# Determinants of 3×3 matrices

Let $A = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$ be a 3×3 matrix.

We define the <u>determinant</u> of A by

$$\det(A) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \underline{u} \cdot (\underline{v} \times \underline{w}),$$

where $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

Thus $\det(A) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \left( \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right)$

$$= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \left( \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} \underline{i} - \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} \underline{j} + \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \underline{k} \right)$$

$$= u_1 \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

$$= u_1 (v_2 w_3 - v_3 w_2) - u_2 (v_1 w_3 - v_3 w_1) + u_3 (v_1 w_2 - v_2 w_1).$$

Example

$$\text{Let } A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 0 & -3 \\ -2 & 1 & 1 \end{pmatrix}.$$

Then $\det(A) = 3\begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix}$

$$= 3(3) - 2(3) - 2(-6) = 15.$$

Many nice properties of determinants of $3 \times 3$ matrices follow easily from known properties of the triple scalar product. This is illustrated by Theorem 7 below and its proof.

Theorem 7

$$\text{Let } A = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}, \text{ let}$$

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

and let $\alpha$ be a scalar. Then:

(1) $\det(A) = \underline{u}\cdot(\underline{v}\times\underline{w}) = \underline{v}\cdot(\underline{w}\times\underline{u}) = \underline{w}\cdot(\underline{u}\times\underline{v})$,

$-\det(A) = \underline{u}\cdot(\underline{w}\times\underline{v}) = \underline{w}\cdot(\underline{v}\times\underline{u}) = \underline{v}\cdot(\underline{u}\times\underline{w})$

(so interchanging two columns of $A$ negates the determinant).

(2) $$\alpha\det(A) = (\alpha\underline{u})\cdot(\underline{v}\times\underline{w})$$
$$= \underline{u}\cdot((\alpha\underline{v})\times\underline{w})$$
$$= \underline{u}\cdot(\underline{v}\times(\alpha\underline{w}))$$

(so multiplying a column of $A$ by $\alpha$ multiplies the determinant by $\alpha$).

(3) $$\det(A) = (\underline{u}+\alpha\underline{v})\cdot(\underline{v}\times\underline{w}) = (\underline{u}+\alpha\underline{w})\cdot(\underline{v}\times\underline{w})$$
$$= \underline{u}\cdot((\underline{v}+\alpha\underline{u})\times\underline{w}) = \underline{u}\cdot((\underline{v}+\alpha\underline{w})\times\underline{w})$$
$$= \underline{u}\cdot(\underline{v}\times(\underline{w}+\alpha\underline{u})) = \underline{u}\cdot(\underline{v}\times(\underline{w}+\alpha\underline{v}))$$

(so adding a scalar multiple of one column of $A$ to another column of $A$ does not change the determinant).

Proof:

(1) $\det(A) = \underline{u}\cdot(\underline{v}\times\underline{w})$ (by definition)

$= \underline{v}\cdot(\underline{w}\times\underline{u})$

$= \underline{w}\cdot(\underline{u}\times\underline{v})$ } by Theorem 5

$-\det(A) = -\underline{u}\cdot(\underline{v}\times\underline{w})$

$= \underline{u}\cdot(\underline{w}\times\underline{v})$ (since $\underline{w}\times\underline{v} = -(\underline{v}\times\underline{w})$)

$= \underline{w}\cdot(\underline{v}\times\underline{u})$

$= \underline{v}\cdot(\underline{u}\times\underline{w})$ } by Theorem 5

(2) First, $(\alpha\underline{u})\cdot(\underline{v}\times\underline{w})$

$= \alpha(\underline{u}\cdot(\underline{v}\times\underline{w}))$ (property of scalar product)

$= \alpha\det(A)$.

Next, $\underline{u}\cdot((\alpha\underline{v})\times\underline{w})$

$= \underline{u}\cdot(\alpha(\underline{v}\times\underline{w}))$ (property of vector product)

$= \alpha(\underline{u}\cdot(\underline{v}\times\underline{w}))$ (property of scalar product)

$= \alpha\det(A)$.

Similarly, $\underline{u}\cdot(\underline{v}\times(\alpha\underline{w})) = \alpha\det(A)$.

(3) $\quad (\underline{u} + \alpha \underline{v}) \cdot (\underline{v} \times \underline{w})$

$= \underline{u} \cdot (\underline{v} \times \underline{w}) + (\alpha \underline{v}) \cdot (\underline{v} \times \underline{w})$ (distributivity of scalar product)

$= \underline{u} \cdot (\underline{v} \times \underline{w}) + 0$ (since $\alpha \underline{v}, \underline{v}, \underline{w}$ coplanar)

$= \det(A)$

Similarly, $(\underline{u} + \alpha \underline{w}) \cdot (\underline{v} \times \underline{w}) = \det(A)$.

Now $\quad \underline{u} \cdot ((\underline{v} + \alpha \underline{u}) \times \underline{w})$

$= (\underline{v} + \alpha \underline{u}) \cdot (\underline{w} \times \underline{u})$ (by Theorem 5)

$= \underline{v} \cdot (\underline{w} \times \underline{u})$ (by preceding arguments)

$= \underline{u} \cdot (\underline{v} \times \underline{w})$ (by Theorem 5)

$= \det(A)$

The other cases are similar.

