Weak chaos, infinite ergodic theory, and anomalous diffusion

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Motivation

(Deadline for applications has passed - sorry)

This talk: focus on anomalous diffusion
setting the scene: **chaos** in a simple map

from chaos to **weak chaos** and to **infinite ergodic theory**

**anomalous diffusion** in a simple map and **continuous time random walk theory**

**fractional diffusion equations** and anomalous biological cell migration
**Bernoulli shift and Ljapunov exponent**

**warmup:** deterministic chaos modeled by a simple 1d map

Bernoulli shift $M(x) = 2x \mod 1$ with $x_{n+1} = M(x_n)$:

![Bernoulli shift diagram](image)

apply small perturbation $\Delta x_0 := \tilde{x}_0 - x_0 \ll 1$ and iterate:

$$\Delta x_n = 2\Delta x_{n-1} = 2^n \Delta x_0 = e^{n \ln 2} \Delta x_0$$

$\Rightarrow$ exponential dynamical instability with Ljapunov exponent

$\lambda := \ln 2 > 0$: Ljapunov chaos
Ljapunov exponent and ergodicity

local definition for one-dimensional maps via *time average*:

$$\lambda(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |M'(x_i)|, \ x = x_0$$

if map is *ergodic*: time average = ensemble average,

$$\lambda = \langle \ln |M'(x)| \rangle_\mu, \text{ cf. Birkhoff’s theorem}$$

with $\langle \ldots \rangle_\mu = \int_I dx \varrho(x) \ldots$ average over the *invariant* probability density $\varrho(x)$ related to the map’s SRB measure via

$$\mu(A) = \int_A dx \varrho(x), \ A \subseteq I$$

Bernoulli shift is *expanding*: $\forall x |M'(x)| > 1$, hence ‘hyperbolic’

*normalizable* pdf exists, here simply $\varrho(x) = 1 \Rightarrow \lambda = \ln 2$
Theorem

For closed $C^2$ Anosov systems the sum of positive Lyapunov exponents is equal to the Kolmogorov-Sinai entropy.

*Pesin (1976), Ledrappier, Young (1984)*

(believed to hold for a wider class of systems)

for one-dimensional hyperbolic maps: $\lambda = h_{ks}$ with

$$h_{ks} := \lim_{n \to \infty} -\frac{1}{n} \sum_{w \in \{W_i^n\}} \mu(w) \ln \mu(w)$$

(if the partition is generating), where $\mu(w)$ is the SRB measure of an element $w$ of the partition $\{W_i^n\}$ and $n$ defines the level of refinement

$h_{ks} > 0$: measure-theoretic chaos
Anomalous dynamics

consider the nonlinear Pomeau-Manneville map

\[ x_{n+1} = M(x_n) = x_n + x_n^z \mod 1, \, z \geq 1 \]

phenomenology of intermittency: long periodic laminar phases interrupted by chaotic bursts; here due to an indifferent fixed point, \( M'(0) = 1 \) (Pomeau, Manneville, 1980)

⇒ map not hyperbolic (\( \not\exists N > 0 \) s.t. \( \forall x \forall n \geq N \mid (M^n)'(x) \mid \neq 1 \)
Infinite invariant measure and dynamical instability

- **invariant density** of this map calculated to

\[ \varrho(x) \sim x^{1-z} \quad (x \rightarrow 0) \]

Thaler (1983)

is non-normalizable for \( z \geq 2 \) yielding an infinite invariant measure \( \rightarrow \) infinite ergodic theory!

\[ \mu(x) = \int_x^1 dy \varrho(y) \rightarrow \infty \quad (x \rightarrow 0) \]

- **dispersion of nearby trajectories** calculated to

\[ \Delta x_n \sim \exp \left( n^{\frac{1}{z-1}} \right) \Delta x_0 \quad (z > 2) \]


grows weaker than exponential yielding \( \lambda = 0: \) weak chaos

Zaslavsky, Usikov (2001)
“weak ergodicity breaking:” choose a ‘nice’ observable \( f(x) \)
• for \( 1 \leq z < 2 \) it is \( \sum_{i=0}^{n-1} f(x_i) \sim n \ (n \to \infty) \)

