Irregular diffusion in the bouncing ball billiard

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2 Frequency locking, diffusion and correlated random walks
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The bouncing ball: experiments

Pieranski (1983ff)
Tufillaro (1986ff)
“Jugend forscht” (2004?)

Luck, Mehta (1993): “chattering” bifurcations into chaotic motion?
Linz (2003?)
The bouncing ball: ‘theory’

**linear stability analysis** of the exact (implicit) equations of motion yields **frequency locking** regions (“tongues”):

- **high bounce approximation:**
  - for displacement amplitude $A \ll y_{\text{max}}$ ball’s max. height
  - eom’s become

$$
\begin{align*}
\theta_{k+1} &= \theta_k + v_k \\
v_{k+1} &= \alpha v_k + \gamma \cos \theta_{k+1}
\end{align*}
$$

- dissipative standard map
  - with $\theta_k$: phase of the table; $v_k$: ball velocity at the $k$th collision
  - and $\gamma = 2\omega^2(1 + \alpha)A/g$

**Tufillaro (1986ff)**

- cp. with driven pendulum and Fermi acceleration

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Hongler et al. (1989)

Luck, Mehta (1993)
The bouncing ball billiard

study gas of granular particles on vibrating surface coated with periodic scatterers:

motivated our one dimensional bouncing ball billiard:

at collision: two friction coefficients $\alpha$ perpendicular and $\beta$ tangential to the surface

Q: $\exists$ frequency locking in diffusion?

Farkas et al. (1999)
Urbach et al. (2002)
**Frequency locking and diffusion**

**parameters:** scatterer radius $R = 25\text{mm}$, amplitude $A = 0.1\text{mm}$, restitution $\alpha = 0.5$, $\beta = 0.99$

**diffusion coefficient** $D(f)$ from MD computer simulations:

- highly irregular $D(f)$, no monotonicity
- frequency locking $\leftrightarrow$ largest maxima of $D(f)$
The bouncing ball billiard

Irregular diffusion

Spiral modes

Summary

Numerical analysis of the dynamics: resonance

∃ two types of attractors; projections at collisions:

attractor 1:

- ∃ 1/1-resonance \(\text{vertically}\), irregular motion \(\text{horizontally}\)
- traces of harmonic oscillator \(\text{separatrix}\)
- fan-shaped structure by \(\text{chaotic}\) scatterers
  \(\Rightarrow\) defines regime (b)(ii)
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The bouncing ball billiard

Introduction

Summary

Irregular diffusion

Spiral modes

- 1/1
- 2/1
- 3/1

(b)(i) (b)(ii)

(b)(i) (b)(ii) (b)(i) (c) (d)

\[ D \]

\[ f \]
Numerical analysis of the dynamics: creeps

attractor 2:

• non-resonant irregular motion in \( x \) and \( y \)
• long creeps: sequences of correlated tiny jumps along the surface: regime (c)

both types of dynamics can be linked to each other ergodically (d) or exist on different attractors non-ergodically (b)(i)
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Simple random walk approximation

Diffusion as a random walk on the line:

\[ D_{\text{rw}}(f) = \frac{d^2}{2\tau(f)} \]

Distance \( d \) between wedges and escape time \( \tau \) out of wedge \( D_{\text{rw}}(f) \) for \( \tau \) numerically:

\[ \tau \simeq \frac{d}{<v_x>} \simeq \frac{d}{\sqrt{2E_x}} \] links \( D_{\text{rw}}(f) \) to kinetic energy \( E_x(f) \)

dotted line: energy balance

\[ E = E_x + E_y + E_{\text{pot}} \]

with

\[ E_{\text{pot}} \simeq g\bar{y} \simeq gA, \quad E \simeq A^2\omega^2/2 \quad \text{and} \quad E_y \simeq 19E_x \]

leads to

\[ D_{\text{stoch}}(f) \simeq \frac{d}{2}\sqrt{2E_x} \simeq \frac{d}{2}\sqrt{\frac{A^2\omega^2}{20}} - \frac{gA}{10} \]
Correlated random walk approximation

diffusion via Taylor-Green-Kubo formula:

