Turán and Ramsey Problems in the Discrete Cube

The discrete $n$-cube is the graph whose vertex set is the set of all $\{0, 1\}$ vectors of length $n$ with two vectors being adjacent if they differ in precisely one coordinate. The $n$-cube is an extremely natural and important combinatorial object (among other things it is the obvious place to work in for many problems from the theory of set systems and coding theory). There are many beautiful theorems concerning it but many basic questions about its structure remain tantalisingly unsolved. This project would involve study of the generalisation of the classical graph theory theorems of Turán and Ramsey to the cube.

A Turán-type result determines the maximum number of edges we can have without the occurrence of a fixed subgraph. In the context of the cube Erdős asked whether a subgraph of the $n$-cube which contains no 4-cycle can have more than $1/2 + o(1)$ of the cube’s edges as $n$ becomes large; this is still an open problem. However, there are partial results and related problems and results by Chung [3]; Alon, Krech and Szabó [1] and others.

A Ramsey-type result determines whether every colouring of the edges of a graph contains a monochromatic copy of a fixed subgraph. A result of Chung [3] implies that, for any $k$, if $n$ is sufficiently large then every colouring of the edges of the $n$-cube with $k$ colours contains a monochromatic 8-cycle (in fact she proved that the same is true for any cycle of length a multiple of 4). More recently, Alon, Radoićić, Sudakov and Vondrák [2] classified all graphs with this Ramsey property.

The project could focus on either Turán or Ramsey type questions or the interplay between them. Having taken a first course in combinatorics and/or graph theory would be an advantage but previous familiarity with Turán’s theorem and Ramsey’s theorem is not essential. For further references and to discuss possible ideas please contact Robert Johnson (r.johnson@qmul.ac.uk).

References

