Stochastic modeling of diffusion in dynamical systems: three examples

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Motivation: dynamical systems, diffusion and stochastic modeling

Diffusion in three random walk-like examples:
1. non-chaotic ‘slicer’ map
2. randomly perturbed dissipative standard map
3. a simple random dynamical system

Conclusion: successes, failures and pitfalls for these three examples when relating the above three layers to each other
Motivation

1. slicer
2. standard map
3. random dynamical system

Summary

Microscopic chaos in a glass of water?

- dispersion of a droplet of ink by diffusion
- chaotic collisions between billiard balls
- chaotic hypothesis:

\[ \text{microscopic chaos} \Downarrow \text{macroscopic diffusion} \]


P. Gaspard et al. (1998): experiment on small colloidal particle in water; diffusion due to microscopic chaos based on positive pattern entropy per unit time

\[ h(\varepsilon, \tau) \leq h_{KS} = \sum_{\lambda_i > 0} \lambda_i \]
The random wind tree model

counterexample:

Ehrenfest, Ehrenfest (1959)

no positive Lyapunov exponent, hence non-chaotic dynamics

Dettmann et al. (1999): generates trajectories and $h(\epsilon, \tau)$ indistinguishable from the colloidal particle dynamics
## Microscopic chaos, diffusion and stochastic modeling

**Conclusion:**
- **Theory:** (chaotic) model $\rightarrow$ diffusion
- **Experiment:** diffusion $\rightarrow$ (chaotic) model?

$\Rightarrow$ non-trivial interplay microscopic model $\leftrightarrow$ diffusion

**Theme of this talk:**
add yet a third layer of stochastic modeling

dynamical system $\xrightarrow{\text{generate}}$ diffusion

stochastic model

**Two questions:**
1. what *type of diffusion* is generated by a dynamical system?
2. can it be *reproduced* by some stochastic model?
in the following only **diffusion in one dimension**

key quantity: **mean square displacement**

\[ < x^2 > := \int dx \ x^2 \rho(x, t) \sim t^\gamma \]

**note:** three basic types of diffusion

1. there is not only ‘Brownian’ (normal) diffusion with \( \gamma = 1 \)
   but also anomalous diffusion:
   2. subdiffusion with \( \gamma < 1 \)
      and
   3. superdiffusion with \( \gamma > 1 \)

(plus more exotic types)
I. The slicer map
Motivation: diffusion in polygonal billiards

Zaslavsky et al. (2001), Jepps et al. (2006)

- **zero Lyapunov exponent**: different points separate *linearly* but not *exponentially* in time, hence non-chaotic dynamics
- **mean square displacement** from simulations: sub-, super- or normal diffusion depending on parameters, with partially conflicting results *(Alonso / Jepps / Sanders et al., 2002ff)*
Pictorial construction

a one-dimensional ‘random walk-like’ but fully deterministic system; diffusion of a density of points from uniform initial density in space \((m)\) - discrete time \((n)\) diagram:

‘slicers’ at points (of Lebesgue measure zero) split the density; no stretching, hence zero Lyapunov exponent: no chaos!
Formal definition

- consider a chain of intervals $\hat{M} := M \times \mathbb{Z}$, $M := [0, 1]$ with point $\hat{X} = (x, m)$ in $\hat{M}$, where $\hat{M}_m := M \times \{m\}$ is the $m$-th cell of $\hat{M}$

- subdivide each $\hat{M}_m$ in subintervals, separated by points called slicers: $\{1/2\} \times \{m\}$, $\{\ell_m\} \times \{m\}$, $\{1 - \ell_m\} \times \{m\}$, where $0 < \ell_m < 1/2$ for every $m \in \mathbb{Z}$ with
  
  \[
  \text{power law } \ell_m(\alpha) = \frac{1}{(|m|+2^{1/\alpha})^\alpha}, \alpha > 0
  \]

- slicer map: $S : \hat{M} \rightarrow \hat{M}$, $\hat{X}_{n+1} = S(\hat{X}_n)$, $n \in \mathbb{N}$ with
  
