

# The Turán number of $F_{3,3}$

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## Abstract

Let  $F_{3,3}$  be the 3-graph on 6 vertices, labelled  $abcxyz$ , and 10 edges, one of which is  $abc$ , and the other 9 of which are all triples that contain 1 vertex from  $abc$  and 2 vertices from  $xyz$ . We show that for all  $n \geq 6$ , the maximum number of edges in an  $F_{3,3}$ -free 3-graph on  $n$  vertices is  $\binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$ . This sharpens results of Zhou [7] and of the second author and Rödl [5].

## 1 Introduction

The *Turán number*  $\text{ex}(n, F)$  is the maximum number of edges in an  $F$ -free  $r$ -graph on  $n$  vertices.<sup>1</sup> It is a long-standing open problem in Extremal Combinatorics to understand these numbers for general  $r$ -graphs  $F$ . For ordinary graphs ( $r = 2$ ) the picture is fairly complete, although there are still many open problems, such as determining the order of magnitude for Turán numbers of bipartite graphs. However, for  $r \geq 3$  there are very few known results. Having solved the problem for the complete graph  $F = K_t$ , Turán [6] posed the natural question of determining  $\text{ex}(n, F)$  when  $F = K_t^r$  is a complete  $r$ -graph on  $t$  vertices. To date, no case with  $t > r > 2$  of this question has been solved, even asymptotically. Despite the lack of progress on the Turán problem for complete hypergraphs, there are certain hypergraphs for which the problem has been solved asymptotically, or even exactly; we refer the reader to the survey [3]. While it would be more satisfactory to have a general theory, we may hope that this will develop out of the methods discovered in solving isolated examples.

The contribution of this paper is a surprisingly short complete solution to the Turán problem for the following 3-graph. Let  $F_{3,3}$  be the 3-graph on 6 vertices, labelled  $abcxyz$ , and 10 edges, one of which is  $abc$ , and the other 9 of which are all triples that contain 1 vertex from  $abc$  and 2 vertices from  $xyz$ . A lower bound for  $\text{ex}(n, F_{3,3})$  is given by the following natural construction. Let  $B(n)$  denote the balanced complete bipartite 3-graph, which is obtained by partitioning a set of  $n$  vertices into parts of size  $\lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$ , and taking as edges all triples that are not contained within either part. Let

$$b(n) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$$

denote the number of edges in  $B(n)$ . We prove the following result.

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<sup>1</sup>An  $r$ -graph (or  $r$ -uniform hypergraph)  $G$  consists of a vertex set and an edge set, each edge being some  $r$ -set of vertices. We say  $G$  is  $F$ -free if it does not have a (not necessarily induced) subgraph isomorphic to  $F$ .

**Theorem 1.1.** *For any  $n \geq 1$ ,  $ex(n, F_{3,3}) = b(n)$ , unless  $n = 5$ , when  $ex(5, F_{3,3}) = 10$ .*

The Turán problem for  $F_{3,3}$  was previously studied by Mubayi and Rödl [5], who obtained the asymptotic result  $ex(n, F_{3,3}) = (1 + o(1))b(n)$ . Another related problem is the following result of Zhou [7]. Say that two vertices  $x, y$  in a 3-graph  $G$  are  $t$ -connected if there are vertices  $a, b, c$  such that every triple with 2 vertices from  $abc$  and 1 from  $xy$  is an edge. Say that  $xyz$  is a  $t$ -triple if  $xyz$  is an edge and each pair in  $xyz$  is  $t$ -connected. For example,  $K_5^3$  is a  $t$ -triple (this is the motivation for the definition). The result of [7] is that the unique largest 3-graph on  $n$  vertices with no  $t$ -triple is complete bipartite. Note that  $F_{3,3}$  is a  $t$ -triple, so Theorem 1.1 strengthens Zhou’s extremal result (but not the classification of the extremal example).