<u>Example</u>

$$\begin{vmatrix} 15 & 100 & 200 \\ -9 & 100 & 201 \\ 6 & 100 & 202 \end{vmatrix}$$

$$= \begin{vmatrix} 15 & 100 & 0 \\ -9 & 100 & 1 \\ 6 & 100 & 2 \end{vmatrix} \quad \text{(by Theorem 7, part (3))}$$

$$= 100 \begin{vmatrix} 15 & 1 & 0 \\ -9 & 1 & 1 \\ 6 & 1 & 2 \end{vmatrix} \quad \text{(by Theorem 7, part (2))}$$

$$= 300 \begin{vmatrix} 5 & 1 & 0 \\ -3 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix} \quad \text{(by Theorem 7, part (2))}$$

$$= -300 \begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & -3 \\ 2 & 1 & 2 \end{vmatrix} \quad \text{(by Theorem 7, part (1))}$$

$$= -300 \left( 0 - \begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 5 \\ 1 & -3 \end{vmatrix} \right)$$

$$= -300 \left( -(-3) + 2(-8) \right)$$

$$= -300 (-13) = 3900$$

Remark

Although we do not prove it in this module, it is true that if $A, B$ are $3 \times 3$ matrices then

$$\det(AB) = \det(A)\det(B),$$

and $A$ is invertible if and only if $\det(A) \neq 0$.

# The vector space $\mathbb{R}^n$

An $n \times 1$ matrix $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is called a

<u>column vector</u> of <u>dimension</u> $n$,

and a $1 \times n$ matrix $(a_1\ a_2 \cdots a_n)$ (or $(a_1, a_2, \ldots, a_n)$) is called a <u>row vector</u> of <u>dimension</u> $n$.

In this module we deal mostly with column vectors, and call a column vector of dimension $n$ simply an <u>$n$-vector</u>. (Thus, a vector $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ given in co-ordinates is a 3-vector.) We let $\mathbb{R}^n$ denote the set of all $n$-vectors. Thus

$$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}.$$

We denote by $0_n$ the $n$-vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ with all entries $0$ (thus $0_n = 0_{n1}$).

(In other contexts, $\mathbb{R}^n$ may be considered to be the set of all row vectors of dimension $n$, but not in this module.)

As a consequence of the properties of matrix addition and scalar multiplication that we have proved, we observe that for all $u, v, w \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$ we have:

(a.1) $u + v \in \mathbb{R}^n$.

(a.2) $\mathbb{R}^n$ contains an element $e$, such that $v + e = e + v = v$ for all $v \in \mathbb{R}^n$. $[e = 0_n]$

(a.3) For every $v \in \mathbb{R}^n$ there is a $-v \in \mathbb{R}^n$, such that $-v + v = v + (-v) = e$.

(a.4) $u + (v + w) = (u + v) + w$.

(a.5) $u + v = v + u$.

(m.1) $\alpha v \in \mathbb{R}^n$.

(m.2) $1 v = v$.

(m.3) $\alpha(\beta v) = (\alpha\beta) v$.

(m.4) $(\alpha + \beta) v = \alpha v + \beta v$.

(m.5) $\alpha(u + v) = \alpha u + \alpha v$.

This shows that $\mathbb{R}^n$ satisfies the rules to be an algebraic structure called a (real) <u>vector space</u>. You will study vector spaces in detail in the module Linear Algebra I.

Here we are mostly concerned with the vector spaces $\mathbb{R}^3$ and $\mathbb{R}^2$.

We have

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}, \text{ and}$$

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\}.$$

Recall that $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$.

We use $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ to denote the vector $a_1 \underline{i} + a_2 \underline{j}$.

With this notation, we identify $\mathbb{R}^2$ with the set of position vectors of the points lying in the plane defined by $z = 0$ (the $(x, y)$-plane). Indeed, $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \underline{i} + a_2 \underline{j} + 0 \underline{k}$ is the position vector of the point $(a_1, a_2, 0)$, which we abbreviate to $(a_1, a_2)$ when working in the $(x, y)$-plane.

## Note on notation

In these notes, an underlined symbol (such as $\underline{v}$, $\underline{w}$, $\underline{0}$, ...) always denotes a (free) vector which can be represented by a bound vector in 3-dimensional space. Such a vector $\underline{v}$ can be given by its length and direction (unless $\underline{v} = \underline{0}$, in which case the length is 0 and the direction undefined), or by co-ordinates, with $\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ meaning $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$. The notation $\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ means $\underline{v} = a\underline{i} + b\underline{j}$, in which case $\underline{v}$ is the position vector of the point $(a, b)$ $(= (a, b, 0))$ in the $(x, y)$-plane.

In contrast, an $n$-dimensional column vector $v \in \mathbb{R}^n$ may or may not be represented by a bound vector in 3-space.

# Linear Transformations

**Definition** A function $t : \mathbb{R}^n \to \mathbb{R}^m$ from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$ is called a linear transformation if for all $u, v \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$ we have:

(i) $t(u+v) = t(u) + t(v)$,

(ii) $t(\alpha u) = \alpha t(u)$.

If $m=n$ we call $t$ a linear transformation of $\mathbb{R}^n$. Linear transformations are also called linear maps.

**Example**

Consider the function $t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $t\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$.

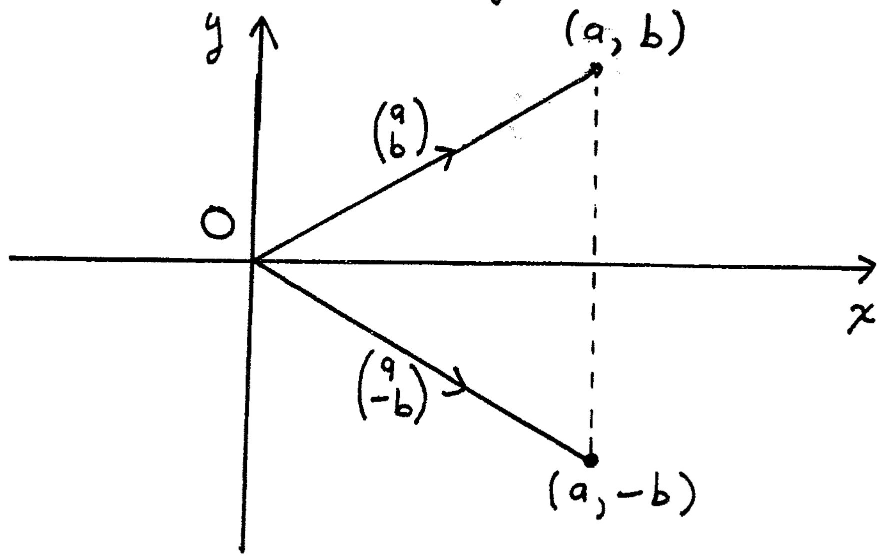
If $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and $\alpha \in \mathbb{R}$, we have

