

The rate of entropy increase at the edge of chaos

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Abstract

Under certain conditions, the rate of increase of the statistical entropy of a simple, fully chaotic, conservative system is known to be given by a single number, characteristic of this system, the Kolmogorov–Sinai entropy rate. This connection is here generalized to a simple dissipative system, the logistic map, and especially to the chaos threshold of the latter, the edge of chaos. It is found that, in the edge-of-chaos case, the usual Boltzmann–Gibbs–Shannon entropy is not appropriate. Instead, the non-extensive entropy $S_q \equiv (1 - \sum_{i=1}^W p_i^q)/(q-1)$, must be used. The latter contains a parameter q , the entropic index which must be given a special value $q^* \neq 1$ (for $q = 1$ one recovers the usual entropy) characteristic of the edge-of-chaos under consideration. The same q^* enters also in the description of the sensitivity to initial conditions, as well as in that of the multifractal spectrum of the attractor. © 2000 Published by Elsevier Science B.V.

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The connection between chaos and thermodynamics has been receiving increased attention lately. A review of the central ideas can be found in [1]. Recent studies have focused on both *conservative* (classical long-range interacting many-body Hamiltonians [2–5], low-dimensional conservative maps [6]) and *dissipative* (low-dimensional maps [7–11],

many-body self-organized criticality [12–15], symbolic sequences [16]) systems. In Ref. [6] the connection between the Kolmogorov–Sinai (KS) entropy rate and the statistical entropy (or thermodynamic entropy) was brought out for simple conservative systems. In Refs. [7,8] the non-extensive entropy introduced some years ago by one of us [17–19]¹ was shown to be the relevant quantity at the chaos

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¹ A regularly updated bibliography on the subject is accessible at <http://tsallis.cat.cbpf.br/biblio.htm>.

threshold (*the edge of chaos*). This entropy contains a parameter q which has been called *the entropic index* and it reduces to the usual Boltzmann–Gibbs–Shannon (BGS) entropy when $q = 1$. In Refs. [7–11] q was determined in two completely different ways: from the sensitivity to initial conditions and from the multifractal spectrum (using $1/(1-q) = 1/\alpha_{\min} - 1/\alpha_{\max}$, where α_{\min} and α_{\max} are respectively the lower and upper values where the multifractal function $f(\alpha)$ vanishes) [30–38].

The purpose of this Letter is to extend the work of [6] to a dissipative case and to focus especially on the edge of chaos. The results are: 1) In the chaotic regime the linear rate of the BGS entropy gives the KS entropy; 2) At the edge of chaos, the non-extensive entropy for one particular value $q \neq 1$ grows linearly with time, as did the usual entropy in [6]; 3) The value of q thus determined at the edge of chaos is identical with that found with the two other independent methods respectively used in Refs. [7,8] and [9–11].

The dissipative system chosen is the simplest possible: the logistic map, a nonlinear one-dimensional dynamical system described by the iterative rule [1]:

$$\begin{aligned} x_{t+1} &= 1 - ax_t^2 \\ (-1 \leq x_t \leq 1; 0 \leq a \leq 2; t = 0, 1, 2, \dots). \end{aligned} \quad (1)$$

It has chaotic behavior (with a positive Lyapunov exponent) for most of the values of the control parameter a above the critical value $a_c \equiv 1.40115519\dots$. This critical value marks the edge of chaos.

For convenience, we recall here the definition of the non-extensive entropy [17–19]. If the phase space \mathbb{R} has been divided into W cells of equal measure, and if the probability of being in cell i is p_i , we define the entropy S_q by

$$S_q \equiv \frac{1 - \sum_{i=1}^W p_i^q}{q-1} \quad (q \in \mathbb{R}). \quad (2)$$