Our proof uses the link multigraph method introduced by de Caen and Füredi [1]. There are now a few examples where this method has been used to obtain asymptotic results, or exact results for  $n$  sufficiently large. We used it in [4] to obtain an exact result for cancellative 3-graphs for all  $n$ , and an exact result for the configuration  $F_5 = \{123, 124, 345\}$  for  $n \geq 33$ . However, until recently there were no known applications to an exact result for all  $n$  with a single forbidden hypergraph. The result in this paper gives such an application; another was given very recently by Goldwasser [2], who obtained an exact result for  $F_5$  for all  $n$ .

In the next section we describe the link multigraph construction and prove a lemma that applies to such multigraphs. We use this to prove Theorem 1.1 in Section 3. The final section contains some concluding remarks and open problems.

## 2 A multigraph lemma

The proof of Theorem 1.1 will use the following construction of a multigraph from a 3-graph  $G$ . Suppose  $S$  is a set of vertices in  $G$ . The ‘link multigraph’ of  $S$  has vertex set  $X = V \setminus S$  and edge set  $M = \sum_{a \in S} G(a)[X]$ . Here we write  $G(a) = \{xy : axy \in G\}$ , denote the restriction to  $X$  by  $[X]$ , and use summation to denote multiset union. Thus we obtain a multigraph  $M$  in which each pair of vertices has multiplicity between 0 and  $|S|$ . Furthermore, we may regard each edge of  $M$  as being ‘coloured’ by a vertex in  $S$  (an edge may have several colours). We write  $w(xy)$  for the multiplicity of the pair  $xy$  in  $M$ .

Now suppose  $M$  is any multigraph on  $n$  vertices (not necessarily as above). Write  $w(xy)$  for the multiplicity of a pair  $xy$  in  $M$ , and write  $e(M)$  for the sum of  $w(xy)$  over all (unordered) pairs of vertices in  $M$ . For any  $S \subseteq V(M)$  let  $i(S)$  denote the sum of  $w(xy)$  over all pairs of vertices that contain at least one vertex of  $S$ . If  $S = \{x\}$  consists of a single vertex then  $i(S) = d(x)$  is the weighted degree of  $x$ . Define

$$m(n) = \begin{cases} \frac{3n^2}{2} - n & \text{if } n \text{ is even,} \\ \frac{3n^2-1}{2} - n & \text{if } n \text{ is odd.} \end{cases}$$

**Lemma 2.1.** *Suppose  $M$  is a multigraph on  $n$  vertices with  $0 \leq w(xy) \leq 4$  for every pair  $xy$  and  $w(xy) + w(xz) + w(yz) \leq 10$  for every triple  $xyz$ . Then  $e(M) \leq m(n)$ .*

**Proof.** We argue by induction on  $n$ . The statement is trivial for  $n = 1$  and  $n = 2$ , and is immediate from the assumption on triples for  $n = 3$ . Now suppose that  $n \geq 4$ . We consider separate cases according to the parity of  $n$ .

Suppose first that  $n$  is even,  $M$  is a multigraph satisfying the hypotheses of the lemma, and suppose for a contradiction that  $e(M) = \frac{3n^2}{2} - n + 1$ . Since  $e(M) > 3\binom{n}{2}$  we can choose a pair  $xy$  with  $w(xy) = 4$ . If we delete  $xy$  then we obtain a multigraph  $M'$  on  $n - 2$  satisfying the hypotheses of the lemma. By induction hypothesis we have  $e(M') \leq \frac{3(n-2)^2}{2} - (n-2)$ . Then  $i(xy) = e(M) - e(M') \geq \frac{3n^2}{2} - n + 1 - \frac{3(n-2)^2}{2} + (n-2) = 6n - 7$ . Now the sum of  $w(xz) + w(yz)$  over  $z$  in  $V(M) \setminus \{x, y\}$  is  $i(xy) - w(xy) \geq 6n - 11 > 6(n-2)$ , so there must be some  $z$  with  $w(xz) + w(yz) \geq 7$ . But then  $w(xy) + w(xz) + w(yz) \geq 11$ , contradiction.