$$t(u+v) = t\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ -u_2-v_2 \end{pmatrix}$$
$$= \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = t(u) + t(v),$$

and $t(\alpha u) = t\begin{pmatrix}\alpha u_1 \\ \alpha u_2\end{pmatrix} = \begin{pmatrix}\alpha u_1 \\ -\alpha u_2\end{pmatrix}$

$$= \alpha\begin{pmatrix}u_1 \\ -u_2\end{pmatrix} = \alpha t(u).$$

Thus $t$ is a linear transformation of $\mathbb{R}^2$. For each point $(a, b)$ in the $(x,y)$-plane, $t$ maps its position vector $\begin{pmatrix}a \\ b\end{pmatrix}$ to $\begin{pmatrix}a \\ -b\end{pmatrix}$, the position vector of $(a, -b)$. We may thus consider $t$ as a mapping of points to points in the $(x,y)$-plane, via their position vectors. Geometrically, $t$ describes a reflection in the $x$-axis:



<u>Example</u> Consider the function $t : \mathbb{R}^n \to \mathbb{R}^m$ defined by $t(u) = 0_m$ for all $u \in \mathbb{R}^n$. Then, for all $u, v \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$, we have:

$$t(u+v) = 0_m = 0_m + 0_m = t(u) + t(v),$$

and $t(\alpha u) = 0_m = \alpha 0_m = \alpha t(u)$.

Thus $t$ is a linear transformation.

<u>Example</u> The function $t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $t\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b+1 \end{pmatrix}$ is <u>not</u> a linear transformation since, for example, $t\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, but

$$t\begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

## Properties of linear transformations

Let $t : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then, for all $u, v \in \mathbb{R}^n$ and all scalars $\alpha, \beta$, we have:

(i) $t(\alpha u + \beta v) = \alpha t(u) + \beta t(v)$;

(ii) $t(0_n) = 0_m$;

(iii) $t(-u) = -t(u)$;

(iv) $t(u - v) = t(u) - t(v)$.

Proof: (i)
$$\left.\begin{aligned} t(\alpha u + \beta v) &= t(\alpha u) + t(\beta v) \\ &= \alpha t(u) + \beta t(v) \end{aligned}\right\} \text{since } t \text{ is a linear transformation}$$

(ii) $t(0_n) = t(0_n + 0_n) = t(0_n) + t(0_n)$

Thus $-t(0_n) + t(0_n) = -t(0_n) + t(0_n) + t(0_n)$,

so
$$0_m = 0_m + t(0_n) = t(0_n).$$

(iii) $t(u) + t(-u) = t(u + (-u)) = t(0_n) = 0_m$ (by (ii))

Thus $-t(u) + t(u) + t(-u) = -t(u) + 0_m$, so $t(-u) = -t(u)$.

(iv)
$$\begin{aligned} t(u - v) &= t(u + (-v)) = t(u) + t(-v) \\ &= t(u) + (-t(v)) \quad \text{(by (iii))} \\ &= t(u) - t(v). \end{aligned}$$

# Matrices represent linear transformations

Let $A$ be an $m \times n$ matrix and let $u$ be an $n$-vector (i.e. an $n \times 1$ matrix).

Then $Au$ is defined, and $Au$ is an $m$-vector (i.e. an $m \times 1$ matrix).

Now define the function

$t_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $t_A(u) = Au$

for all $u \in \mathbb{R}^n$. By properties of matrices we have proved, we have, for all $u, v \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$:

$$t_A(u+v) = A(u+v) = Au + Av = t_A(u) + t_A(v),$$

$$t_A(\alpha u) = A(\alpha u) = \alpha(Au) = \alpha t_A(u).$$

Thus $t_A$ is a linear transformation.

We call $t_A$ the linear transformation <u>represented by</u> $A$.

<u>Example</u> Let $t = t_{I_n}$. Then $t$ is a linear transformation of $\mathbb{R}^n$, with $t(u) = I_n u = u$ for all $u \in \mathbb{R}^n$.

<u>Is every linear transformation $t: \mathbb{R}^n \to \mathbb{R}^m$ represented by some matrix?</u> The answer is yes, and we illustrate this in the special case of a linear transformation $t: \mathbb{R}^3 \to \mathbb{R}^3$. Suppose

$$t\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}, \quad t\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}, \quad t\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}.$$

If $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ then $\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}a\\0\\0\end{pmatrix} + \begin{pmatrix}0\\b\\0\end{pmatrix} + \begin{pmatrix}0\\0\\c\end{pmatrix}$

$$= a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix},$$ and so

$$t\begin{pmatrix}a\\b\\c\end{pmatrix} = t\left(a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= at\begin{pmatrix}1\\0\\0\end{pmatrix} + bt\begin{pmatrix}0\\1\\0\end{pmatrix} + ct\begin{pmatrix}0\\0\\1\end{pmatrix}$$ (since $t$ is a linear trans.)

$$= a\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} + b\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} + c\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$$

$$= \begin{pmatrix}au_1 + bv_1 + cw_1\\au_2 + bv_2 + cw_2\\au_3 + bv_3 + cw_3\end{pmatrix} = \begin{pmatrix}u_1 & v_1 & w_1\\u_2 & v_2 & w_2\\u_3 & v_3 & w_3\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}.$$

Thus, if $A = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$ then

$t\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for all $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, and so $t$ is represented by $A$.

