Probability III – 2009/10

Exercise Sheet 8 JRJ

Please hand in your answer to Question 4 only to the red box on the second floor of the maths building by 10am on Thursday 18 March.

You are strongly encouraged to attempt all the questions on this sheet.

If you want feedback on your answers to any of the questions then see me in a class or office hour.

1. Let $X(t)$ be a Poisson process of rate λ. Show that for all $n > 0$ the probability density function of T_1 conditioned on $X(t) = n$ is

$$f_{T_1|X(t)=n}(u) = \frac{n}{t}\left(1 - \frac{u}{t}\right)^{n-1}.$$

for $0 < u \leq t$.

[Hint: Work out the cdf first.]

Use this to find $\mathbb{E}(T_1|X(t) = n)$.

2. Let $T_1, T_2, \ldots$ be the arrival times of a Poisson process $X(t)$ of rate λ. Find the following

i) $\mathbb{E}(T_1 + T_2 + T_3|X(10) = 3)$,

ii) $\mathbb{E}(T_1 T_2 \ldots T_4|X(1) = 4)$,

iii) $\mathbb{E}(T_1^3 + T_2^3 + T_3^3 + T_4^3 + T_5^3|X(2) = 5)$,

3. Requests arrive at a server as a Poisson process of rate λ per minute. Every T minutes the requests are processed regardless of how many there are (even if there are none). Suppose that processing costs £k (regardless of how many requests are processed). In addition each request incurs a cost of £c for each minute it waits before processing.

a) Show that the expected cost per minute to run the server is

$$\frac{k}{T} + \frac{c\lambda T}{2}.$$

[Hint: First work it out conditioned on there being n requests waiting at time T.]

b) How should T be chosen to minimize the cost of running the server?

4. An island is initially uninhabited. Creatures arrive as a Poisson process of rate α per month. Once there each creature produces offspring as a Poisson process of rate β per month. Assume that the Poisson processes for arrivals and births are all independent. Assume also that there are no deaths and no creatures leave the island. Let $X(t)$ be the number of ceatures on the island at time t months and

$$p_n(t) = \mathbb{P}(X(t) = n).$$

a) What is the probability that the island is inhabited by time 3 months?

b) Explain why $X(t)$ is a birth process with parameters $\lambda_n = \alpha + n\beta$.

c) Write down equations for $p_0'(t), p_1'(t)$ and $p_2'(t)$ in terms of $p_0(t), p_1(t)$ and $p_2(t)$.

For the remainder of the question assume that $\alpha = 1$ and $\beta = 2$.

d) Solve the equations for part c).

e) An ecologist counts the number of creatures on the island at time 3 months. What is the probability that exactly 2 creatures are found?

f) What is the expectation of the time at which the population size first reaches 10?

5. Let $X(t)$ be a linear birth process (that is $X(0) = 1$ and $\lambda_n = n\lambda$ for $n \geq 1$). Let $p_n(t) = \mathbb{P}(X(t) = n)$ for $n \geq 1$.

a) Write down the differerential equations for the $p_n(t)$.

b) Show that they have solution $p_n(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$.

c) Use this to write down the distribution of $X(t)$ and find its expectation and variance.