most 2 if Γ is not strongly regular.

Let $Q := V(\Gamma) \setminus V(\Delta)$, and for $j = 0, 1, 2$, let $\lambda_j := \lambda_j^{(1)}(\Gamma, S, Q)$. We have

$$\lambda_0 = |Q| = v - s$$

and

$$\lambda_1 = s^{-1} \sum_{i=1}^{s} (k - d_i) = k - s^{-1} \sum_{i=1}^{s} d_i.$$

Let e be the number of edges of Δ, n the number of non-edges of Δ, and p the number of simple undirected paths of length 2 in Δ. Then

$$\lambda_2 = \binom{s}{2}^{-1} (e\lambda + n\mu - p). \tag{11}$$

Now $e = \frac{1}{2} \sum_{i=1}^{s} d_i$, $n = \binom{s}{2} - e = \binom{s}{2} - \frac{1}{2} \sum_{i=1}^{s} d_i$, and $p = \sum_{i=1}^{s} \binom{d_i}{2}$. Substituting these into (11), and sinplifying, we obtain

$$\lambda_2 = \frac{1}{s(s-1)} \left((\lambda - \mu + 1) \sum_{i=1}^{s} d_i + s(s-1)\mu - \sum_{i=1}^{s} d_i^2 \right).$$

Let

$$B(x) := B(x, [0^{s+1}], [\lambda_0, \lambda_1, \lambda_2]).$$

Applying (8), we have

$$B(x) = x(x+1)(v-s) - 2xsk + (2x+\lambda-\mu+1)\sum_{i=1}^{s} d_i + s(s-1)\mu - \sum_{i=1}^{s} d_i^2.$$

Now applying Corollary 3.2, we have that $B(m) \geq 0$ for every integer m, and that $B(m) = 0$ for some integer m if and only if each vertex in Q is adjacent to exactly m or $m+1$ vertices of S. In this case, we may apply formula (10), substituting the values $m, m+1$ respectively for that formula's ℓ, m, to determine that the number of vertices in Q adjacent to just m vertices of S must be $0 + B(m+1)/P(-1,2) = B(m+1)/2$. ∎

Example 2 Let Γ be a strongly regular graph with parameters $(76, 30, 8, 14)$. It is unknown (to the author) whether such a graph exists, although these are "feasible" parameters for a strongly regular graph (see [3]).

Now suppose Γ contains an induced subgraph Δ isomorphic to (the 1-skeleton of) an octahedron, i.e. the strongly regular graph with parameters $(6, 4, 2, 4)$. Let $S := V(\Delta)$, $Q := V(\Gamma) \setminus S$, and for $j = 0, 1, 2$, let $\lambda_j := \lambda_j^{(1)}(\Gamma, S, Q)$. We have $\lambda_0 = 70$, $\lambda_1 = 26$, and $\lambda_2 = 34/5$. We calculate

$$B(x) := B(x, [0^7], [70, 26, 34/5]) = 70(x-2)(x-51/35),$$

so in particular, $B(2) = 0$. Hence, exactly $B(3)/2 = 54$ vertices not in Δ are joined to exactly 2 vertices of Δ, and the remaining 16 vertices not in Δ are joined to exactly 3 vertices of Δ.

5 Applications to block designs

Some applications of block intersection polynomials to block designs were given in [5]. Here we give further applications, and start with the proof of Theorem 1.3.

Proof. (of Theorem 1.3) Let t be an even positive integer, let D be a t-(v, k, λ) design, let S be a block of D, and let $n_i := n_i^{(1)}(D, S)$. Then n_i is the number of blocks of D meeting S in exactly i points. Now suppose that $I(D, S) \subseteq \{m, m+1, \ldots, m+t-1\}$ for some integer m, and let $m_0 = \ldots = m_{k-1} = 0$ and $m_k = 1$. Then $m_i \leq n_i$ for all i, and $m_i = n_i$ for all

$i \notin \{m, m+1, \ldots, m+t-1\}$, and so, where $c_j := \lambda_j^{(1)}(D,S) = \lambda\binom{v-j}{t-j}/\binom{k-j}{t-j}$ $(j = 0, \ldots, t)$, and

$$B(x) := B(x, [0^k, 1], [c_0, \ldots, c_t]),$$

we have, by Corollary 3.2, that $B(m) = 0$.

Now let E be any t-(v, k, λ) design, and let T be any block of E. Then $m_i \le n_i^{(1)}(E,T)$ for $i = 0, \ldots, k$ and $\lambda_j^{(1)}(E,T) = c_j$ for $j = 0, \ldots, t$, and since $B(m) = 0$, by Corollary 3.2, we have that

$$[n_0^{(1)}(E,T), \ldots, n_k^{(1)}(E,T)] = [n_0, \ldots, n_k] = f([0^k, 1], [c_0, \ldots, c_t]),$$

for some function f. ∎

In some sense, this result is best possible, for consider the 2-$(8,4,3)$ design with points $1, \ldots, 8$, and blocks (with commas and set brackets omitted):

$$1234, 1238, 1256, 1357, 1458, 1467, 1678, 2367, 2457, 2468, 2578, 3456, 3478, 3568.$$

The sizes of the intersections of the block 1234 with the other blocks are the three consecutive integers $1, 2, 3$, and the sizes of the intersections of the block 1357 with the other blocks are the two nonconsecutive integers $0, 2$.

We now give a somewhat similar application to arbitrary block designs.