For $q = 1$ this is $S_1 = -\sum_{i=1}^W p_i \ln p_i$, the usual entropy. This generalized entropy was proposed a decade ago [17–19] to allow statistical mechanics to

cover certain anomalies due to a possible (multi)-fractal structure of the relevant phase space (for example, whenever we have long-range interactions, long-range microscopic memory, multifractal boundary conditions, some dissipative process, etc). A review of the existing theoretical, experimental and computational evidence and connections is now available [20,21] (very recent verifications in fully developed turbulence and in electron-positron annihilation producing hadronic jets are exhibited in Ref. [22], see also Refs. [23,24] and [25] respectively). We also recall the two main results of Refs. [7–11]. The first [7,8] concerns the sensitivity to initial conditions. In a truly chaotic system, the separation between nearby trajectories, suitably averaged over phase space, $\xi(t)$ diverges in time like the exponential $\exp(\lambda_1 t)$, where $\lambda_1 > 0$ is the Lyapunov exponent. At the edge of chaos, on the other hand, the upper bound growth of the separation follows a power law which may be written

$$\xi(t) \propto [1 + (1-q)\lambda_q t]^{\frac{1}{1-q}} \quad (q \in \mathbb{R}) \quad (3)$$

in terms of a certain parameter q . The exponential is recovered in the limit $q = 1$. For the logistic map at the edge of chaos, the Lyapunov exponent λ_1 is found to vanish, but the growth of the separation is fitted well² by Eq. (3) with $q = q^* \equiv 0.2445\dots$. The second result concerns the geometrical description of the multifractal attractor existing at a_c . See Refs. [9–11] for details. This gives a different method for finding a special value of q which fits the results, and this value turns out to be again 0.2445.

The *power-law* sensitivity to the initial conditions has already been noticed in the literature [26–28]. We shall refer to it as *weak sensitivity*, as opposed to the *exponential* law which we call *strong sensitivity*. The weak case is characteristic of *the edge of chaos*. The conclusion which seems to emerge from our work is that the various manifestations of the edge of

² At the chaos threshold the separation has a very fluctuating behavior. The time series is a set of scattered points organized in a regular but complicated way. Eq. (3) describes the upper bound of this complicated structure, i.e. the quickest possible mixing. See details in Refs. [7,8].

chaos all contain in their description a certain parameter q^* , which has the same value for all of them; and that moreover this q^* is the entropic index which must be used (instead of the usual value 1) in a thermodynamic description of an edge-of-chaos system.

We shall now present the numerical work we have done, in which q^* is calculated in a third, completely different way, which involves rates of increase of entropies. We use the analysis developed in Ref. [6] for conservative systems. In order to do so we partition the interval $-1 \leq x \leq 1$ into W equal cells; we choose (randomly or not) one of them and we select (randomly or uniformly) N values of x (inside that cell) which will be considered as initial conditions for the logistic map at a given value of a . As t evolves, the N points spread within the $[-1, 1]$ interval in such a way that we have a set $\{N_i(t)\}$ with $\sum_{i=1}^W N_i(t) = N \forall t$, and a set of probabilities $\{p_i(t) \equiv N_i(t)/N\}$. Differently from Ref. [6], we consider here the general entropy (3), which for $q = 1$ reduces to the entropy used in [6].

At $t = 0$, all probabilities but one are zero, hence $S_q(0) = 0$. And, as t evolves, $S_q(t)$ tends to increase, in all cases bounded by $(W^{1-q} - 1)/(1-q)$ ($\ln W$ when $q = 1$), which corresponds to equiprobability. Fluctuations are of course present and can be reduced by considering averages over the initial conditions. As last step, we define the following rate of increase

$$\kappa_q \equiv \lim_{t \rightarrow \infty} \lim_{W \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{S_q(t)}{t} \quad (4)$$

where κ_1 , in the case of chaotic conservative systems, is expected to coincide with the standard KS entropy rate [6]. Our expectations, based on the work of Refs. [7,8,39], are:

- (i) A special value q^* exists such that $S_q(t)$ increases linearly with time. κ_q is then finite for $q = q^*$, vanishes for $q > q^*$ and diverges for $q < q^*$.
- (ii) When the system is *strongly sensitive* to initial conditions ($\lambda_1 > 0$), q^* is 1 and the results of Ref. [6] can be extended to *dissipative* systems.

- (iii) At the edge of chaos ($\lambda_1 = 0$, *weakly sensitive* systems), q^* is different from 1 and coincides with the value determined from the sensitivity to initial conditions (Eq. (3)) and from the multifractal spectrum.