  $S(x, m) = \begin{cases} 
  (x, m - 1) & \text{if } 0 \leq x < \ell_m \text{ or } \frac{1}{2} < x \leq 1 - \ell_m, \\
  (x, m + 1) & \text{if } \ell_m \leq x \leq \frac{1}{2} \text{ or } 1 - \ell_m < x \leq 1.
  \end{cases}$

$\Rightarrow$ interval exchange transformation lifted onto the real line
Main result: diffusive properties

**Proposition:** Salari et al., 2015

Given $\alpha \geq 0$ and a uniform initial distribution in $\hat{M}_0$, we have

1. $\alpha = 0$: ballistic motion with MSD $\langle \hat{X}_n^2 \rangle \sim n^2$
2. $0 < \alpha < 1$: superdiffusion with MSD $\langle \hat{X}_n^2 \rangle \sim n^{2-\alpha}$
3. $\alpha = 1$: normal diffusion with linear MSD $\langle \hat{X}_n^2 \rangle \sim n$
   non-chaotic normal diffusion with non-Gaussian density
4. $1 < \alpha < 2$: subdiffusion with MSD $\langle \hat{X}_n^2 \rangle \sim n^{2-\alpha}$
   subdiffusion with ballistic peaks
5. $\alpha = 2$: logarithmic subdiffusion with MSD $\langle \hat{X}_n^2 \rangle \sim \log n$
   a bit exotic
6. $\alpha > 2$: localisation in the MSD with $\langle \hat{X}_n^2 \rangle \sim \text{const.}$
   non-trivial phenomenon
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Summary

Higher order moments

**Theorem:** Salari et al., 2015

For $\alpha \in (0, 2]$ the moments $\langle \hat{X}_n^p \rangle$ with $p > 2$ even and uniform initial distribution in $\hat{M}_0$ have the asymptotic behavior

$$\langle \hat{X}_n^p \rangle \sim n^{p-\alpha}$$

while the odd moments ($p = 1, 3, ...$) vanish.
Matching to stochastic dynamics?

- one-dimensional stochastic Lévy Lorentz gas:

  point particle moves ballistically between static point scatterers on a line from which it is transmitted / reflected with probability $\frac{1}{2}$

  distance $r$ between two scatterers is a random variable iid from the Lévy distribution

  $$\lambda(r) := \beta r_0^\beta \frac{1}{r^{\beta+1}}, \ r \in [r_0, \infty), \ \beta > 0$$

  with cutoff $r_0$

  $\rightarrow$ model exhibits only superdiffusion

  $\rightarrow$ all moments scale with the slicer moments for $\alpha \in (0, 1]$ (piecewise linearly depending on parameters)
Matching to stochastic dynamics?

- Lévy walk modeled by CTRW theory:
  - \( \text{moments calculated to } \sim t^{p+1-\beta} \) for \( p > \beta, \ 1 < \beta < 2 \):
  - match to slicer superdiffusion with \( \beta = 1 + \alpha \)
  - but conceptually a totally different process

- correlated Gaussian stochastic processes:
  - modeled by a generalized Langevin equation with a power law memory kernel
  - formal analogy in the subdiffusive regime
  - but Gaussian distribution and a conceptual mismatch
  - slicer might help to explain a controversy about different stochastic models for diffusion in polygonal billiards
II. The dissipative randomly perturbed standard map
The standard map and diffusion

- paradigmatic Hamiltonian dynamical system:

  **standard map**

  \[
  \begin{align*}
  x_{n+1} &= x_n + y_n \mod 2\pi \\
  y_{n+1} &= y_n + K \sin x_{n+1}
  \end{align*}
  \]

  derived from *kicked rot(at)or* where \( x_n \in \mathbb{R} \) is an angle, \( y_n \in \mathbb{R} \) the angular velocity with \( n \in \mathbb{N} \) and \( K > 0 \) the kick strength.