Similarly, and more generally, if $t : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $A$ is the $m \times n$ matrix

$$\left( t\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\; t\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\; \cdots\cdots \;\; t\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)$$

(i.e. the $j$-th column of $A$ is the image under $t$ of the $n$-vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← 1 in row $j$; 0 elsewhere),

then $t(u) = Au$ for all $u \in \mathbb{R}^n$,

and so $t$ is represented by $A$.

<u>Example</u> The linear transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ with $t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ is represented by the matrix $\begin{pmatrix} 2 & 2 \\ -3 & 7 \end{pmatrix}$. Thus

$t\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + 2b \\ -3a + 7b \end{pmatrix}$, for all $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$.

# Composition of linear transformations and multiplication of matrices

Suppose $s : \mathbb{R}^n \to \mathbb{R}^m$ and $t : \mathbb{R}^p \to \mathbb{R}^n$ are linear transformations. Define the function $s \circ t : \mathbb{R}^p \to \mathbb{R}^m$ by

$(s \circ t)(u) = s(t(u))$ for all $u \in \mathbb{R}^p$

( $s \circ t$ is called the <u>composition</u> of $s$ with $t$).

Now suppose $s$ is represented by the $m \times n$ matrix $A$ and $t$ is represented by the $n \times p$ matrix $B$. Then for all $u \in \mathbb{R}^p$ we have

$$(s \circ t)(u) = s(t(u)) = A(Bu)$$
$$= (AB)u \quad \text{(by the associativity of matrix multiplication).}$$

We conclude that $s \circ t$ is a linear transformation represented by the $m \times p$ matrix $AB$. (Now you see why matrix multiplication is defined how it is!)

<u>Example</u> Let $s: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$s\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \quad s\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}, \quad s\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix},$$

and let $t: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with

$$t\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}, \quad t\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Then $s$ is represented by $A = \begin{pmatrix}1 & 1 & -1\\2 & -3 & 0\end{pmatrix}$,

and $t$ is represented by $B = \begin{pmatrix}2 & 1\\0 & 2\\1 & 3\end{pmatrix}$.

Moreover, $s \circ t : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by $AB = \begin{pmatrix}1 & 0\\4 & -4\end{pmatrix}$.

Thus, if $\begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{R}^2$, then

$$s\left(t\begin{pmatrix}a\\b\end{pmatrix}\right) = (s \circ t)\begin{pmatrix}a\\b\end{pmatrix} = (AB)\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1 & 0\\4 & -4\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\4a-4b\end{pmatrix}.$$

For example, $(s \circ t)\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}1 & 0\\4 & -4\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}-2\\-20\end{pmatrix}$.

# Rotations and reflections in the $(x, y)$-plane

The linear transformation $r_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

represented by the matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

describes a counterclockwise <u>rotation</u> through an angle $\theta$ about the origin in the $(x,y)$-plane. This is illustrated by the figure below from the course text:

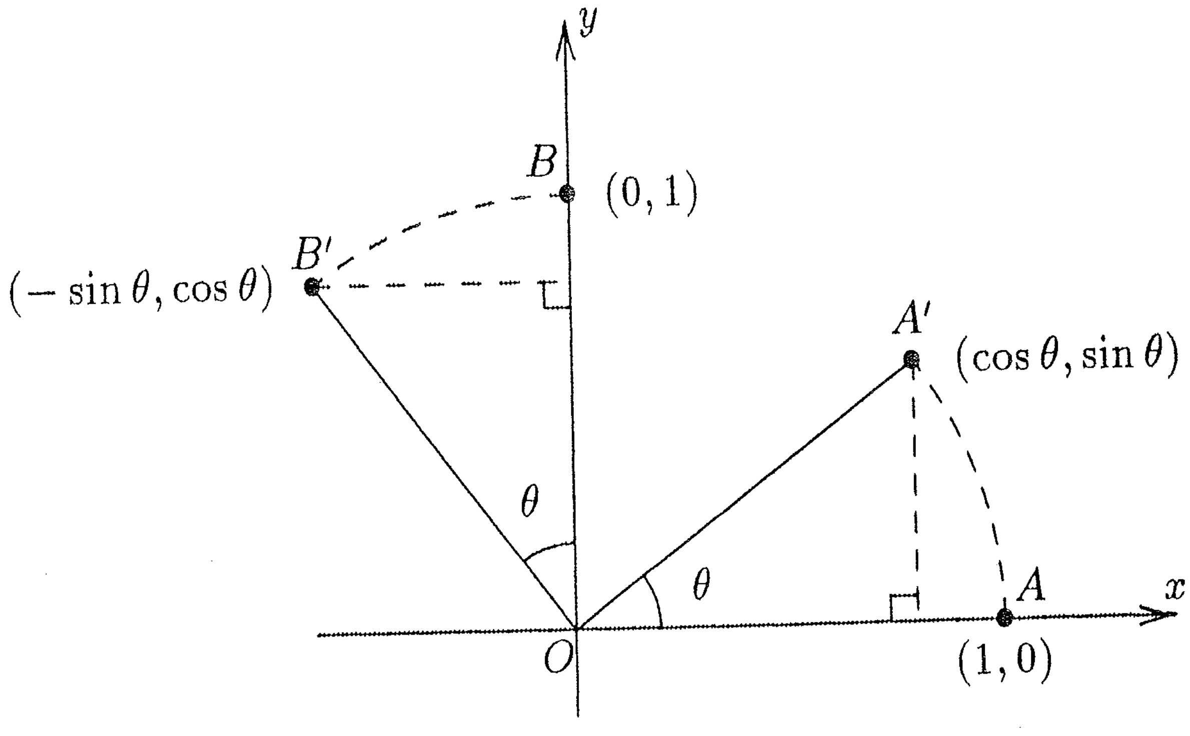



Fig 6.2
(Copyright 1995 A.E. Hirst)

We check that $(R_\theta)^{-1} = R_{-\theta}$.