The following results confirm these expectations.

In Fig. 1 we present the case $a = 2$, for which the system is chaotic (with $\lambda_1 = \ln 2$), and then we expect $q^* = 1$. We show the time evolution of S_q for three different values of q . Only the curves for $q = 1$ show a clear linear behavior before we reach the asymptotic constant value which characterizes the equilibrium distribution in the available part of phase space. The slope in the intermediate time stage does not depend on W and it is equal to the KS entropy rate $\ln 2$ (for any one-dimensional system the KS entropy is given by the positive Lyapunov exponent [29]). This is clear in Fig. 2 which shows $S_1(t)$ for two different cases $a = 2$ and $a = 1.6$. In both cases the fitted slope agrees with the predicted KS entropy rates, respectively $\ln 2$ and 0.3578.

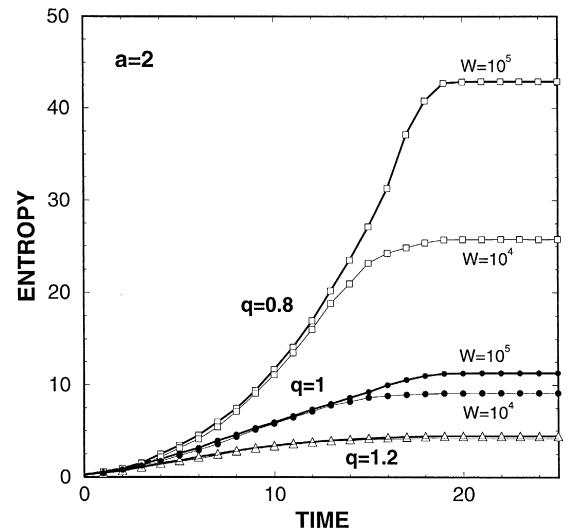


Fig. 1. Time evolution of S_q for $a = 2$. The interval $-1 \leq x \leq 1$ is partitioned into W equal cells. The initial distribution consists of $N = 10^6$ points placed at random inside a cell picked at random anywhere on the map. We consider three different values of q and the two cases $W = 10^4$ and $W = 10^5$. Results are averages over 100 runs.

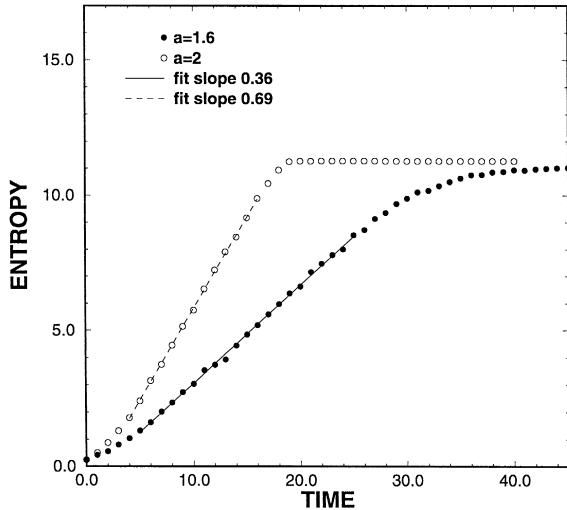


Fig. 2. Time evolution of S_1 for $a = 2$ and $a = 1.6$; $W = 10^5$ and averages of 100 runs. The slopes of the fits shown are respectively equal to the averaged Lyapunov exponents.

So far we have shown that q^* is 1 for all the cases in which the logistic curve is chaotic, i.e. strongly sensitive to the initial conditions. Now we want to study the same system at its chaos threshold $a = a_c = 1.40115519\dots$ for which we expect $q^* = q_c = 0.2445\dots$. For this value of a , considerable