- define **diffusion coefficient** as

  \[
  D(K) = \lim_{n \to \infty} \frac{1}{n} \langle (y_n - y_0)^2 \rangle
  \]

  with ensemble average over the initial density

  \[
  \langle \ldots \rangle = \int dx \, dy \, \rho(x, y) \ldots , \ x \in [0, 2\pi) , \ y = y_0 \in [0, 2\pi)
  \]
Diffusion in the standard map

analytical (Rechester, White, 1980) and numerical studies of parameter-dependent diffusion $D_{\text{eff}}(K)$:

$D_{\text{eff}}(K)$ is highly irregular

for some $K$ there is superdiffusion with mean square displacement $\langle y_n^2 \rangle \sim n^\gamma$, $\gamma > 1$ due to accelerator modes

Manos, Robnik, PRE (2014)
The dissipative standard map

model damping in the standard map by

\[ x_{n+1} = x_n + y_n \mod 2\pi \]
\[ y_{n+1} = (1 - \nu)y_n + f_0 \sin x_{n+1} \]

with \( \nu \in [0, 1] \):

Feudel, Grebogi, Hunt, Yorke, PRE (1996)

- islands in phase space for \( \nu = 0 \) (left) become coexisting periodic attractors (right): 150 found for \( \nu = 0.02 \), \( f_0 = 4 \)
- simple argument yields \( |y_n| < y_{\text{max}} \): quick trapping
**Question:** What happens to dissipative deterministic dynamics \( \mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) \) under random perturbations?

Consider the dissipative standard map with additive noise:

\[
\begin{align*}
\mathbf{x}_{n+1} &= \mathbf{x}_n + \mathbf{y}_n + \epsilon_{x,n} \mod 2\pi \\
\mathbf{y}_{n+1} &= (1 - \nu)\mathbf{y}_n + f_0 \sin \mathbf{x}_{n+1} + \epsilon_{y,n}
\end{align*}
\]

with iid random variables \( \epsilon_n = (\epsilon_{x,n}, \epsilon_{y,n}) \) drawn from uniform distribution bounded by \( ||\epsilon_n|| < \xi \) of noise amplitude \( \xi \)

- beyond a noise threshold \( \xi \geq \xi_0 \) the noise induces a hopping process between all coexisting pseudo attractors
- the resulting dynamics is intermittent because of stickiness to pseudo attractors
Continuous time random walk theory

match simulation results to **CTRW theory** (Montroll, Weiss, Scher, 1973): define stochastic process by master equation with *waiting time distribution* $w(t)$ and *jump distribution* $\lambda(x)$

$$\varrho(x, t) = \int_{-\infty}^{\infty} dx' \lambda(x - x') \int_0^t dt' w(t - t') \varrho(x', t') +$$

$$+(1 - \int_0^t dt' w(t')) \delta(x)$$

*structure:* jump + no jump for points starting at $(x, t) = (0, 0)$

Fourier-\(\tilde{L}\)aplace transform yields **Montroll-Weiss eqn (1965)**

$$\hat{\varrho}(k, s) = \frac{1 - \tilde{W}(s)}{s} \frac{1}{1 - \hat{\lambda}(k) \tilde{w}(s)}$$

with mean square displacement

$$\langle x^2(s) \rangle = -\left. \frac{\partial^2 \hat{\varrho}(k, s)}{\partial k^2} \right|_{k=0}$$
Predictions of CTRW theory according to CTRW theory solving the MW eqn. for

1. a power law waiting time distribution \( w(t) \sim t^{-(\gamma+1)} \)

with jump distribution \( \lambda(x) = \delta(|x| - \text{const.}) \)

2. yields a mean square displacement of \( <x^2(t)> \sim t^\gamma \)

and

3. a stretched exponential position pdf, approximately given by \( P_n(y) \sim \exp\left(-cx^2/(2-\gamma)\right) \)

crucial fit parameter: \( \gamma \); check these three predictions in numerical experiments
**CTRW theory and mean square displacement**

$< y^2(n) >$ for different noise amplitudes $\xi$ at $\nu = 0.002$:

- Transient subdiffusion $< y^2(n) > \sim n^\gamma$ up to $n < 1000$
- Only small variation of $0.85 < \gamma < 0.95$ for different $\xi$; for $\xi = 0.06$ we have $\gamma \approx 0.95$
probability distributions $P(t)$ of escape times $t$ from pseudo attractors; dissipation $\nu = 0.002$ with different noise strength $\xi$:

- transition from power law (stickiness) to exponential
- transition takes longer when $\xi \to 0$
- the dashed red line represents the CTRW theory prediction of $P(t) \sim t^{-1.95}$ corresponding to $< y^2(n) > \sim n^{0.95}$
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**Summary**

CTRW theory and position pdf

\[ P_n(y) \] for position \( y \) at different time steps \( n \):

![Graph showing CTRW theory and position pdf](image)

- \( \xi = 0.06, \nu = 0.002 \)
- ‘Gaussian-like’ diffusive spreading up to \( n < 1000 \)
- Localization trivially due to boundedness of pseudo attractors
- CTRW theory pdf (green lines) for \( \gamma = 0.95 \) corrects mismatch in tails
III. A random dynamical system
Constructing a random dynamical system

three time series for position \( x_t \) of a particle at discrete time \( t \):

- **upper left:** deterministic dynamical system \( D \) yielding normal diffusion
- **upper right:** deterministic dynamical system \( L \) where all particles localize in space.
- **bottom:** random dynamical system \( R \) that mixes these two types of dynamics at time \( t \) with probability \( p \); the result is intermittent motion
Our model

**equation of motion**

\[ x_{t+1} = M_a(x_t) \] with discrete time

\[ t \in \mathbb{N}_0, \ a > 0 \] and

**one-dimensional piecewise linear map**

\[ M_a(x) = \begin{cases} 
ax, & 0 \leq x < \frac{1}{2} \\
ax + 1 - a, & \frac{1}{2} \leq x < 1 
\end{cases} \]

**lift** \( M_a(x + 1) = M_a(x) + 1 \);

**Lyapunov exponent** \( \lambda(a) = \ln a \)

**random map** \( R = M_a(x) \): at any \( t \) choose \( a \) iid with probability

\( p \in [0,1] \) from \( a = 1/2 \) and with \( 1 - p \) from \( a = 4 \)
Diffusion in a simple random dynamical system

- **left**: $\langle x_t^2 \rangle$ for $p = 0.6, \ldots, 0.7$ (top to bottom); subdiffusion with zero Lyapunov exponent at $p_c = 2/3$

- **right**: $\langle x_t^2 \rangle$ at $p_c$ with *same* random sequence for each particle (colors), cp. to *different* random sequence (black); MSD is a random variable breaking self-averaging and ergodicity
Ageing and weak ergodicity breaking

- **left:** $\langle x_t^2 \rangle$ at $p_c$ by starting the computations after different ageing times $t_a = 0, 10^2, 10^3, 10^4$ (top to bottom) displays ageing, cp. to CTRW theory (Barkai, 2003; bold lines)
- **right:** corresponding waiting time distribution $\eta(t)$ (for particles leaving a unit cell at $t_a$), again matching to CTRW theory
- both results imply weak ergodicity breaking (Bouchaud, 1992)
Connection with dynamical systems theory

- mixing ‘expanding’/chaotic with contracting/non-chaotic dynamics randomly in time generates intermittent motion

- the underlying microscopic mechanism is called on-off intermittency (Pikovsky (1984), Fujisaka et al. (1985)); transition called blowout bifurcation (Ott et al. (1994))
central theme: interplay between dynamical systems, diffusion and stochastic modeling

main results:
1. dynamical systems can feature novel types of (anomalous) diffusion
2. naive matching to stochastic models can be misleading and difficult

outlook: perhaps dynamical systems theory can inspire stochastic theory to invent new stochastic processes?
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- **random dynamical system**: Y. Sato, RK, arXiv:1810.02674; resubmitted to PRL