1. *The harmonic oscillator*

Consider the differential equation

\[ \ddot{x} + \omega^2 x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \]

where \( \omega > 0 \) is a parameter.

(a) Solve this differential equation for the initial conditions \( x(0) = 0, \dot{x}(0) = v_0 \).

(b) Depict your solution graphically by drawing trajectories in the phase space of the system for different values of \( v_0 \) (this is called a *phase portrait*).

(c) Let us assume that ‘chaos’ is a subset of complicated dynamics. According to the Poincaré-Bendixson theorem, is chaos possible in this dynamical system?

2. *The Poincaré-Bendixson theorem again*

(a) Is chaos possible in the chain of coupled harmonic oscillators defined by

\[
\begin{align*}
    m_1 \ddot{x}_1 &= -k_{12}(x_1 - x_2) \\
    m_2 \ddot{x}_2 &= -k_{12}(x_2 - x_1) - k_{23}(x_2 - x_3) \\
    m_3 \ddot{x}_3 &= -k_{23}(x_3 - x_2),
\end{align*}
\]

where \( x \in \mathbb{R}, \ t \geq 0 \), where \( m_i, \ k_{ij} \ > 0, \ i, j = 1, 2, 3 \) are all parameters?

(b) Consider the three maps

\[
T(x) = x/2 \quad (x \in \mathbb{R}), \ \ V(x) = 2x \mod 1 \quad \text{and} \quad W(x) = 4x \quad \text{for} \ -0.5 \leq x < 0.5 \ \text{with} \ W(x+1) = W(x) + 1 \quad (x \in \mathbb{R}).
\]

Can they exhibit chaos? Draw graphs of these maps, including cobweb plots for initial conditions of your choice, to illustrate your answers.

(c) Let

\[
\begin{align*}
    x_{n+1} &= x_n + y_n \\
    y_{n+1} &= y_n + k \sin x_{n+1}, \quad n \in \mathbb{N}, \quad (x_n, y_n) \in \mathbb{R}^2,
\end{align*}
\]

be the (standard) map, where \( k > 0 \) is a parameter. Is this map invertible? Justify your answer. According to the Poincaré-Bendixson theorem, is chaos possible in this dynamical system?

(d) Let us consider a vector field on the unit square in the plane with periodic boundary conditions, where each vector has slope \( q \in \mathbb{R} \). According to the Poincaré-Bendixson theorem, is chaos possible? Let us consider this vector field on a torus, i.e., by gluing together the left and right edges of the square, and likewise the top and bottom ones. Is chaos possible? (*hint: in case of trouble with this question the book by Alligood et al. might help*)
3. Cobweb plots and periodic orbits

(a) Consider the map $C : \mathbb{R} \to \mathbb{R}$, $x_{n+1} = C(x_n)$, $n \in \mathbb{N}$ defined by the function $C(x) = -x^2 + x + 2$, $x \in \mathbb{R}$. Calculate the set $\text{Per}_2(C)$ of all period 2 points for this map. Draw the graph of $C(x)$ and mark the positions of all period 2 points. Include cobweb plots for all period two orbits and illustrate the stability of the fixed points by cobweb plots.

(b) Draw a cobweb plot for a one-dimensional map of your choice showing a prime period three orbit and an eventually periodic orbit.

4. Rotation on the circle

Let $S^1$ be the unit circle in the plane. Let denote a point in $S^1$ by its angle $\theta$ such that a point on the circle is determined by any angle of the form $\theta + 2k\pi$ for an integer $k$. Now let $R_\lambda(\theta) = \theta + 2\pi\lambda$ be a rotation on the circle. Show that if $\lambda$ is rational then every $\theta \in S^1$ is a periodic point. Prove by contradiction that there are no periodic points if $\lambda$ is irrational.

Model solutions will be on the course webpage starting from Thursday, October 25, 2007.