We have $R_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$

$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and so

$$R_\theta R_{-\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} (\cos\theta)^2 + (\sin\theta)^2 & 0 \\ 0 & (\sin\theta)^2 + (\cos\theta)^2 \end{pmatrix} = I_2 .$$

Similarly, $R_{-\theta} R_\theta = I_2$.

The linear transformation $s_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

represented by the matrix $S_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

describes a reflection (in the $(x,y)$-plane) in the line through the origin at (counterclockwise) angle $\theta$ from the $x$-axis. This is illustrated by the figure below from the course text:

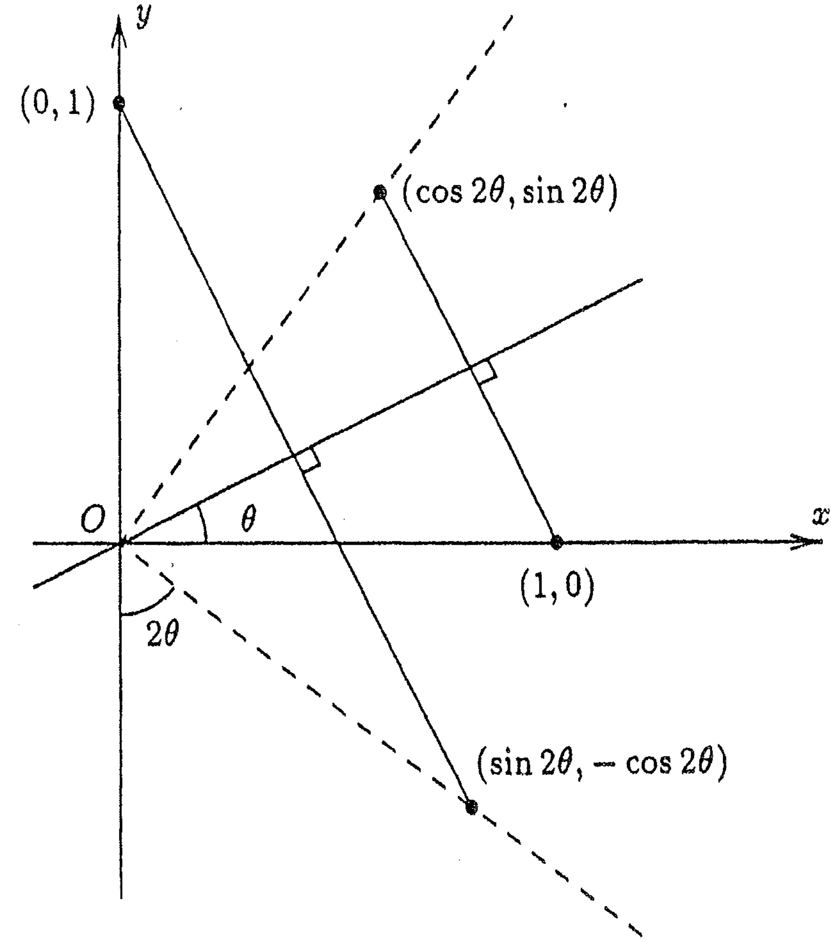



Fig 6.3

Indeed, the preceding figure illustrates:

$$s_\theta\begin{pmatrix}1\\0\end{pmatrix} = r_{2\theta}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos 2\theta\\ \sin 2\theta\end{pmatrix}, \quad \text{and}$$

$$s_\theta\begin{pmatrix}0\\1\end{pmatrix} = r_{-2(\frac{\pi}{2}-\theta)}\begin{pmatrix}0\\1\end{pmatrix} = r_{-\pi+2\theta}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}-\sin(-\pi+2\theta)\\ \cos(-\pi+2\theta)\end{pmatrix} = \begin{pmatrix}\sin 2\theta\\ -\cos 2\theta\end{pmatrix}.$$

Exercise

Check that $(S_\theta)^{-1} = S_\theta$, by showing that $S_\theta S_\theta = I_2$. [This verifies that applying the same reflection twice fixes every point in the $(x,y)$-plane.]

