Queen Mary
University of London

M. Sci. Examination by course unit 2009

MTH711U Extremal Combinatorics

Duration: 3 hours

Date and time: May 13, 10am-1pm

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Keevash

Question 1 (a) Define the Turán number $\mathrm{ex}(n,H)$. [5]

(b) State and prove Mantel's theorem. [10]

(c) Let H be the graph with 4 vertices and 5 edges. Determine $\mathrm{ex}(n,H)$ for all $n \geq 1$. [10]

Question 2 (a) Define the projective plane over the finite field $\mathbb{F}_q$ and show that the Fano plane is an example of a projective plane. [5]

(b) Show that for any prime power q there is a C_4-free graph with $2(q^2+q+1)$ vertices and $(q+1)(q^2+q+1)$ edges. [5]

(c) Define the term *bipartite* for a general hypergraph and prove that the Fano plane is not bipartite. [5]

(d) Prove that any k-graph with fewer than 2^{k-1} edges is bipartite. [10]

Question 3 (a) Define the term *intersecting family*. [5]

(b) State and prove a formula for the maximum size of an intersecting family on a ground set of size n. [5]

(c) State and prove a formula for the maximum size of an intersecting k-uniform family on a ground set of size n. [10]

(d) Prove that any intersecting family is contained in a maximum size intersecting family on the same ground set. [5]

Question 4 (a) Suppose $d \geq 1$ and let G be the graph with $V(G) = \mathbb{R}^d$ and $E(G) = \{xy : x,y \in \mathbb{R}^d, \|x-y\| = 1\}$. Prove that the chromatic number of G is at most 9^d. [10]

(b) Suppose L is a set of integers and p is a prime number. Define the terms *L-intersecting* and *L-intersecting mod p*. Suppose k is an integer with $k \notin L$ mod p. State (without proof) the best possible upper bound for the size of a k-uniform family on a set of size n that is L-intersecting mod p. [5]

(c) Prove that the chromatic number of the graph G in part (a) is at least $(1.1)^d$ for d sufficiently large. [10]

End of Paper