The argument for  $n$  odd is similar. Suppose for a contradiction that  $M$  satisfies the hypotheses of the lemma but  $e(M) = \frac{3n^2-1}{2} - n + 1$ . Choose  $xy$  with  $w(xy) = 4$ . The induction hypothesis gives  $i(xy) \geq \frac{3n^2-1}{2} - n + 1 - \frac{3(n-2)^2-1}{2} + (n-2) = 6n - 7$ . Then, as in the case of  $n$  even, we have  $i(xy) - w(xy) > 6(n-2)$ , so there must be some  $z$  with  $w(xz) + w(yz) \geq 7$ , contradiction.  $\square$

Note the following two examples where equality holds in Lemma 2.1. (We do not claim that these are the only cases of equality.)

1. Define a multigraph  $M_1(n)$  on  $n$  vertices as follows. Let  $A \cup B$  be a balanced partition of the vertex set. Let crossing pairs have multiplicity 4 and pairs inside each part have multiplicity 2. If  $n$  is even then  $e(M_1(n)) = 2\binom{n}{2} + 2(n/2)^2 = 3n^2/2 - n$ . If  $n$  is odd then  $e(M_1(n)) = 2\binom{n}{2} + 2\frac{n^2-1}{4} = \frac{3n^2-1}{2} - n$ .
2. Define a multigraph  $M_2(n)$  on  $n$  vertices as follows. Let all pairs have multiplicity 3 except for a maximum size matching of multiplicity 4. If  $n$  is even then  $e(M_2(n)) = 3\binom{n}{2} + n/2 = 3n^2/2 - n$ . If  $n$  is odd then  $e(M_2(n)) = 3\binom{n}{2} + \frac{n-1}{2} = \frac{3n^2-1}{2} - n$ .

The following two calculations will also be useful. Note that  $M_2(n-1)$  can be obtained from  $M_2(n)$  by deleting a vertex, which can be any vertex if  $n$  is even, but must be the vertex not incident to an edge of multiplicity 4 when  $n$  is odd. Then  $m(n) - m(n-1)$  is equal to the number of edges removed, which is  $3(n-2) + 4 = 3(n-1) + 1$  when  $n$  is even, or  $3(n-1)$  when  $n$  is odd.

Next consider a copy of  $B(n)$  with parts  $A$  and  $B$ . Construct a copy of  $B(n-4)$  by removing vertices  $wx$  from  $A$  and  $yz$  from  $B$ . Then we have  $b(n) - b(n-4) = i(wxyz)$ , where similarly to our multigraph notation, we let  $i(S)$  denote the number of edges that contain at least one vertex of  $S$ . We can count  $i(wxyz)$  as follows. There are 4 edges of  $B(n)$  contained in  $wxyz$ . Next consider edges with 2 vertices in  $wxyz$ . The 4 crossing pairs  $wy, wz, xy, xz$  form an edge with each of the  $n-4$  vertices of  $V \setminus \{w, x, y, z\}$ , so contribute  $4(n-4)$  edges. The pairs  $wx$  and  $yz$  form an edge with any of the vertices in the other part, so contribute  $n-4$  edges (this holds whether  $n$  is even or odd). Finally, note that the link multigraph of  $wxyz$  in  $B(n)$  is precisely  $M_1(n-4)$ . Thus the number of edges with 1 vertex in  $wxyz$  is  $m(n-4)$ . Then we have

$$b(n) - b(n-4) = m(n-4) + 5(n-4) + 4.$$

### 3 Proof of Theorem 1.1

We start with the lower bound. We have already described the construction  $B(n)$ , but we also need to check that it is  $F_{3,3}$ -free. To see this, suppose for a contradiction that there is a copy of  $F_{3,3}$  in