<u>Example</u> We calculate

$$S_\theta S_0 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

$= R_{2\theta}$. Thus $s_\theta \circ s_0 = r_{2\theta}$,

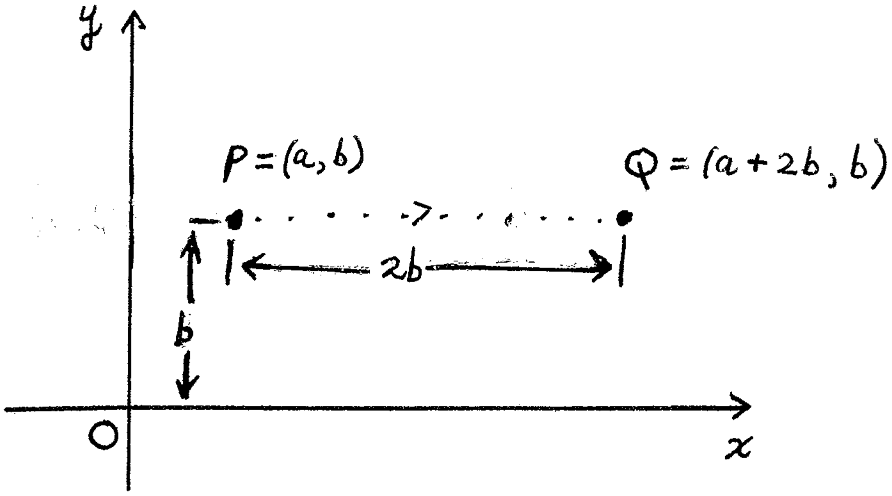
and so a reflection in the $x$-axis followed by a reflection in the line through the origin at angle $\theta$ from the $x$-axis has the same effect as the rotation $r_{2\theta}$ through a (counterclockwise) angle of $2\theta$, and more generally, a reflection in any line $l$ through the origin followed by a reflection in the line through the origin at angle $\theta$ from $l$ also has the same effect as $r_{2\theta}$.

# More linear transformations of $\mathbb{R}^2$

Consider the linear transformation $t$ represented by $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\left(= \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)$; so $t\begin{pmatrix} a \\ b \end{pmatrix} = A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \beta b \end{pmatrix}$. If $\alpha, \beta \neq 0$ then $t$ is a <u>stretch parallel to the $x$-axis with scale factor</u> $\alpha$ (represented by $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$) composed with a <u>stretch parallel to the $y$-axis with scale factor</u> $\beta$ (represented by $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$). In the very special case when $\alpha = \beta = 1$, $t$ is the <u>identity transformation</u>, and is represented by $I_2$.

A linear transformation of $\mathbb{R}^2$ is called a <u>shear</u> if it fixes all the points on a line $l$ through the origin and moves each point $P$ not on $l$ in a direction parallel to $l$, such that the distance moved by $P$ is proportional to its distance from $l$.

In the special case where $l$ is the $x$-axis, a shear $s$ is represented by a matrix of the form $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, so that, if $P = (a, b)$ is a point then $s$ maps $P$ to $Q = (a + \alpha b, b)$ (as $s$ maps the position vector $\begin{pmatrix} a \\ b \end{pmatrix}$ of $P$ to the position vector $\begin{pmatrix} a + \alpha b \\ b \end{pmatrix}$ of $Q$).




Example showing shear represented by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

A linear transformation of $\mathbb{R}^2$ is called <u>singular</u> if it is represented by a non-invertible matrix [equivalently, if it is represented by a matrix with determinant 0]. For example, let $t$ be the linear transformation ~~represented~~ by $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$. Then $t$ is singular, as $\det(A) = 2 - 2 = 0$. We have $t\begin{pmatrix} a \\ b \end{pmatrix} = A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ -2a+2b \end{pmatrix}$.

Thus $t$ maps each point $(a, b)$ to a point lying on the line (in the $(x, y)$-plane) defined by $y = -2x$. Moreover, $t$ maps each point $(a, a)$ (on the line defined by $y = x$) to the origin $(0, 0)$.

Indeed, a singular transformation of $\mathbb{R}^2$ will <u>always</u> map some non-zero vector to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Suppose $t$ is represented by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) = ad - cb = 0$. If $a \neq 0$ or $b \neq 0$, then $\begin{pmatrix} -b \\ a \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ \det(A) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, if $c \neq 0$ or $d \neq 0$ then $\begin{pmatrix} d \\ -c \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} \det(A) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and if

$a = b = c = d = 0$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

for <u>every</u> vector $\begin{pmatrix} e \\ f \end{pmatrix} \in \mathbb{R}^2$.

## Eigenvectors, eigenvalues, and fixed lines

<u>Definition</u> Let $t$ be a linear transformation of $\mathbb{R}^n$ represented by the $n \times n$ matrix $A$. We call $v \in \mathbb{R}^n$ an <u>eigenvector</u> of $t$ (and of $A$) if $v \neq 0_n$ and $t(v) = Av = \lambda v$ for some scalar $\lambda$, in which case $\lambda$ is called the <u>eigenvalue</u> of $t$ (and of $A$) corresponding to $v$.

<u>Example</u> Let $t$ be the linear transformation of $\mathbb{R}^2$ represented by $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Then

$t\begin{pmatrix} 1 \\ 2 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and so $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $t$ (and of $A$) with corresponding eigenvalue 5.

<u>Definition</u> Let $n = 2$ or $3$ and let $t$ be a linear transformation of $\mathbb{R}^n$. We say that a line $l$ in $\mathbb{R}^n$ is <u>fixed</u> by $t$ if $t$ maps each point $P$ on $l$ to a point also on $l$. (In other words, the line $l$ is fixed by $t$ if whenever $\underline{p}$ is the position vector of a point on $l$ then $t(\underline{p})$ is also the position vector of some point on $l$.)

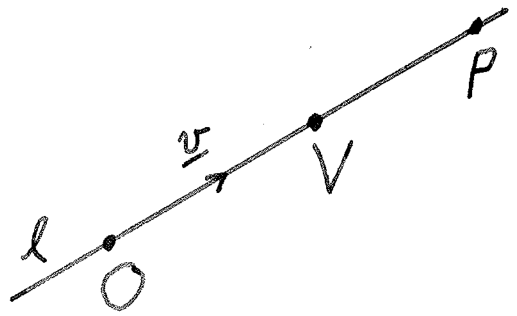
<u>Example</u> Let $t$ be the linear transformation of $\mathbb{R}^2$ represented by $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Then the line $l$ in the $(x, y)$-plane defined by $y = 2x$ is fixed by $t$, because if $P$ is on $l$ then $P = (a, 2a)$ for some $a \in \mathbb{R}$, and $t\begin{pmatrix} a \\ 2a \end{pmatrix} = A\begin{pmatrix} a \\ 2a \end{pmatrix} = \begin{pmatrix} 5a \\ 10a \end{pmatrix}$ is the position vector of the point $(5a, 10a)$ also on $l$.

