Queen Mary
University of London

B. Sc. Examination by course unit 2009

MTH6130 Probability III

Duration: 2 hours

Date and time: SAMPLE PAPER

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): Robert Johnson

Question 1 [20]

(a) State and prove the Chapman-Kolmogorov relations for a discrete time Markov chain.

Let $X_0, X_1, \ldots$ be the discrete time Markov chain on state space $\{1,2,3,4,5\}$ with $X_0 = 2$ and transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/6 & 1/6 & 1/6 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Identify the absorbing states of the chain.

(c) For each absorbing state calculate the probability that the chain is eventually absorbed in that state.

(d) Calculate the expected number of visits to state 4 before absorption.

Question 2 [20]

Let $X_0, X_1, \ldots$ be a discrete time Markov chain on state space S.

(a) What does it mean to say that a state $i \in S$ is recurrent?

(b) State, without proof, a necessary and sufficient condition for i to be recurrent in terms of the t-step transition probabilities $p_{i,i}^{(t)}$.

Adam tosses a fair coin. If it shows heads he takes one step North. If it shows tails he takes one step South. This process is performed repeatedly.

(c) Explain how to model Adam's position as a Markov chain. Is this chain irreducible? Is it regular?

(d) Show that the probability that after $2k$ coin tosses Adam is back where he started is

$$2^{-2k}\binom{2k}{k}.$$

(e) Is the starting state of the chain recurrent? Are the other states recurrent?

You may use Stirling's approximation that

$$\frac{n^n\sqrt{2\pi n}}{e^n} \leq n! \leq \frac{11}{10}\frac{n^n\sqrt{2\pi n}}{e^n}.$$

Question 3 [20]

A fair die is thrown repeatedly. Let S_n be the sum of the outcomes, and let R_n be the remainder when S_n is divided by 4 (that is R_n is the sum of the first n throws reduced modulo 4).

(a) Show that R_n is a Markov chain on state space $\{0, 1, 2, 3\}$.

(b) Draw the transition diagram for the chain.

(c) Write down the transition matrix for the chain.

(d) Show that $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is an equilibrium distribution for the chain.

(e) How does the probability that the sum of the outcomes of the first n throws is divisible by 4 behave as n becomes large?

The die is replaced with a biased die for which different outcomes may have different probabilities. However, the same die is used for each throw and the outcome of different throws are independent. Again R_n is defined to be the remainder when the sum of the first n throws is divided by 4.

(f) What is the equilibrium distribution for this chain?

Question 4 [20]

(a) Say what it means for a continuous-time stochastic process $X(t)$ to be a pure birth process.

(b) Let $X(t)$ be a pure birth process with $X(0) = 1$ and parameters $\lambda_k = k$. Let

$$p_n(t) = \mathbb{P}(X(t) = n).$$

Show that the $p_n(t)$ satisfy the equations

$$p_n'(t) = (n-1)p_{n-1}(t) - np_n(t)$$

for all $n \geq 1$.

(c) Use the equations from part (b) to find $p_1(t)$.

(d) Hence determine the distribution of the time of the first birth.

(e) Use the equations from part (b) to find $p_2(t)$.

(f) Hence find $\mathbb{P}(X(1) = 2)$.

Question 5 [20]

A supermarket has 2 checkouts. Customers form a single queue and when one of the checkouts becomes available the customer at the front of the queue moves to it. Assume that customers arrive at the checkouts according to a Poisson process of rate 3 per minute and that the service time of each customer is exponentially distributed with parameter 2 per minute independent of all other events. Let $Q(t)$ be the number of customers in the queue at time t minutes.

(a) What is the name of the queuing system which models this process? Explain the notation which you use.

(b) Explain why $Q(t)$ forms a birth-death process, specifying the parameters.

(c) Find the equilibrium distribution $(w_0, w_1, \ldots)$ for the system. You may assume that, if it exists, the equilibrium distribution for a birth-death process with birth parameters λ_i and death parameters μ_i satisfies

$$w_i = \frac{\lambda_0 \lambda_1 \ldots \lambda_{i-1}}{\mu_1 \mu_2 \ldots \mu_i} w_0.$$

(d) In the long run what proportion of customers are served without having to wait for a checkout to become free?

End of Paper