$B(n)$ , labelled  $abcxyz$  as above. Label the parts of  $B(n)$  as  $X$  and  $Y$ , and suppose without loss of generality that  $a \in X$ . The edges  $axy$ ,  $axz$ ,  $ayz$  can only be simultaneously realised by putting all of  $x, y, z$  in  $Y$ , or 2 of  $x, y, z$  in  $Y$  and 1 in  $X$ . Either way, the edges  $bxy$ ,  $bxz$ ,  $byz$  imply that  $b$  is in  $X$ , and the edges  $cxy$ ,  $cxz$ ,  $cyz$  imply that  $c$  is in  $X$ . But this contradicts the fact that  $abc$  is an edge. Thus  $B(n)$  is  $F_{3,3}$ -free. This shows that  $\text{ex}(n, F_{3,3}) \geq b(n)$ . This bound can be improved when  $n = 5$ , as  $b(5) = 9$ , but the complete 3-graph  $K_5^3$  is  $F_{3,3}$ -free. It follows that  $\text{ex}(5, F_{3,3}) = 10$  (obviously it cannot be larger).

The main task in the proof is to establish the upper bound. We prove the following statement by induction on  $n$ :

- Suppose  $G$  is an  $F_{3,3}$ -free 3-graph on  $n \geq 1$  vertices. Then  $e(G) \leq b(n)$ , unless  $n = 5$ , in which case  $e(G) \leq 10$ .

Note that this statement is trivial for  $n \leq 5$ , as  $B(n)$  is complete for  $n \leq 4$ , and for  $n = 5$  the statement allows  $e(K_5^3) = 10$  edges. Furthermore, the bound holds for  $n = 6$ , as  $B(6)$  is obtained by deleting 2 edges from  $K_6^3$ , and it is clear that if one only deletes one edge from  $K_6^3$  then there is a copy of  $F_{3,3}$ . Moreover,  $B(6)$  is the unique  $F_{3,3}$ -free 3-graph on 6 vertices with  $\binom{6}{3} - 2 = 18$  edges. To see this, we exhibit appropriate edges in the complement of  $F_{3,3}$  as defined above:  $xab$  and  $yac$  are not in  $F_{3,3}$  and intersect in 1 vertex, whereas  $xab$  and  $xac$  are not in  $F_{3,3}$  and intersect in 2 vertices.

Now suppose for a contradiction that  $n \geq 7$  with  $n \neq 9$  and  $G$  is an  $F_{3,3}$ -free 3-graph on  $n$  vertices with  $e(G) = b(n) + 1$ . (We will need to modify the argument in the case  $n = 9$ .) We start by finding a copy of  $K_4^3$  in  $G$ . For this we use the following averaging argument. Given a 3-graph  $H$  on  $m$  vertices, let  $d(H) = e(H) \binom{m}{3}^{-1}$  denote the density of  $H$ . A simple calculation shows that  $d(G)$  is the average of  $d(G \setminus v)$  over all vertices  $v$  of  $G$ . Note that deleting a vertex from  $B(n)$  leaves a complete bipartite 3-graph on  $n - 1$  vertices; it is not necessarily balanced, but certainly has at most  $b(n - 1)$  edges. It follows that  $d(B(n)) \leq d(B(n - 1))$ , i.e.  $d(B(n))$  is non-increasing in  $n$ . Since  $d(B(n)) \rightarrow 3/4$  as  $n \rightarrow \infty$  we have  $d(B(n)) \geq 3/4$  for all  $n$ . Since  $e(G) > b(n)$  we have  $d(G) > 3/4$ . Averaging again, we see that there is a set  $abcd$  of 4 vertices where  $d(G[abcd]) > 3/4$ . This implies that all 4 triples in  $abcd$  are edges of  $G$ , as desired.

