Probability III – 2009/10

Exercise Sheet 9 (Revision sheet) JRJ

This sheet contains 7 questions in two sections. In each section the questions are in roughly increasing order of difficulty.

Reflect on how you are finding the course and based on this choose one of the questions to hand in. Hand in your answer to your chosen question only to the red box on the second floor of the maths building by 10am on Thursday 25 March. (Serious attempts at the harder questions will be marked generously so don't let the fact that the sheet is marked limit your ambition.)

As usual I am happy to discuss your answers to any of the questions you don't hand in during office hours or classes.

Section 1: Discrete Time Markov Chains

Question 1.1 is straightforward and is there for those who are having trouble with the basic concepts. Question 1.2 is a little more testing and would be a reasonable (but rather long) exam question. Questions 1.3 and 1.4 are harder than anything I would put on an exam and are there for those who are comfortable with the course and want a challenge.

1.1. Let $X_0, X_1, \dots$ be the discrete time Markov chain on state space $\{1,2,3,4,5,6\}$ with transition matrix

$$\begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/5 & 2/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a) Draw the transition graph of the chain.

b) Calculate the following probabilities:

i) $\mathbb{P}(X_1 = 5 | X_0 = 2)$,

ii) $\mathbb{P}(X_2 = 5 | X_1 = 2)$,

iii) $\mathbb{P}(X_2 = 5 | X_1 = 2, X_0 = 1)$,

iv) $\mathbb{P}(X_2 = 5, X_1 = 2 | X_0 = 1)$.

c) i) Which states are absorbing?

ii) For each absorbing state find the probability that the process is absorbed at that state given that $X_0 = 1$.

d) i) Find the communicating classes of the chain.

ii) Clasify each state as recurrent or transient giving a brief explanation.

1.2. A bag contains 5 balls. Initially 1 of them is red and the remaining 4 are blue. At each time step I choose a ball at random, remove it from the bag, and replace it with a ball of the opopsite colour. Let X_t be the number of red balls in the bag after t steps of the process.

a) Explain why this process is a Markov chain and give the transition matrix.

b) Find the equilibrium distribution for this chain.

c) In the long run how does the proportion of time for which the bag contains only blue balls behave?

d) Let E_t be the event that after t steps of the process the bag contains only blue balls. Does $\lim_{t\to\infty}\mathbb{P}(E_t)$ exist? If it does exists then say what is it, if it does not then explain why not.

e) Suppose that after 6 steps the bag contains only blue balls. What can you say about the expectation of the next value of t for which the bag contains only blue balls.

f) The process is modified so that at each step the chosen ball is replaced with a ball of the same colour with probability p, and is replaced with a ball of the opposite colour with probability $1-p$ where $0<p<1$. How does the transition matrix for this Markov chain differ from the original one?

g) Let F_t be the event that after t steps of the new process the bag contains only blue balls. Does $\lim_{t\to\infty}\mathbb{P}(F_t)$ exist? If it does exists say what is it, if it does not then explain why not.

1.3. Let $X_0, X_1, \dots$ be the random walk on a finite graph G. Fix two vertices x and y and define

$$P_i = \mathbb{P}(\text{the walk reaches } x \text{ before } y | X_0 = i).$$

a) Consider a flow of some substance around the graph with each edge ab carrying a flow of $P_a - P_b$ units from a to b. Show that for every vertex v apart from x and y the flow into v is equal to the flow out of v.

b) Give a probabilistic interpretation of the total flow into vertex y.

Recalling some basic physics, part a) means that if we think of the graph as an electrical network with each edge having unit resistance then the P_i define a potential function. The total flow into vertex y is then the reciprocal of the effective resistance between x and y and so we have a surprising connection between random walks and electrical networks.

Let G be an infinite graph in which all degrees are finite. For each $n \geq 1$ form a finite graph G_n by identifying all vertices at distance at least n from y to a single vertex x_n. Let R_n be the effective resistance between x_n and y.

c) By considering part b) give a probabilistic interpretation of $\lim_{n\to\infty} R_n^{-1}$ (you may assume that the limit exists). Hence characterise recurrence/transience of y in terms of the limit of the R_n.

1.4. Discuss what the Law of Large Numbers and the Central Limit Theorem imply about the random walk on $\mathbb{Z}$ with $p = 1/2$? What about when $p \neq 1/2$? How does this compare with looking at long-run behaviour via a limiting distribution?

Section 2: Continuous Time Stochastic Processes

Question 2.1 is straightforward and is there for those who are having trouble with the basic concepts. Question 2.2 is a little more testing and would be a reasonable exam question. Question 2.3 is harder than anything I would put on an exam and is there for those who are comfortable with the course and want a challenge.

2.1. Let $X(t)$ be the number of visits to my webpage in the time interval $[0,t]$ and suppose that this forms a Poisson process of rate 4 per hour.

a) State the distribution of $X(2)$.

b) State the distribution of $X(5)$ conditional on $X(3) = 10$.

c) State the distribution of $X(3)$ conditional on $X(5) = 15$.

d) State the distribution of the time of the first visit to my webpage.

Suppose now that in the first hour visits occur at times $2, 20, 35, 36, 49, 55$ minutes.

e) Sketch $X(t)$ for the first hour of the process and find the associated arrival times and interarrival times.

f) What can you say about the process $Y(t) = X(t+1) - 6$?

2.2. Let $X(t)$ to be a (pure) birth process with parameters λ_i, $i \geq 0$ and $X(0) = 0$. Let $p_n(t) = \mathbb{P}(X(t) = n)$.

a) Define what it means to say that $X(t)$ is be a birth process.

b) Write down differential equations for $p_0(t)$ and $p_1(t)$.

c) Suppose that $\lambda_0 \neq \lambda_1$. Solve the equations of part b) and hence find $\mathbb{P}(X(t) \geq 2)$ as a function of t.

d) Let i be the time of the ith birth and $S_i = T_i - T_{i-1}$ for $i \geq 1$ (where by convention we let $T_0 = 0$. State the distribution of the S_1 and prove your assertion. State (without proof) the distribution of the S_i.

e) Find the expectation of T_i.

Let $Y(t)$ be a Poisson process of rate 2. Let $Z(t)$ be a a birth process with parameters $\lambda_i = 2^i$ and $Z(t) = 0$. Suppose that $Y(t)$ and $Z(t)$ are independent processes.

f) For each k determine whether the expected time of the kth birth in $Y(t)$ is smaller or larger that the expected time of the kth birth in $Z(t)$.

2.3. Recalling Question 4 on sheet 8, let $X(t)$ be a birth process with $X(0) = 0$ and parameters $\lambda_n = \alpha + n\beta$ for $n \geq 0$. Let $p_n(t) = \mathbb{P}(X(t) = n)$ and $e(t) = \mathbb{E}(X(t))$.

a) Write down the differential equations for $p_n'(t)$.

b) Without solving these equations show that $e'(t) = \alpha + \beta e(t)$ and hence find $e(t)$.

c) Describe (in as much detail as you have enthusiasm for) how to find the variance of $X(t)$ by a similar method.

d) Solve the equations of part b) for some small values of n and try to guess at a pattern.