Extremal Combinatorics – 2009/10

Exercise Sheet 6 JRJ

I am happy to comment on any work you hand in.

1. For $r = 5, t = 3$ write down the families $\mathcal{A}_{n,r,t,k}$ (the possible maximal t-intersecting r-uniform families as proved by Ahlswede and Khachatrian) and calculate their sizes. Which is largest for each n?

2. Show that if $0 \leq k \leq n/2$, and $\mathcal{A} \subseteq \mathcal{P}([n])$ is such that $|A \cup B| \leq 2k$ for all $A, B \in \mathcal{A}$ then $|\mathcal{A}| \leq \sum_{i=0}^{k} \binom{n}{i}$.

3.

a) Prove that ij-compressions preserve the property of being t-intersecting. That is show that if $\mathcal{A}$ is t-intersecting then $C_{ij}(\mathcal{A})$ is t-intersecting.

b) Use part a) and induction to give another proof of Katona's t-intersection theorem in the case that $n+t$ is even.

[Hint: Split $\mathcal{A}$ into sets which contain 1 and sets which do not contain 1. What does the fact that $\mathcal{A}$ is compressed tell you about the sets which do not contain 1?]

4. Let $\mathcal{A} \subseteq \mathcal{P}([n])$ satisfy that $|A| \not\equiv 0 \mod k$ for all $A \in \mathcal{A}$ and $|A \cap B| \equiv 0 \mod k$ for all distinct $A, B \in \mathcal{A}$.

a) Show that if k is prime then $|\mathcal{A}| \leq n$.

b) Show that for any k we have $|\mathcal{A}| \leq c_k n$ where c_k is a constant which does not depend on n.

5. How large can $\mathcal{A} \subseteq \mathcal{P}([n])$ be if $|A|$ is even for all $A \in \mathcal{A}$ and $|A \cap B|$ is even for all $A, B \in \mathcal{A}$.
[Hint: Consider the orthogonal complement of the subspace of $\mathbb{F}_2^n$ spanned by the characteristic vectors of the sets in $\mathcal{A}$.]

Please let me know if you have any comments or corrections