Note that  $G \setminus \{a, b, c, d\}$  is an  $F_{3,3}$ -free 3-graph on  $n - 4$  vertices with  $e(G) - i(abcd)$  edges. By induction this is at most  $b(n - 4)$  (since  $n \neq 9$ ), so we obtain  $i(abcd) \geq b(n) - b(n - 4) + 1$ . Now we count the edges incident to  $abcd$  according to the number of vertices of  $abcd$  they contain. There are 4 such edges contained in  $abcd$ . To estimate edges with one vertex in  $abcd$  let  $M$  be the link multigraph of  $abcd$  in  $G$ . Note that there is no triangle  $xyz$  in  $M$  such that each pair  $xy$ ,  $xz$ ,  $yz$  is coloured by the same set of 3 colours from  $abcd$ : this would give a copy of  $F_{3,3}$ . This implies that  $w(xy) + w(xz) + w(yz) \leq 10$  for every triple  $xyz$  in  $M$ . Thus we can apply Lemma 2.1 to get  $e(M) \leq m(n - 4)$ . We conclude that the number of edges with 2 vertices in  $abcd$  is at least  $b(n) - b(n - 4) + 1 - 4 - m(n - 4) = 5(n - 4) + 1$ . It follows that there is some  $e \in V(G) \setminus \{a, b, c, d\}$  such that all 6 pairs from  $abcd$  form an edge with  $e$ . Thus  $abcde$  forms a copy of  $K_5^3$  in  $G$ .

For each  $x \in abcde$  we have  $i(abcde \setminus x) \geq b(n) - b(n - 4) + 1$ , so

$$\Sigma := \sum_{x \in abcde} i(abcde \setminus x) \geq 5(b(n) - b(n - 4) + 1) = 5(m(n - 4) + 5(n - 3)).$$

We can also count  $\Sigma$  according to the intersection of edges with  $abcde$ . Edges with at least 2 vertices in  $abcde$  are counted 5 times, and edges with 1 vertex in  $abcde$  are counted 4 times. By Lemma 2.1,

for each  $x \in abcde$  the link multigraph of  $abcde \setminus x$  restricted to  $V(G) \setminus \{a, b, c, d, e\}$  has at most  $m(n-5)$  edges. By averaging, there are at most  $\frac{5}{4}m(n-5)$  edges with 1 vertex in  $abcde$ . Since these are counted 4 times they contribute at most  $5m(n-5)$  to  $\Sigma$ . Also,  $abcde$  is complete, so we have 10 edges inside  $abcde$ .

Writing  $Z$  for the number of edges with 2 vertices in  $abcde$ , we obtain

$$5(m(n-4) + 5(n-3)) = 5(b(n) - b(n-4) + 1) \leq \Sigma \leq 5(10 + Z + m(n-5)),$$

so  $Z \geq m(n-4) - m(n-5) + 5(n-5)$ . Recall that  $m(n-4) - m(n-5)$  is  $3(n-5) + 1$  when  $n$  is even, or  $3(n-5)$  when  $n$  is odd. Thus  $Z \geq 8(n-5)$ . It follows that there is some  $f \in V(G) \setminus \{a, b, c, d, e\}$  such that at least 8 pairs from  $abcde$  form an edge with  $f$ . Thus  $abcdef$  is obtained from  $K_6^3$  by deleting at most 2 edges, and if 2 edges are deleted they cannot be disjoint, as they both contain  $f$ . As noted above, this implies that  $abcdef$  contains  $F_{3,3}$ , so we have a contradiction.