fluctuations are observed which require an efficient and careful averaging over the initial conditions. Consider that the attractor occupies only a tiny part of phase space (844 cells out of the $W = 10^5$ of our partition). We adopt the following criterion: we choose the initial distribution (made of $N = 10^6$ points) in one of the $W = 10^5$ cells of the partition and we study the number of occupied cells during the time evolution; the integrated number of occupied cells, i.e., the sum of the numbers of occupied cells for all time steps from iteration 1 to iteration 50, is a measure of how good this initial condition is at spreading itself. We repeat the same study for each one of the $W/2$ cells in the interval $0 \leq x \leq 1$ and, in Fig. 3, we plot the integrated number of occupied cells versus the position of the initial distribution. The cells for which this number is larger than a fixed cutoff ($= 5000$ in Fig. 3) are selected for inclusion in the averaging process (in Fig. 3 this is 1251 cells, out of a total 10^5). In Fig. 4 we plot $S_q(t)$ for four different values of q ; the curves are an average over the 1251 initial cells selected by Fig. 3. The growth of $S_q(t)$ is found to be linear when $q = q_c = 0.2445$, while for $q < q_c$ ($q > q_c$) the curve is concave (convex). This behavior is similar to the one in Fig. 1, with a major difference: the linear

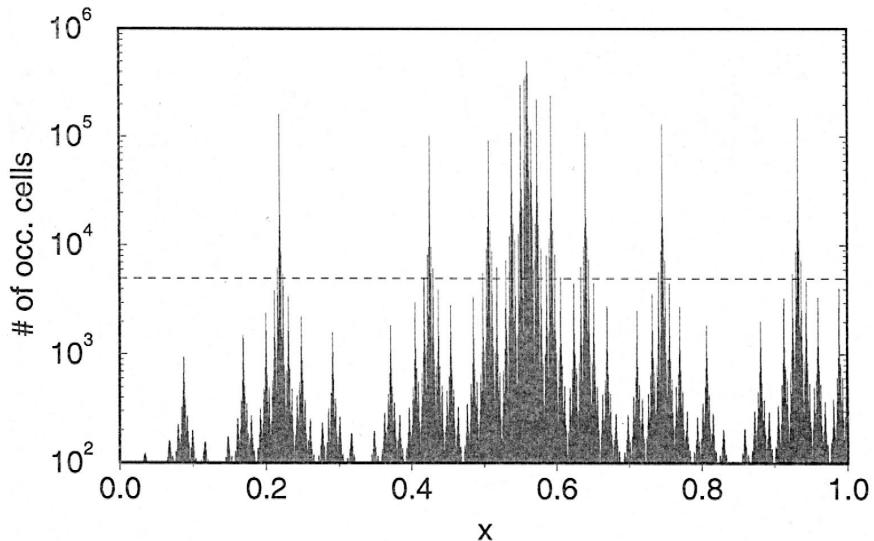


Fig. 3. Integrated number of occupied cells vs. position of the initial cell. The horizontal line selects the best initial conditions (see text). If we increase or decrease the value of this (numerically convenient, but dispensable) cut-off, the value for q^* remains the same; what changes is the proportionality coefficient between S_{q^*} and time.

