Queen Mary
University of London

M. Sci. Examination by course unit 2010

MTH711U Extremal Combinatorics

Duration: 3 hours

Date and time: Sample Paper

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): Robert Johnson

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Question 1 [25 marks]

(a) Define the Turán number $\mathrm{ex}(n,H)$ for a graph H.

(b) Define the Turán graph $T_r(n)$.

(c) State and prove Turán's theorem on the maximum number of edges in a K_r-free graph.

(d) Write down a graph H with chromatic number r but $\mathrm{ex}(n,H) > \mathrm{ex}(n,K_r)$ for all sufficiently large n.

Question 2 [25 marks]

(a) Define the lower shadow of a set system $\mathscr{A} \subset [n]^{(r)}$.

(b) Define the ij-compression C_{ij} for a set and a set system.

(c) State (without proof) the Kruskal-Katona theorem and indicate briefly how to prove it using ij-compressions.

(d) Let $\mathscr{A} \subset \mathscr{P}([n])$ be an antichain which contains exactly $\binom{k}{r}$ sets of size r for some $k \in \mathbb{N}$. What is the maximum number of sets of size $r-1$ that $\mathscr{A}$ can contain?

Question 3 [25 marks]

(a) What does it mean for A to be less than B in the cube order on $\mathscr{P}([n])$?

(b) State and prove Harper's theorem (the vertex isoperimetric inequality in Q_n).

(c) Let $\mathscr{A}.\mathscr{B} \subset \mathscr{P}([n])$ be two set systems with $|\mathscr{A}| = |\mathscr{B}| = \sum_{i=0}^{k} \binom{n}{i}$ for some $0 \le k < n/2$. Let the distance between $\mathscr{A}$ and $\mathscr{B}$ be

$$d(\mathscr{A},\mathscr{B}) = \min\{|A \triangle B| : A \in \mathscr{A}, B \in \mathscr{B}\}.$$

How large can $d(\mathscr{A},\mathscr{B})$ be?

Question 4 [25 marks]

(a) Say what it means for a set system to be t-intersecting

(b) State and prove the Erdős-Ko-Rado theorem (giving the largest r-uniform 1-intersecting system).

(c) Show that there is an r-uniform t-intersecting system of size $\binom{n-t}{r-t}$.

(d) Does the largest r-uniform, t-intersecting system of subsets of $[n]$ always have size $\binom{n-t}{r-t}$? Justify your answer with a proof or counterexample.

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Question 5 [25 marks]

(a) Prove that if $\mathscr{A} \subset \mathscr{P}([n])$ is a set system with $|A \cap B| = 1$ for all distinct $A, B \in \mathscr{A}$ then $|\mathscr{A}| \leq n$.

(b) Give two non-isomorphic examples which show that the bound in part (a) can be achieved.

(c) Let $\mathscr{B} \subset \mathscr{P}([n])$ be a set system with $|A \cap B| \leq 1$ for all distinct $A, B \in \mathscr{B}$. Prove an upper bound for $|\mathscr{B}|$ and show that it can be achieved.

(d) Let $\mathscr{C} \subset \mathscr{P}([7])$ be a set system with $|A \cap B| \leq 1$ for all distinct $A, B \in \mathscr{C}$ and which contains no sets of size 2. Prove an upper bound for $|\mathscr{C}|$ and show that it can be achieved.

End of Paper

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