Extremal Combinatorics – 2009/10

Warm-ups **JRJ**

Most of this module is concerned with graphs and sets. We will require some techniques from probability, linear algebra and analysis. Here are a few simple questions to check you know the basic definitions in these areas and can work with them.

A. Graphs

1. What is a graph? Make sure you know the meaning of the following terms: edges, vertices, degree of a vertex, complete graph, bipartite graph, regular graph.

2. What is the maximum number of edges that a bipartite graph with n vertices can have?

3. Let G be a graph with 8 vertices and 14 edges. Take a vertex of minimum degree in G and delete it and all incident edges to form a new graph H with 7 vertices. What are the maximum and minimum number of edges which H could have?

B. Sets

1. Make sure you know the meaning of the following terms: intersection, union, complement, symmetric difference, disjoint, empty set.

2.

i) How many subsets of $\{1,2,\dots,n\}$ are there?

ii) How many of them have even size?

iii) How many of them have size k?

iv) What is the probability that a randomly chosen subset of $\{1,2,\dots,n\}$ of size k contains 1?

3.

i) Can you find 3 sets with the property that any 2 of them intersect but the intersection of all 3 is empty?

ii) Can you find 4 sets with the property that any 3 of them intersect but the intersection of all 4 is empty?

C. Miscellaneous

1.

i) Let $A_1, A_2, \ldots, A_n$ be events and let N be the event that none of the A_i occur. Show that if each of the A_i has probability p of occurring and $p < 1/n$ then the probability of N is positive. Under what circumstances can we have $p = 1/n$ but the probability of N being 0.

ii) Let $B_1, B_2, \ldots, B_n$ be events, M be the event that none of the B_i occur, and S be the random variable the number of the B_i which occur. Show that if $\mathbb{E}(S) < 1$ then the probability of N is positive,

2.

i) What is a vector space? Make sure you know the meaning of the terms: linearly independent, basis, dimension, subspace.

ii) Show that if a set of vectors in $\mathbb{R}^n$ is orthogonal then it is linearly independent.

3. Order the following functions of n by how fast they grow for large n:

$$\log n; \quad 2^{0.9n}; \quad 1000n^2; \quad n/1000; \quad 2^n/n.$$

4. Given n real numbers which sum to S how small can the sum of their squares be?