**Birkhoff’s theorem:** if \( M \) is ergodic then \( \frac{1}{n} \sum_{i=0}^{n-1} f(x_i) = <f>_{\mu} \)
• but for \( 2 \leq z \) we have the **Aaronson-Darling-Kac theorem**, 

\[
\frac{1}{a_n} \sum_{i=0}^{n-1} f(x_i) \overset{d}{\to} \mathcal{M}_\alpha <f>_{\mu} \ (n \to \infty)
\]

\( \mathcal{M}_\alpha \): random variable with normalized **Mittag-Leffler** pdf
for \( M \) it is \( a_n \sim n^\alpha \), \( \alpha := 1/(z - 1) \); integration wrt to Lebesgue measure \( m \) suggests

\[
\frac{1}{n^\alpha} \sum_{i=0}^{n-1} <f(x_i)>_m \sim <f(x)>_{\mu}
\]

**note:** for \( z < 2 \Rightarrow \alpha = 1 \ \exists \) absolutely continuous invariant measure \( \mu \), and we have an equality; for \( z \geq 2 \ \exists \) infinite invariant measure, and it remains a **proportionality**
This motivates to define a **generalized Ljapunov exponent**

\[
\Lambda(M) := \lim_{n \to \infty} \frac{\Gamma(1 + \alpha)}{n^\alpha} \sum_{i=0}^{n-1} < \ln |M'(x_i)| >_m
\]

and analogously a **generalized KS entropy**

\[
h_{KS}(M) := \lim_{n \to \infty} -\frac{\Gamma(1 + \alpha)}{n^\alpha} \sum_{w \in \{W_i^n\}} \mu(w) \ln \mu(w)
\]

**new result:** Howard, RK, tbp

Shown analytically for a piecewise linearization of \( M \) that

\[
h_{KS}(M) = \Lambda(M)
\]

cf. **Rokhlin’s formula**, generalizing **Pesin’s theorem** to anomalous dynamics; see also **Korabel, Barkai (2009)**
An intermittent map with anomalous diffusion

continue map \( M(x) = x + ax^2, \ 0 \leq x \leq 1/2, \ a \geq 1 \) by \( M(-x) = -M(x) \) and \( M(x + 1) = M(x) + 1 \)

Geisel, Thomae (1984); Zumofen, Klafter (1993)

deterministic random walk on the line; classify diffusion in terms of the mean square displacement

\[
\langle x^2 \rangle = Kn^\alpha (n \to \infty)
\]

if \( \alpha \neq 1 \) one speaks of anomalous diffusion; here subdiffusion:

\[
\alpha = \begin{cases} 
1, & 1 \leq z \leq 2 \\
\frac{1}{z-1} < 1, & 2 < z
\end{cases}
\]

goal: calculate generalized diffusion coefficient \( K = K(z, a) \)
Montroll, Weiss, Scher, 1973:

master equation for a stochastic process defined by *waiting time distribution* $w(t)$ and *distribution of jumps* $\lambda(x)$:

$$
\varrho(x, t) = \int_{-\infty}^{\infty} dx' \lambda(x - x') \int_0^t dt' \ w(t - t') \ \varrho(x', t') +
+(1 - \int_0^t dt' \ w(t')) \ \delta(x)
$$

*structure*: jump + no jump for particle starting at $(x, t) = (0, 0)$

Fourier-Laplace transform yields **Montroll-Weiss eqn (1965)**

$$
\hat{\varrho}(k, s) = \frac{1 - \tilde{w}(s)}{s} \frac{1}{1 - \hat{\lambda}(k) \tilde{w}(s)}
$$

with mean square displacement $\langle x^2(s) \rangle = -\left. \frac{\partial^2 \hat{\varrho}(k, s)}{\partial k^2} \right|_{k=0}$
apply CTRW to maps: need $w(t), \lambda(x)$ (Klafter, Geisel, 1984ff)