It remains to prove the bound for  $n = 9$ . Again we start by choosing  $abcd$  as a copy of  $K_4^3$  in  $G$ . Since  $b(5) = \binom{5}{3} - 1$ , we obtain  $i(abcd) \geq b(n) - b(n-4)$ . Then the same calculation as above shows that there are at least  $5(n-4)$  edges with 2 vertices in  $abcd$ . Furthermore, equality can only hold if deleting  $abcd$  leaves  $\binom{5}{3}$  edges, i.e. a copy of  $K_5^3$ . If equality does not hold then we can find a copy of  $K_5^3$  as above, so either way we have a copy of  $K_5^3$ . Let  $X$  be the vertex set of this  $K_5^3$ . Now note that for any  $Y$  spanning a copy of  $K_4^3$  we either have  $i(Y) \geq b(n) - b(n-4) + 1$  or  $G \setminus Y$  spans  $K_5^3$ . Also, there cannot be 3 vertices  $x_1, x_2, x_3$  in  $X$  such that  $G \setminus (X \setminus x_i)$  spans  $K_5^3$  for  $1 \leq i \leq 3$ , as then  $x_1, x_2, x_3$  together with any 3 vertices of  $G \setminus X$  spans a copy of  $F_{3,3}$ . Now we can modify the second calculation above to get  $3(b(n) - b(n-4) + 1) + 2(b(n) - b(n-4)) \leq 5(10 + Z + m(n-5))$ , so  $Z \geq 8(n-5) - 2 = 30 > 7(n-5)$ . It follows that there is some  $f \in V(G) \setminus \{a, b, c, d, e\}$  such that at least 8 pairs from  $abcde$  form an edge with  $f$ . As above, this creates a copy of  $F_{3,3}$ , so we have a contradiction. This proves the theorem.  $\square$

## 4 Concluding remarks

The obvious unanswered question from this paper is to characterise the extremal examples for the problem: is it true that for  $n \geq 6$ , equality can only be achieved by  $B(n)$ ? It may be that there is a simple proof of this statement, but if not, one might still hope to prove it for  $n$  sufficiently large by the stability method. The idea would be to show that any  $F_{3,3}$ -free graph  $G$  on  $n$  vertices with  $e(G) \sim b(n)$  is ‘structurally close’ to  $B(n)$ . Then one would hope to show that any construction  $B'(n)$  that is sufficiently close to  $B(n)$  is suboptimal, unless  $B'(n) = B(n)$ . (See [3, Section 5] for further discussion of this method.)

Consideration of this stability question leads us in turn to the question of what constructions are (near) extremal for the multigraph result, Lemma 2.1. We have already seen two very different constructions that achieve the maximum  $m(n)$ . However, we can rule out  $M_2(n)$  by returning to the 3-graph world. Suppose that  $G$  is an  $F_{3,3}$ -free 3-graph and  $M$  is the link multigraph of a  $K_4^3$   $abcd$  in  $G$ . Let  $J$  be the set of pairs of multiplicity at least 3 in  $M$ . We claim that  $J$  is  $K_t$ -free, where  $t = R_4(3)$  is the 4-colour Ramsey number for triangles. To see this, colour the pairs of multiplicity 3 according to which colour from  $abcd$  is *not* available, and colour the pairs of multiplicity 4 arbitrarily from  $abcd$ . If  $J$  contains  $K_t$  then by definition we can find a monochromatic triangle  $xyz$ . Without loss

of generality  $d$  is the missing colour for  $xy$ ,  $xz$  and  $yz$ . Then  $abcxyz$  is a copy of  $F_{3,3}$ , contradiction. Thus  $J$  is  $K_t$ -free.

It follows that  $M_2(n)$ , or indeed any ‘sufficiently close’ construction, cannot be realised as the link multigraph of  $G$ . However, we can give other multigraph constructions that are not ruled out on these grounds. For example, take a balanced partition of  $n$  vertices into 4 parts  $W, X, Y, Z$ , such that all pairs within a part have multiplicity 2, all pairs between  $W$  and  $X$  or between  $Y$  and  $Z$  have multiplicity 4, and all pairs between the other four pairs of parts have multiplicity 3. One can check that this construction satisfies the hypotheses of Lemma 2.1, and the number of edges is approximately  $m(n)$ . However, as far as we are aware, it does not seem to arise as the link multigraph of a near extremal  $F_{3,3}$ -free 3-graph. The potentially large variety of multigraph constructions suggests that this may be a difficult approach to proving stability for  $F_{3,3}$ , so perhaps other ideas are needed.

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