## The connection between eigenvectors and fixed lines through the origin

Let $n = 2$ or $3$ and let $t$ be a linear transformation of $\mathbb{R}^n$.

Suppose $\underline{v}$ is an eigenvector of $t$ with corresponding eigenvalue $\lambda$, let $V$ be the point with position vector $\underline{v}$, and let $l$ be the line through the origin and $V$. Then $l$ is fixed by $t$.

Proof:



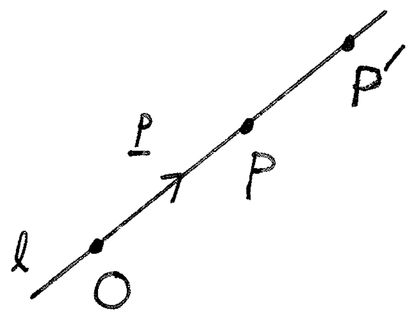

Note that $l$ has vector equation $\underline{r} = \underline{0} + \mu \underline{v}$.

Suppose $P$ is a point on $l$ and $\underline{p}$ is its position vector. Then $\underline{p} = \alpha \underline{v}$ for some scalar $\alpha$, and so $t(\underline{p}) = t(\alpha \underline{v}) = \alpha\, t(\underline{v}) = \alpha \lambda \underline{v}$ is the position vector of a point also on $l$.

[Note also that if $\underline{p} \neq \underline{0}$ then $\underline{p}$ is an eigenvector of $t$ with corresponding eigenvalue $\lambda$, for $t(\underline{p}) = \alpha \lambda \underline{v} = \lambda(\alpha \underline{v}) = \lambda \underline{p}$.]

We have seen how an eigenvector of $t$ gives rise to a line through the origin fixed by $t$. Conversely, let $l$ be any line through the origin fixed by $t$, let $P$ be any point on $l$, other than the origin, and let $\underline{p}$ be the position vector of $P$. Then $\underline{p}$ is an eigenvector of $t$.

<u>Proof</u>: Since $l$ is fixed by $t$, $\underline{p}' = t(\underline{p})$ is the position vector of some point $P'$ on $l$, and since $l$ goes through the origin, we have $\underline{p}' = \alpha\underline{p}$ for some scalar $\alpha$.




Thus $\underline{p} \neq \underline{0}$ (since $P$ is not the origin) and $t(\underline{p}) = \alpha\underline{p}$, and so $\underline{p}$ is an eigenvector of $t$ with corresponding eigenvalue $\alpha$.

<u>Example</u> We have already seen that the linear transformation $t$ of $\mathbb{R}^2$ represented by $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ has an eigenvector $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with corresponding eigenvalue 5. We have also seen that the line $l$ in the $(x, y)$-plane defined by $y = 2x$ is fixed by $t$. This is the line through the origin and $V = (1, 2)$.

<u>Matrices having $\lambda = 0$ as an eigenvalue</u>

Let $A$ be an invertible $n \times n$ matrix. Then there is no $v \in \mathbb{R}^n$ with $v \neq 0_n$ and $Av = 0v$, because if there were, then we would have

$$Av = 0_n$$
$$A^{-1}Av = A^{-1}0_n$$
$$I_n v = 0_n$$
$$v = 0_n, \text{ a contradiction!}$$

This proves that no invertible $n \times n$ matrix has an eigenvector with corresponding eigenvalue $0$.

What if $A$ is not invertible?

In our study of singular linear transformations of $\mathbb{R}^2$, we saw that if $A$ is a non-invertible $2 \times 2$ matrix then there must be a vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ with $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and so $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector of $A$ with corresponding eigenvalue $0$.

Although we do not prove it in this module, it is more generally true that every non-invertible $n \times n$ matrix has some eigenvector with corresponding eigenvalue $0$.

# Finding all eigenvalues

Let $n = 2$ or $3$. We define the <u>characteristic polynomial</u> of an $n \times n$ matrix $A$ to be $f(x) = \det(A - x I_n)$.

<u>Example</u> Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. Then the characteristic polynomial of $A$ is

$$f(x) = \det(A - x I_2)$$

$$= \det\left(\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right)$$

$$= \begin{vmatrix} 1-x & 2 \\ 4 & 3-x \end{vmatrix} = (1-x)(3-x) - 8$$

$$= x^2 - 4x - 5 = (x+1)(x-5)$$

<u>Example</u>

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$. Then the characteristic polynomial of $A$ is

$$f(x) = \det(A - xI_3)$$

$$= \begin{vmatrix} 1-x & 2 & -1 \\ 0 & 1-x & 4 \\ 0 & 0 & 3-x \end{vmatrix}$$

$$= (1-x)\begin{vmatrix} 1-x & 4 \\ 0 & 3-x \end{vmatrix} - 0 + 0$$

$$= (1-x)\big((1-x)(3-x) - 0\big)$$

$$= -(x-1)^2(x-3)$$

Theorem 8 Let $n = 2$ or $3$, and let $A$ be an $n \times n$ matrix. Then a scalar $\lambda$ is an eigenvalue of $A$ if and only if it is a zero of the characteristic polynomial of $A$. [Recall that $\lambda$ is a zero of the polynomial $f(x)$ just means $f(\lambda) = 0$.]

