Probability III – 2009/10

Exercise Sheet 7 JRJ

Please hand in your answer to Question 3 only to the red box on the second floor of the maths building by 10am on Thursday 11 March.

You are strongly encouraged to attempt all the questions on this sheet.

If you want feedback on your answers to any of the questions then see me in a class or office hour.

1. Remind yourself of the following results from Probability I and/or Probability II. In each case you should be able to find the result in your lecture notes or on a problem sheet or to prove it yourself using ideas from those modules.

a) If A has a $\text{Po}(\lambda)$ and B has a $\text{Po}(\mu)$ distribution and A and B are independent random variables then $A+B$ has a $\text{Po}(\lambda+\mu)$ distribution.

b) If the number of arrivals in a time interval has a $\text{Po}(\lambda)$ distribution and each arrival has probability p of being registered by a counter independently of all other arrivals then the number of arrivals registered has a $\text{Po}(p\lambda)$ distribution.

c) If E has an $\text{Exp}(\lambda)$ distribution and then for all $s,t \geq 0$

$$\mathbb{P}(E > s+t|E > s) = \mathbb{P}(E > t).$$

(This is sometimes described as the memoryless property of the Exponential distribution.)

2. At Stepney Green Underground station the arrival of Hammersmith and City Line trains forms a Poisson process of rate 10 per hour and the arrival of District Line trains forms a Poisson process of rate 15 per hour. These processes are independent. Suppose also that each train is full with probability $1/10$ independently of all other trains.

a) Show that the arrival of Underground trains at Stepney Green station forms a Poisson process of rate 25 per hour.

b) Show that the arrival of full trains at Stepney Green station forms a Poisson process of rate $5/2$ per hour.

(These properties are called superposition and thinning respectively).

3. Customers enter a shop according to a Poisson process of rate $1/2$ per minute. Let $C(t)$ be the number of customers who have entered the shop after it has been open for t minutes.

a) Calculate the following:

   i) $\mathbb{P}(C(10) = 3)$,
   ii) $\mathbb{P}(C(10) = 3 | C(5) = 0)$,
   iii) $\mathbb{P}(C(10) = 3, C(5) = 0)$,
   iv) $\mathbb{P}(C(10) = 0 | C(5) = 3)$,
   v) $\mathbb{P}(C(5) = 0 | C(10) = 3)$.

   (Your answer should be left involving powers of e but simplified in all other ways. You do not have to state which properties of the Poisson process you are using but you should be able to give this information if you were asked.)

b) Let T_{50} be the time at which the 50th customer enters the shop. Express T_{50} in terms of the interarrival times and hence find its expectation.

c) Suppose that each customer spends exactly 5 minutes in the shop. Find the distribution of the number of customers in the shop after it has been open for 1 hour.

d) What is the probability that the first and second customers to arrive do so within 1 minute of each other?

4.

a) A hitchhiker arrives at the side of a remote road at a random time. What is the expectation of the time he waits before a car passes under each of the following assumptions:

   i) cars pass along the road as a Poisson process of rate 1 per hour,
   ii) cars are uniformly spaced 1 hour apart.

b) How would you explain the difference in your answers to the two parts to non-mathematician?