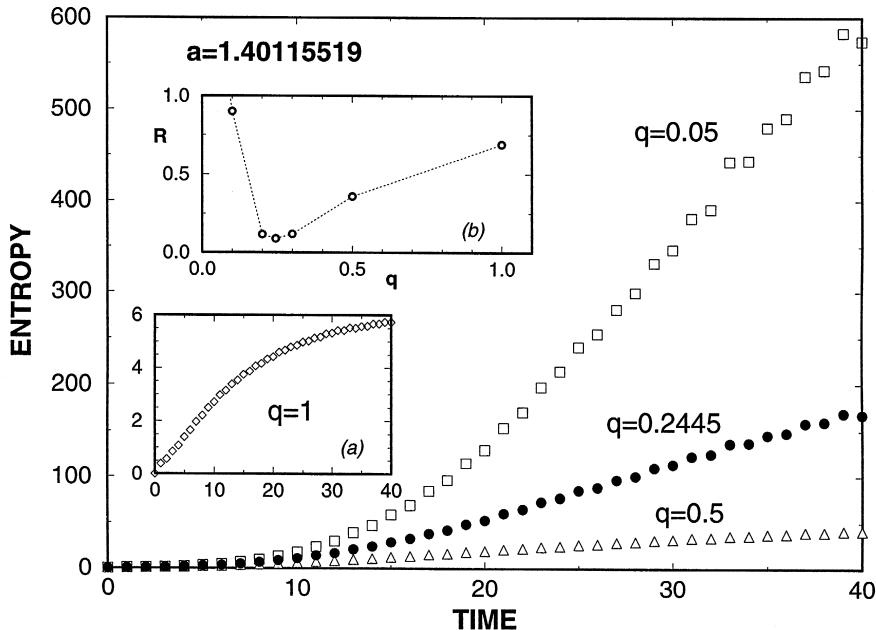


Fig. 4. Time evolution of S_q for $a = a_c$. We consider four different values of q and $W = 10^5$. The case $q = 1$ is reported in the inset (a) with a different scale. Results are averages over 1251 runs. We show the coefficient of nonlinearity R versus q in the inset (b). See text. The 4-digit precision for q^* was not attained through the present numerical procedure, but using the scaling $1/(1-q) = 1/\alpha_{\min} - 1/\alpha_{\max}$. The present procedure does not provide higher precision than $q^* = 0.24\dots$.

growth is not at $q = 1$ (see inset (a) in Fig. 4), but at a particular value of the entropic index, which happens to coincide with $q^* = 0.2445$. In order to make this result much more convincing, we fitted the curves $S_q(t)$ in the time interval $[t_1, t_2]$ with the polynomial $S(t) = a + bt + ct^2$.³ We define $R = |c| \cdot (t_1 + t_2)/b$ as a measure of the importance of the nonlinear term in the fit: if the points were on a perfect straight line, R should be zero. We choose $t_1 = 15$ and $t_2 = 38$ for all q 's, so that the factor $(t_1 + t_2)$ is just a normalizing constant. Fig. 4(b) shows the minimum of R for $q = q_c = 0.2445$. These results are not sensitive to small changes in t_1 and t_2 .

Precisely the same behavior, in all of its aspects, that we find here for the standard logistic map has

also been found recently [40] for its extended version where the term x_i^2 is generalized into $|x_i|^z$ ($z \in \mathcal{R}$). As expected from well known universality class considerations for this family of maps, it was found that q^* depends on z . The same values for $q^*(z)$ were also found [40] for a periodic family of maps which belongs to the same universality class as the logistic-like family. Finally, for the usual logistic map, the value $q^* = 0.24\dots$ was found through a different algorithm [41], closer in fact to the original definition of the Kolmogorov–Sinai entropy.

To summarize, we have illustrated for the logistic map the connections between the sensitivity to initial conditions, the geometrical support in phase space, and the linear growth in time of the entropy S_q . In the case of strong sensitivity (exponential divergence between trajectories), the geometrical support is Euclidean, and the relevant entropy with linear growth is S_1 , the usual entropy. In the case of weak sensitivity (power-law divergence), the geometrical support is multifractal, and the relevant entropy whose growth is linear is the non-extensive entropy S_q , with a

³ This method is more sensitive than the standard correlation coefficient of the linear regression analysis in determining the best linear slope among different values of q .

special value $q^* \neq 1$ of the entropic index. This q^* is the same parameter which enters also in the power-law divergence and in the multifractal support: the same q^* describes all three phenomena. Thus strong sensitivity and weak sensitivity to initial conditions become unified in a single description, the difference residing in the particular value of q ($= 1$ for the strong case). The KS entropy rate, which is an average loss of information, is also indexed by q . We believe that the scenario herein exhibited is valid for vast classes of nonlinear dynamical systems, whose full and rigorous characterization would be very welcome. We conclude, as a final remark, that the still unclear foundation of statistical mechanics on microscopic dynamics seems more than ever to follow along the lines pioneered by Krylov [42]. Indeed, the crucial concept appears to be the *mixing* (and not only ergodicity): if the mixing is *exponential* (*strong mixing*), then $q = 1$ and the standard thermodynamical extensivity is the adequate hypothesis for those physical phenomena, whereas at the edge of chaos the mixing is only *algebraic* (*weak mixing*) and then $q \neq 1$ and thermodynamical nonextensivity is expected to emerge.

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