Proof: Let $f(x) = \det(A - xI_n)$. If $\lambda$ is a zero of $f(x)$ then $f(\lambda) = 0$, i.e. $\det(A - \lambda I_n) = 0$, so $A - \lambda I_n$ is not invertible and there is a non-zero $v \in \mathbb{R}^n$ with

$$(A - \lambda I_n)v = 0_n.$$ [only proved when $n = 2$ in this module]

Then

$$Av - \lambda I_n v = 0_n$$
$$Av - \lambda v = 0_n$$
$$Av = \lambda v,$$

and so $v$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$.

Conversely, suppose $v$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda$. Then $v \neq 0_n$ and $Av = \lambda v$. Then

$$Av - \lambda v = 0_n$$

$$Av - \lambda I_n v = 0_n$$

$$(A - \lambda I_n) v = 0_n = 0v$$

Thus $A - \lambda I_n$ has $0$ as an eigenvalue, and so $A - \lambda I_n$ is not invertible, so $\det(A - \lambda I_n) = 0$ [proved only when $n=2$ in this module], i.e. $f(\lambda) = \det(A - \lambda I_n) = 0$.

Example

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. We found that the characteristic polynomial of $A$ is $f(x) = (x+1)(x-5)$. The only zeros of $f(x)$ are $-1$ and $5$, so the only eigenvalues of $A$ are $-1$ and $5$.

## Remarks

Let $n = 2$ or $3$ and let $A$ be an $n \times n$ matrix with characteristic polynomial $f(x)$. It is not difficult to see that $f(x)$ really is a polynomial (with coefficients in $\mathbb{R}$), and that the degree of $f(x)$ is $n$ (do this first for $n = 2$). Such an $f(x)$ has at most $n$ real zeros, and so $A$ has at most $n$ (real, distinct) eigenvalues.

A polynomial of odd degree with coefficients in $\mathbb{R}$ always has at least one real zero. Thus a $3 \times 3$ matrix always has at least one (real) eigenvalue, and ~~so~~ a linear transformation of $\mathbb{R}^3$ always fixes at least one line through the origin.

# Finding eigenvectors

Let $n=2$ or $3$ and let $A$ be an $n \times n$ matrix with an eigenvalue $\lambda$. How do we find one (or all) eigenvectors $v$ of $A$ with $Av = \lambda v$?

Answer: we solve a certain system of linear equations in $n$ unknowns.

We have $$Av = \lambda v$$

$$\iff Av - \lambda v = 0_n$$

(means "if and only if")

$$\iff Av - \lambda I_n v = 0_n$$

$$\iff (A - \lambda I_n) v = 0_n$$

If $n=2$, we solve

$$(A - \lambda I_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, to obtain the eigenvectors with eigenvalue $\lambda$.

If $n=3$, we solve

$$(A-\lambda I_3)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

for $\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3$, $\begin{pmatrix}x\\y\\z\end{pmatrix}\neq\begin{pmatrix}0\\0\\0\end{pmatrix}$.

<u>Example</u>

Let $A=\begin{pmatrix}1&2\\4&3\end{pmatrix}$. We found that the eigenvalues of $A$ are $-1$ and $5$. We now determine the eigenvectors with corresponding eigenvalue $-1$. We solve

$$(A-(-1)I_2)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\left(\begin{pmatrix}1&2\\4&3\end{pmatrix}-\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}2&2\\4&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\qquad(*)$$

This is equivalent to the following system of linear equations in $x, y$:

$$\begin{cases} 2x + 2y = 0 \\ 4x + 4y = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} 2x + 2y = 0 \\ \qquad\quad 0 = 0 \end{cases}$$

which is equivalent to

$$\{ 2x + 2y = 0 .$$

Thus $y$ can be any real number $r$; then $2x + 2r = 0$, so $2x = -2r$, $x = -r$.

The set of solutions to the matrix equation $(*)$ is thus $\left\{ \begin{pmatrix} -r \\ r \end{pmatrix} : r \in \mathbb{R} \right\}$, and so the set of all eigenvectors of $A$ with corresp. eigenvalue $-1$ is $\left\{ \begin{pmatrix} -r \\ r \end{pmatrix} : 0 \neq r \in \mathbb{R} \right\}$.

One such eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Check: $A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Example

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$. We have found that the characteristic polynomial of $A$ is $f(x) = -(x-1)^2(x-3)$. Thus the eigenvalues of $A$ are $1$ and $3$ (the (real) zeros of $f(x)$). We now find all eigenvectors of $A$ with corresponding eigenvalue $3$. We solve:

$$(A - 3I_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-3 & 2 & -1 \\ 0 & 1-3 & 4 \\ 0 & 0 & 3-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

This is equivalent to the following system of linear equations in $x, y, z$:

$$\begin{cases} -2x+2y-z=0 \\ \quad\quad -2y+4z=0 \\ \quad\quad\quad\quad 0=0 \end{cases}$$

which happens to be in echelon form. Applying back substitution, $z$ can be any real number $r$; then $-2y+4r=0$, so $-2y=-4r$, $y=2r$; then $-2x+2(2r)-r=0$, so $-2x=-3r$, $x=\frac{3}{2}r$.

The set of solutions to the matrix equation (*) is thus $\left\{\begin{pmatrix}\frac{3}{2}r\\2r\\r\end{pmatrix} : r\in\mathbb{R}\right\}$, and so the set of all eigenvectors of $A$ with corresp. eigenvalue 3 is $\left\{\begin{pmatrix}\frac{3}{2}r\\2r\\r\end{pmatrix} : 0\neq r\in\mathbb{R}\right\}$.

One such eigenvector is $\begin{pmatrix}3\\4\\2\end{pmatrix}$ (check this!).

Thus the line through the origin with vector equation $\underline{r}=\mu\begin{pmatrix}3\\4\\2\end{pmatrix}$ is fixed by the linear transformation $t_A$ represented by $A$.