\[ D(f) = \frac{d^2}{2\tau} + \frac{1}{\tau} \sum_{k=1}^{\infty} < h(x_0) \cdot h(x_k) > \]

with lattice vectors \( h(x_k) = \pm d \) and equilibrium ensemble average \(< \ldots >\) (R.K., Korabel, 2002)

truncate series and express it by conditional probabilities

\[ D_n(f) = \frac{d^2}{2\tau} + \frac{1}{\tau} \sum_{s_1\ldots s_n} p(s_1 s_2 \ldots) h \cdot h(s_1 s_2 \ldots) \]

examples: 1st order approximation by forward- and backward scattering: \( D_1 = D_0 + 2D_0(p_f - p_b) = D_0 + 2D_0(1 - 2p_b) \)

2nd order approximation: \( D_2 = D_1 + 2D_0(p_{ff} - p_{fb} + p_{bf} - p_{bb}) \)
Understanding correlations in deterministic diffusion

compute probabilities numerically and check convergence of higher-order terms to $D(f)$:

→ irregularities on fine scales are real and due to dynamical correlations

Hamiltonian billiard without vibrations and friction:

Harayama, Gaspard (2001) fractal diffusion coefficient in energy $E$, cp. to yesterday’s talk
Irregular diffusion for other parameters

2nd set of parameters closer to experiments: \( R = 15 \text{mm}, \ A = 0.1 \text{mm}, \ \alpha = 0.7 , \ \beta = 0.99 \)

\( D(f) \) from simulations:

- highly irregular diffusion coefficient, but very different from previous one

projections of velocities \( v_y^+ \) around \( y = 0 \):

- local extrema \( \leftrightarrow \) frequency locking?
- cp. ‘bifurcations’ \( \leftrightarrow \) local extrema!
projections of orbits onto the \((y, v_y^+)-plane:\)

(A) **onset of diffusion:**
particles oscillate harmonically with the surface

(B) **onset of 1/1-resonance:**
enhancement of diffusion; coexistence with creeping orbits
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(C) **destruction of 1/1-resonance:** existence of a local minimum in the diffusion coefficient

(D) **new type of resonance:** a virtual harmonic oscillator mode (VHO) is forming; explains the second peak in $D(f)$; unstable around $f \approx 62$
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50 55 60 65 70 75 80 85 90 95

D

0 500 1000 1500 2000 2500 3000

50 55 60 65 70 75 80 85 90 95

f
Spiral modes and diffusion

(E) the VHO spirals out: further enhancement of diffusion

(F) two-loop spiral
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Spiral modes and diffusion 4

(G) onset of a third loop around $f \approx 76$: explains third local maximum

(H) onset of a fourth loop: related to fourth local maximum
note: diffusion coefficient is also irregular with respect to other control parameters $\alpha, \beta, R$
Spiral modes quantitatively

**frequency locking condition:** $k := \frac{T_p}{T_f} = 2\nu_y^+ f / g$ with $T_p$
particle time of flight and $T_f$ period of vibration

**numerical finding:** $D(f)$ has local maxima with complete VHO loops at half-integer $k$

**spiral equation:** assume flat surface and no correlations between collisions; from eom’s (Luck, Mehta, 1993):

$$y = -A \sin(2\pi ft_1), \quad \nu_y = \alpha g / 2(t_1 - t_0) - A2\pi f(1 + \alpha) \cos(2\pi ft_1)$$
with particle launched at time $t_0$ and first collision at $t_1$, cp. with simulations for $f = 72, 78$:
Outlook: Vibro-transporters

for agricultural material, also corrugated (Persson et al. (1992))

current generated by symmetry breaking;

current reversals under asymmetric vibrations:

reproduced in simulations with frequency locking (Elhor, Linz, 2003); cf. also Hongler et al. (1989); Han, Lee (2001)
Summary

- **bouncing ball billiard** models diffusion of a granular particle on a vibrating corrugated floor

- computer simulations show a **highly irregular frequency-dependent diffusion coefficient**; main impact by frequency locking and spiral modes

- **highly correlated nonlinear dynamics** yields further irregularities on fine scales, understood by correlated random walk approximations

- **fractal transport coefficients** in experiments?

**References:**