**sketch:**
- $w(t)$ calculated from $w(t) \sim \varrho(x_0) \left| \frac{dx_0}{dt} \right|$ with density of initial positions $\varrho(x_0) \sim 1$, $x_0 = x(0)$; for waiting times $t(x_0)$ solve the continuous-time approximation of the PM-map $x_{n+1} - x_n \sim \frac{dx}{dt} = ax^z$, $x \ll 1$ with $x(t) = 1$
- revised (Korabel, RK et al., 2007) ansatz for jumps: $\lambda(x) = \frac{p}{2} \delta(|x| - \ell) + (1 - p) \delta(x)$ with average jump length $\ell$ and escape probability $p$

plug results into MW eqn: CTRW machinery ... yields ...
Transition from normal to anomalous diffusion

\[
\langle x^2 \rangle \sim \begin{cases} 
\frac{n}{\ln n}, & n < n_{cr} \text{ and } z < 2 \\
\frac{n}{\ln n}, & z = 2 \\
\frac{n^\alpha}{\ln n}, & n < \tilde{n}_{cr} \text{ and } z > 2 \\
\frac{n^\alpha}{\ln n}, & n > \tilde{n}_{cr} \text{ and } z > 2 
\end{cases}
\]

∃ suppression of diffusion due to logarithmic corrections compare CTRW approximation to simulations for \( K(z, 5) \):

Korabel, RK et al. (2007)

note: ∃ fractal structure → details see poster by G.Knight
Time-fractional equation for subdiffusion

For the lifted \textbf{PM map} \( M(x) = x + ax^2 \mod 1 \), the MW equation in long-time and large-space asymptotic form reads

\[
\mathcal{s}^\gamma \hat{\rho} - \mathcal{s}^{\gamma-1} = -\frac{p\ell^2 a^\gamma}{2\Gamma(1-\gamma)\gamma}\hat{\kappa}^2 \hat{\rho}, \quad \gamma := \frac{1}{(z-1)}
\]

LHS is the Laplace transform of the \textbf{Caputo fractional derivative}

\[
\frac{\partial^{\gamma} \hat{\rho}}{\partial t^{\gamma}} := \begin{cases} \frac{\partial \rho}{\partial t} & \gamma = 1 \\ \frac{1}{\Gamma(1-\gamma)} \int_0^t dt' (t - t')^{-\gamma} \frac{\partial \rho}{\partial t'} & 0 < \gamma < 1 \end{cases}
\]

transforming the Montroll-Weiss eq. back to real space yields the \textbf{time-fractional (sub)diffusion equation}

\[
\frac{\partial^{\gamma} \rho(x, t)}{\partial t^{\gamma}} = K \frac{\Gamma(1+\alpha)}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2}
\]
Deterministic vs. stochastic density

initial value problem for fractional diffusion equation can be solved exactly; compare with simulation results for $P = \varrho_n(x)$:

- **fine structure** due to density on the unit interval
  $r = \varrho_n(x)$ ($n \gg 1$) (see inset)
- Gaussian and non-Gaussian envelopes (blue) reflect intermittency (Korabel, RK et al., 2007)
Anomalous dynamics in biological cell migration

**outlook:** single biological cell crawling on a substrate; trajectory recorded with a video camera

Dieterich, RK et al., PNAS (2008)
pdf $P(x, t)$ of cell positions $x$ (in 1d) at time $t$ from experiment

green lines: Gaussians; red lines: solution of the fractional (super)diffusion equation (Schneider, Wyss, 1989)

$$\frac{\partial P(x, t)}{\partial t} = _0D_t^{1-\alpha}K_\alpha \frac{\partial^2}{\partial x^2} P(x, t)$$

with generalized diffusion coefficient $K_\alpha$ and Riemann-Liouville fractional derivative $_0D_t^{1-\alpha}$; for more evidence that this works see Dieterich, RK et al., PNAS (2008)
Summary

1. **weak chaos quantities** inspired by infinite ergodic theory
2. simple **weakly chaotic map** exhibiting complex subdiffusion
3. **continuous time random walk theory** and **fractional diffusion equations**
4. cross-link to **anomalous biological cell migration**
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literature:

for cell migration: Dieterich et al., PNAS 105, 459 (2008)