Theorem 5.1 *Let $D = (V, \mathcal{B})$ be a block design, let S be an s-subset of V, and let r and t be positive integers, with t even, $t \le s$, such that the set of those i for which $n_i^{(r)}(D,S) > 0$ is contained in a set of t consecutive integers. Suppose $E = (W, \mathcal{C})$ is a block design, with $S' \subseteq W$ and $|S'| = s$. Then $n_i^{(r)}(E,S') = n_i^{(r)}(D,S)$ for $i = 0, \ldots, s$ if and only if $\lambda_j^{(r)}(E,S') = \lambda_j^{(r)}(D,S)$ for $j = 0, \ldots, t$.*

Proof. Let $\lambda_j := \lambda_j^{(r)}(D,S)$ for $j = 0, \ldots, t$, and let

$$B(x) := B(x, [0^{s+1}], [\lambda_0, \ldots, \lambda_t]).$$

Then, by Corollary 3.2, applied with $m_i = 0 \le n_i^{(r)}(D,S)$ for $i = 0, \ldots, s$, we have that $B(m) = 0$ for some integer m.

Suppose now that $n_i^{(r)}(E,S') = n_i^{(r)}(D,S)$ for $i = 0, \ldots, s$. Then, by Corollary 2.2, we have $\lambda_j^{(r)}(E,S') = \lambda_j$ for $j = 0, \ldots, t$.

Conversely, suppose that for $j = 0, \ldots, t$ we have $\lambda_j^{(r)}(E, S') = \lambda_j$. Then, since $B(m) = 0$, by Corollary 3.2, we have that

$$[n_0^{(r)}(E, S'), \ldots, n_s^{(r)}(E, S')] = [n_0^{(r)}(D, S), \ldots, n_s^{(r)}(D, S)] = f([0^{s+1}], [\lambda_0, \ldots, \lambda_t]),$$

for some function f. ∎

This theorem certainly applies when $S = V$, the point-set of the block design $D = (V, \mathcal{B})$. Now $n_i^{(1)}(D, V)$ is the number of blocks of D of size i, and more generally, $n_i^{(r)}(D, V)$ is the number of collections of r blocks of D whose intersection has size i. Such numbers were studied for t-designs in [1, 2], and were found to be useful isomorphism-class invariants. Note that D is the dual of a r-wise balanced design if and only if there is just one i for which $n_i^{(r)}(D, V) > 0$.

5.1 An application to design search

Here we give an example illustrating how a block intersection polynomial can give useful information to aid in the search for a block design with certain given properties.

In [6], Dobcsányi, Preece and Soicher study the problem of when a 2-(v, k, λ) design can have a repeated block, and they present a summary of their catalogue of 2-(v, k, λ) designs with repeated blocks and having parameters (v, b, r, k, λ) satisfying

$$r \leq 20, \quad 3 \leq k \leq v/2, \quad \gcd(b, r, \lambda) = 1, \tag{12}$$

where $r = \lambda(v-1)/(k-1)$ is the number of blocks containing a given point and $b = vr/k$ is the number of blocks.

There was just one parameter list in [6] satisfying (12) for which the existence of a 2-design with repeated blocks was unknown, but this has now been settled by Vedran Krčadinac [9], who found a 2-$(31, 5, 2)$ design with exactly four pairs of repeated blocks, making use of an assumed group of automorphisms of order 5 with just one fixed point, and tactical decompositions. Examining further the catalogue in [6], we find just two parameter lists satisfying (12) for which there is a 2-design with repeated blocks, but no design is given meeting the block multiplicity bound given in [5], the open cases being whether there is a 2-$(20, 5, 4)$ design with a block of multiplicity 3 and whether there is a 2-$(17, 5, 5)$ design also with a block of multiplicity 3.

We thus want to find a 2-$(20, 5, 4)$ design D with at least one block having multiplicity 3, and we start by assuming a group G of automorphisms of order 10, generated by

$$(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)(16, 17, 18, 19, 20)$$

and

$$(2, 5)(3, 4)(7, 10)(8, 9)(12, 15)(13, 14)(17, 20)(18, 19).$$

Since D will have 76 blocks, it is reasonable to assume that D will have two disjoint G-invariant blocks, $B_1 := \{1, 2, 3, 4, 5\}$ and $B_2 := \{6, 7, 8, 9, 10\}$, say, each with multiplicity 3. Now we calculate the block intersection polynomial

$$B(x) := B(x, [3, 0^4, 3], [76, 19, 4]) = 70(x - 1)(x - 2/7).$$

Since $B(1) = 0$, we have that a block of multiplicity 3 disjoint from three blocks must meet every further block in exactly 1 or 2 points, and so any block not having the same underlying set as B_1 or B_2 must meet each of B_1 and B_2 in exactly 1 or 2 points. This provides restrictions on the possible G-orbits of the other blocks of D, and using the function `BlockDesigns` in the DESIGN package [14], we find a set of blocks that complete B_1^3 and B_2^3 to the block multiset of a 2-$(20, 5, 4)$ design. This set is the union of the G-orbits of $\{1, 2, 6, 7, 14\}$, $\{1, 3, 6, 8, 17\}$, $\{1, 7, 11, 12, 14\}$, $\{1, 7, 11, 16, 17\}$, $\{1, 7, 15, 18, 19\}$, $\{1, 8, 12, 16, 19\}$, $\{1, 8, 13, 14, 20\}$, $\{1, 8, 15, 18, 20\}$.

In a similar way, we have found a 2-$(17, 5, 5)$ design having one block of multiplicity 3, and no other repeated blocks.

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